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A CERTAIN FAMILY OF BI-UNIVALENT FUNCTIONS ASSOCIATED WITH THE PASCAL DISTRIBUTION SERIES BASED UPON THE HORADAM POLYNOMIALS

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Abstract. The purpose of this article is to introduce a new subclass $\mathcal{H}_\Sigma(\delta, \lambda, m, \theta, x)$ of analytic and bi-univalent functions by using the Horadam polynomials, which is associated with the Pascal distribution series and to investigate the bounds for $|a_2|$ and $|a_3|$, where a_2, a_3 are the initial Taylor-Maclaurin coefficients. Further we obtain the Fekete-Szegö inequality for functions in the class $\mathcal{H}_\Sigma(\delta, \lambda, m, \theta, x)$ which we have introduced here.

1 Introduction

We denote by $\mathcal{A}$ the class of functions which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$$

and have the following normalized form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

We also denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions which are also univalent in $\mathbb{U}$. According to the Koebe one-quarter theorem [5], every function $f \in \mathcal{S}$ has an inverse f^{-1} defined by

$$f^{-1}\big(f(z)\big) = z \qquad (z \in \mathbb{U})$$

and

$$f\big(f^{-1}(w)\big) = w \qquad \left(|w| < r_0(f);\ r_0(f) \geqq \frac{1}{4}\right),$$

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where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \quad (1.2)$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ stand for the class of bi-univalent functions in $\mathbb{U}$ given by (1.1). For a brief historical account and for several interesting examples of functions in the class Σ; see the pioneering work on this subject by Srivastava *et al.* [22], which actually revived the study of bi-univalent functions in recent years. From the work of Srivastava *et al.* [22], we choose to recall the following examples of functions in the class Σ :

$$\frac{z}{1-z}, \quad -\log(1-z) \quad \text{and} \quad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right).$$

We notice that the class Σ is not empty. However, the Koebe function is not a member of Σ.

In a considerably large number of sequels to the aforementioned work of Srivastava *et al.* [22], several different subclasses of the bi-univalent function class Σ were introduced and studied analogously by the many authors (see, for example, [1, 2, 3, 7, 15, 16, 17, 18, 19, 23, 25, 26, 27, 28, 29, 30]), but only non-sharp estimates on the initial coefficients $|a_2|$ and $|a_3|$ in the Taylor Maclaurin expansion (1.1) were obtained in several recent papers. The problem to find the general coefficient bounds on the Taylor-Maclaurin coefficients

$$|a_n| \qquad (n \in \mathbb{N};\ n \geqq 3)$$

for functions $f \in \Sigma$ is still not completely addressed for many of the subclasses of the bi-univalent function class Σ (see, for example, [17, 23, 25]). The Fekete-Szegö functional $\left|a_3 - \mu a_2^2\right|$ for $f \in \mathcal{S}$ is well known for its rich history in the field of Geometric Function Theory. Its origin was in the disproof by Fekete and Szegö [6] of the Littlewood-Paley conjecture that the coefficients of odd univalent functions are bounded by unity. The functional has since received great attention, particularly in the study of many subclasses of the family of univalent functions. This topic has become of considerable interest among researchers in Geometric Function Theory (see, for example, [20, 21, 24]).

Let the functions f and g be analytic in $\mathbb{U}$. We say that the function f is subordinate to g, if there exists a Schwarz function ω, which is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \qquad \text{and} \qquad |\omega(z)| < 1 \quad (z \in \mathbb{U}),$$

such that

$$f(z) = g\big(\omega(z)\big).$$

This subordination is denoted by

$$f \prec g \qquad \text{or} \qquad f(z) \prec g(z) \quad (z \in \mathbb{U}).$$

It is well known that (see [13]), if the function g is univalent in $\mathbb{U}$, then

$$f \prec g \quad (z \in \mathbb{U}) \iff f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subseteq g(\mathbb{U}).$$

The Horadam polynomials $h_n(x)$ are defined by the following recurrence relation (see [9]):

$$h_n(x) = pxh_{n-1}(x) + qh_{n-2}(x) \quad (x \in \mathbb{R};\ n \in \mathbb{N} = \{1, 2, 3, \cdots\}) \tag{1.3}$$

with

$$h_1(x) = a \quad \text{and} \quad h_2(x) = bx,$$

for some real constants a, b, p and q. The characteristic equation of the recurrence relation (1.3) is given by

$$t^2 - pxt - q = 0.$$

This equation has the following two real roots:

$$\alpha = \frac{px + \sqrt{p^2x^2 + 4q}}{2} \qquad \text{and} \qquad \beta = \frac{px - \sqrt{p^2x^2 + 4q}}{2}.$$

Remark 1. By selecting the particular values of a, b, p and q, the Horadam polynomial $h_n(x)$ reduces to several known polynomials. Some of these special cases are recorded below.

1. Taking $a = b = p = q = 1$, we obtain the Fibonacci polynomials $F_n(x)$.

2. Taking $a = 2$ and $b = p = q = 1$, we get the Lucas polynomials $L_n(x)$.

3. Taking $a = q = 1$ and $b = p = 2$, we have the Pell polynomials $P_n(x)$.

4. Taking $a = b = p = 2$ and $q = 1$, we find the Pell-Lucas polynomials $Q_n(x)$.

5. Taking $a = b = 1$, $p = 2$ and $q = -1$, we obtain the Chebyshev polynomials $T_n(x)$ of the first kind.

6. Taking $a = 1$, $b = p = 2$ and $q = -1$, we have the Chebyshev polynomials $U_n(x)$ of the second kind.

These polynomials, the families of orthogonal polynomials and other special polynomials, as well as their extensions and generalizations, are potentially important in a variety of disciplines in many branches of science, especially in the mathematical, statistical and physical sciences. For more information associated with these polynomials, see [8, 9, 11, 12]. The generating function of the Horadam polynomials $h_n(x)$ is given as follows (see [10]):

$$\Pi(x,z) = \sum_{n=1}^{\infty} h_n(x) z^{n-1} = \frac{a + (b-ap)xz}{1 - pxz - qz^2}. \tag{1.4}$$

A variable τ is said to be a Pascal distribution, if it takes on the values $0, 1, 2, 3, \cdots$ with the probabilities

$$(1-\theta)^m, \quad \frac{\theta m (1-\theta)^m}{1!}, \quad \frac{\theta^2 m(m+1)(1-\theta)^m}{2!}, \quad \frac{\theta^3 m(m+1)(m+2)(1-\theta)^m}{3!}, \cdots,$$

respectively, where θ and m are called the parameters of the Pascal distribution τ. Hence

$$\mathrm{Prob}(\tau = k) = \binom{k+m-1}{m-1} \theta^k (1-\theta)^m \qquad (k = 0, 1, 2, 3, \cdots).$$

Recently, El-Deeb *et al.* [4] introduced the following power series whose coefficients are probabilities of the Pascal distribution τ:

$$\Psi_\theta^m(z) = z + \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} \theta^{n-1} (1-\theta)^m z^n \qquad (z \in \mathbb{U};\ m \geqq 1;\ 0 \leqq \theta \leqq 1).$$

We note by the familiar Ratio Test that the radius of convergence of the above series is infinity. More recently, Murugusundaramoorthy *et al.* [14] introduced a linear operator $\mathcal{I}_\theta^m(z) : \mathcal{A} \longrightarrow \mathcal{A}$ which is defined as follows:

$$\mathcal{I}_\theta^m f(z) = \Psi_\theta^m(z) * f(z) = z + \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} \theta^{n-1} (1-\theta)^m a_n z^n \qquad z \in \mathbb{U},$$

where $*$ indicate the Hadamard product (or convolution) of two series.

The object of the present paper is to introduce a new subclass of Σ involving the Pascal distribution associated with Horadam polynomials $h_n(x)$. We obtain the estimates on the initial Taylor-Maclaurin coefficients and the Fekete-Szegö inequalities for this subclass of the bi-univalent function class Σ defined by means of the Horadam polynomials. We also give several illustrative examples of the bi-univalent function class which we introduce here.

2 A Set of Main Results

We begin this section by defining the new subclass $\mathcal{H}_\Sigma(\delta, \lambda, m, \theta, x)$ associated with Pascal distribution.

Definition 2. For $\delta \in \mathbb{C} \setminus \{0\} = \mathbb{C}^*$, $0 \leqq \lambda \leqq 1$, $m \geqq 1$, $0 \leqq \theta \leqq 1$ and $x \in \mathbb{R}$, a function $f \in \Sigma$ is said to be in the class $\mathcal{H}_\Sigma(\delta, \lambda, m, \theta, x)$ if it satisfies the following subordination conditions:

$$1 + \frac{1}{\delta}\left(\frac{\lambda z^2\big(\mathcal{I}_\theta^m f(z)\big)'' + z\big(\mathcal{I}_\theta^m f(z)\big)'}{\lambda z\big(\mathcal{I}_\theta^m f(z)\big)' + (1-\lambda)\mathcal{I}_\theta^m f(z)} - 1\right) \prec \Pi(x,z) + 1 - a$$

and

$$1 + \frac{1}{\delta}\left(\frac{\lambda w^2\big(\mathcal{I}_\theta^m g(w)\big)'' + w\big(\mathcal{I}_\theta^m g(w)\big)'}{\lambda w\big(\mathcal{I}_\theta^m g(w)\big)' + (1-\lambda)\mathcal{I}_\theta^m g(w)} - 1\right) \prec \Pi(x,w) + 1 - a,$$

where a is real constant and the function $g = f^{-1}$ is given by (1.2).

Example 3. For $\delta \in \mathbb{C}^*$, $\lambda = 0$, $m \geqq 1$, $0 \leqq \theta \leqq 1$ and $x \in \mathbb{R}$, a function $f \in \Sigma$ is said to be in the class $\mathcal{H}_\Sigma(\delta, 0, m, \theta, x) =: \mathcal{S}_\Sigma(\delta, m, \theta, x)$ if it satisfies the following subordination conditions:

$$1 + \frac{1}{\delta}\left(\frac{z\big(\mathcal{I}_\theta^m f(z)\big)'}{\mathcal{I}_\theta^m f(z)} - 1\right) \prec \Pi(x,z) + 1 - a$$

and

$$1 + \frac{1}{\delta}\left(\frac{w\big(\mathcal{I}_\theta^m g(w)\big)'}{\mathcal{I}_\theta^m g(w)} - 1\right) \prec \Pi(x,w) + 1 - a,$$

where a is real constant and the function $g = f^{-1}$ is given by (1.2).

Example 4. For $\delta \in \mathbb{C}^*$, $\lambda = 1$, $m \geqq 1$, $0 \leqq \theta \leqq 1$ and $x \in \mathbb{R}$, a function $f \in \Sigma$ is said to be in the class $\mathcal{H}_\Sigma(\delta, 1, m, \theta, x) =: \mathcal{K}_\Sigma(\delta, m, \theta, x)$ if it satisfies the following subordination conditions:

$$1 + \frac{1}{\delta}\left(\frac{z\big(\mathcal{I}_\theta^m f(z)\big)''}{\big(\mathcal{I}_\theta^m f(z)\big)'}\right) \prec \Pi(x,z) + 1 - a$$

and

$$1 + \frac{1}{\delta}\left(\frac{w\big(\mathcal{I}_\theta^m g(w)\big)''}{\big(\mathcal{I}_\theta^m g(w)\big)'}\right) \prec \Pi(x,w) + 1 - a,$$

where a is real constant and the function $g = f^{-1}$ is given by (1.2).

Our first main result is asserted by Theorem 5 below.

Theorem 5. *For $\delta \in \mathbb{C}^*$, $0 \leqq \lambda \leqq 1$, $m \geqq 1$, $0 \leqq \theta \leqq 1$ and $x \in \mathbb{R}$, let $f \in \mathcal{A}$ be in the class $\mathcal{H}_\Sigma(\delta, \lambda, m, \theta, x)$. Then*

$$|a_2| \leqq \frac{|\delta b x| \sqrt{|bx|}}{\sqrt{\left|\left(b\delta m\theta^2 (1-\theta)^m \varphi(\lambda, m, \theta) - pm^2\theta^2 (\lambda+1)^2 (1-\theta)^{2m}\right) bx^2 - qam^2\theta^2 (\lambda+1)^2 (1-\theta)^{2m}\right|}}$$

and

$$|a_3| \leqq \frac{1}{m\theta^2\,(1-\theta)^m}\left(\frac{|\delta b x|}{(m+1)(2\lambda+1)}+\frac{|\delta b x|^2}{m\,(\lambda+1)^2\,(1-\theta)^m}\right),$$

where

$$\varphi(\lambda, m, \theta) = (m+1)(2\lambda+1) - m\,(\lambda+1)^2\,(1-\theta)^m. \tag{2.1}$$

Proof. Let $f \in \mathcal{H}_\Sigma(\delta, \lambda, m, \theta, x)$. Then there are two analytic functions

$$u, v : \mathbb{U} \longrightarrow \mathbb{U}$$

given by

$$u(z) = u_1 z + u_2 z^2 + u_3 z^3 + \cdots \qquad (z \in \mathbb{U}) \tag{2.2}$$

and

$$v(w) = v_1 w + v_2 w^2 + v_3 w^3 + \cdots \qquad (w \in \mathbb{U}), \tag{2.3}$$

with

$$u(0) = v(0) = 0 \qquad \text{and} \qquad \max\{|u(z)|, |v(w)|\} < 1 \quad (z, w \in \mathbb{U}),$$

such that

$$\frac{1}{\delta}\left(\frac{\lambda z^2\big(\mathcal{I}_\theta^m f(z)\big)'' + z\big(\mathcal{I}_\theta^m f(z)\big)'}{\lambda z\big(\mathcal{I}_\theta^m f(z)\big)' + (1-\lambda)\mathcal{I}_\theta^m f(z)} - 1\right) = \Pi(x, u(z)) - a$$

and

$$\frac{1}{\delta}\left(\frac{\lambda w^2\big(\mathcal{I}_\theta^m g(w)\big)'' + w\big(\mathcal{I}_\theta^m g(w)\big)'}{\lambda w\big(\mathcal{I}_\theta^m g(w)\big)' + (1-\lambda)\mathcal{I}_\theta^m g(w)} - 1\right) = \Pi(x, v(w)) - a$$

or, equivalently, that

$$\frac{1}{\delta}\left(\frac{\lambda z^2\big(\mathcal{I}_\theta^m f(z)\big)'' + z\big(\mathcal{I}_\theta^m f(z)\big)'}{\lambda z\big(\mathcal{I}_\theta^m f(z)\big)' + (1-\lambda)\mathcal{I}_\theta^m f(z)} - 1\right) = h_1(x) + h_2(x)u(z) + h_3(x)u^2(z) + \cdots - a \tag{2.4}$$

and

$$\frac{1}{\delta}\left(\frac{\lambda w^2\big(\mathcal{I}_\theta^m g(w)\big)'' + w\big(\mathcal{I}_\theta^m g(w)\big)'}{\lambda w\big(\mathcal{I}_\theta^m g(w)\big)' + (1-\lambda)\mathcal{I}_\theta^m g(w)} - 1\right) = h_1(x) + h_2(x)v(w) + h_3(x)v^2(w) + \cdots - a. \tag{2.5}$$

Combining (2.2), (2.3), (2.4) and (2.5), we find that

$$\frac{1}{\delta}\left(\frac{\lambda z^2\big(\mathcal{I}_\theta^m f(z)\big)'' + z\big(\mathcal{I}_\theta^m f(z)\big)'}{\lambda z\big(\mathcal{I}_\theta^m f(z)\big)' + (1-\lambda)\mathcal{I}_\theta^m f(z)} - 1\right) = h_2(x)u_1 z + \left[h_2(x)u_2 + h_3(x)u_1^2\right] z^2 + \cdots \tag{2.6}$$

and

$$\frac{1}{\delta}\left(\frac{\lambda w^2\big(\mathcal{I}_\theta^m g(w)\big)''+w\big(\mathcal{I}_\theta^m g(w)\big)'}{\lambda w\big(\mathcal{I}_\theta^m g(w)\big)'+(1-\lambda)\mathcal{I}_\theta^m g(w)}-1\right)=h_2(x)v_1w+\left[h_2(x)v_2+h_3(x)v_1^2\right]w^2+\cdots. \tag{2.7}$$

It is well-known that, if

$$\max\{|u(z)|,|v(w)|\}<1 \qquad (z,w\in\mathbb{U}),$$

then

$$|u_j|\leqq 1 \qquad \text{and} \qquad |v_j|\leqq 1 \qquad (\forall\, j\in\mathbb{N}). \tag{2.8}$$

Now, by comparing the corresponding coefficients in (2.6) and (2.7), and after some simplification, we have

$$\frac{m\theta(\lambda+1)\,(1-\theta)^m}{\delta}\,a_2=h_2(x)u_1, \tag{2.9}$$

$$\frac{m\theta^2(m+1)(2\lambda+1)\,(1-\theta)^m}{\delta}\,a_3-\frac{m^2\theta^2\,(\lambda+1)^2\,(1-\theta)^{2m}}{\delta}\,a_2^2$$
$$=h_2(x)u_2+h_3(x)u_1^2, \tag{2.10}$$

$$-\frac{m\theta(\lambda+1)\,(1-\theta)^m}{\delta}\,a_2=h_2(x)v_1 \tag{2.11}$$

and

$$\frac{m\theta^2(m+1)(2\lambda+1)\,(1-\theta)^m}{\delta}\,(2a_2^2-a_3)-\frac{m^2\theta^2\,(\lambda+1)^2\,(1-\theta)^{2m}}{\delta}\,a_2^2$$
$$=h_2(x)v_2+h_3(x)v_1^2. \tag{2.12}$$

It follows from (2.9) and (2.11) that

$$u_1=-v_1 \tag{2.13}$$

and

$$\frac{2m^2\theta^2\,(\lambda+1)^2\,(1-\theta)^{2m}}{\delta^2}\,a_2^2=h_2^2(x)(u_1^2+v_1^2). \tag{2.14}$$

If we add (2.10) to (2.12), we find that

$$\frac{2m\theta^2\,(1-\theta)^m\,\varphi(\lambda,m,\theta)}{\delta}\,a_2^2=h_2(x)(u_2+v_2)+h_3(x)(u_1^2+v_1^2), \tag{2.15}$$

where $\varphi(\lambda,m,\theta)$ is given by (2.1).

Upon substituting the value of $u_1^2 + v_1^2$ from (2.14) into the right-hand side of (2.15), we deduce that

$$a_2^2 = \frac{\delta^2 h_2^3(x)(u_2+v_2)}{2\left(\delta m\theta^2 (1-\theta)^m \varphi(\lambda,m,\theta)h_2^2(x) - m^2\theta^2 (\lambda+1)^2 (1-\theta)^{2m}\ h_3(x)\right)}. \tag{2.16}$$

By further computations using (1.3), (2.8) and (2.16), we obtain

$$|a_2| \leqq \frac{|\delta|\,|bx|\sqrt{|bx|}}{\sqrt{\left|\left(b\delta m\theta^2 (1-\theta)^m \varphi(\lambda,m,\theta) - pm^2\theta^2 (\lambda+1)^2 (1-\theta)^{2m}\right) bx^2 - qam^2\theta^2 (\lambda+1)^2 (1-\theta)^{2m}\right|}}.$$

Next, if we subtract (2.12) from (2.10), we can easily see that

$$\frac{2m\theta^2(m+1)(2\lambda+1)(1-\theta)^m}{\delta}(a_3 - a_2^2) = h_2(x)(u_2 - v_2) + h_3(x)(u_1^2 - v_1^2). \tag{2.17}$$

In view of (2.13) and (2.14), we find from (2.17) that

$$a_3 = \frac{\delta h_2(x)(u_2 - v_2)}{2m\theta^2(m+1)(2\lambda+1)(1-\theta)^m} + \frac{\delta^2 h_2^2(x)(u_1^2+v_1^2)}{2m^2\theta^2(\lambda+1)^2(1-\theta)^{2m}}.$$

Thus, by applying (1.3), we obtain

$$|a_3| \leqq \frac{1}{m\theta^2(1-\theta)^m}\left(\frac{|\delta bx|}{(m+1)(2\lambda+1)} + \frac{|\delta bx|^2}{m(\lambda+1)^2(1-\theta)^m}\right).$$

This completes the proof of Theorem 5. □

In the next theorem, we present the Fekete-Szegö inequality for $f \in \mathcal{H}_\Sigma(\delta,\lambda,m,\theta,x)$.

Theorem 6. *For $\delta \in \mathbb{C}^*$, $0 \leqq \lambda \leqq 1$, $m \geqq 1$, $0 \leqq \theta \leqq 1$ and $x,\mu \in \mathbb{R}$, let $f \in \mathcal{A}$ be in the class $\mathcal{H}_\Sigma(\delta,\lambda,m,\theta,x)$. Then*

$$\left|a_3 - \mu a_2^2\right| \leqq \begin{cases} \frac{|\delta||bx|}{m\theta^2(m+1)(2\lambda+1)(1-\theta)^m}; \\ \left(|\mu-1| \leqq \frac{\left|\left(b\delta m\theta^2(1-\theta)^m\varphi(\lambda,m,\theta)-pm^2\theta^2(\lambda+1)^2(1-\theta)^{2m}\right)bx^2 - qam^2\theta^2(\lambda+1)^2(1-\theta)^{2m}\right|}{b^2x^2|\delta|m\theta^2(m+1)(2\lambda+1)(1-\theta)^m}\right) \\ \\ \frac{|\delta|^2|bx|^3|\mu-1|}{\left|\left(b\delta m\theta^2(1-\theta)^m\varphi(\lambda,m,\theta)-pm^2\theta^2(\lambda+1)^2(1-\theta)^{2m}\right)bx^2 - qam^2\theta^2(\lambda+1)^2(1-\theta)^{2m}\right|}; \\ \left(|\mu-1| \geqq \frac{\left|\left(b\delta m\theta^2(1-\theta)^m\varphi(\lambda,m,\theta)-pm^2\theta^2(\lambda+1)^2(1-\theta)^{2m}\right)bx^2 - qam^2\theta^2(\lambda+1)^2(1-\theta)^{2m}\right|}{b^2x^2|\delta|m\theta^2(m+1)(2\lambda+1)(1-\theta)^m}\right). \end{cases}$$

Proof. It follows from (2.16) and (2.17) that

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{\delta h_2(x)(u_2 - v_2)}{2m\theta^2(m+1)(2\lambda+1)\,(1-\theta)^m} + (1-\mu)\, a_2^2 \\ &= \frac{\delta h_2(x)(u_2 - v_2)}{2m\theta^2(m+1)(2\lambda+1)\,(1-\theta)^m} \\ &\quad + \frac{\delta^2 h_2^3(x)(u_2+v_2)\,(1-\mu)}{2\left(\delta m\theta^2\,(1-\theta)^m\,\varphi(\lambda,m,\theta)h_2^2(x) - m^2\theta^2\,(\lambda+1)^2\,(1-\theta)^{2m}\ \ h_3(x)\right)} \\ &= \frac{h_2(x)}{2}\left[\left(\psi(\mu,x) + \frac{\delta}{m\theta^2(m+1)(2\lambda+1)\,(1-\theta)^m}\right)u_2 \right. \\ &\quad \left. + \left(\psi(\mu,x) - \frac{\delta}{m\theta^2(m+1)(2\lambda+1)\,(1-\theta)^m}\right)v_2\right], \end{aligned}$$

where

$$\psi(\mu,x) = \frac{\delta^2 h_2^2(x)\,(1-\mu)}{\delta m\theta^2\,(1-\theta)^m\ \ \varphi(\lambda,m,\theta)h_2^2(x) - m^2\theta^2\,(\lambda+1)^2\,(1-\theta)^{2m}\ \ h_3(x)}.$$

Thus, according to (1.3), we have

$$\left|a_3 - \mu a_2^2\right| \leqq \begin{cases} \frac{|\delta||bx|}{m\theta^2(m+1)(2\lambda+1)(1-\theta)^m}; \\ \left(0 \leqq |\psi(\mu,x)| \leqq \frac{|\delta|}{m\theta^2(m+1)(2\lambda+1)(1-\theta)^m}\right) \\ \left(|bx|\,|\psi(\mu,x)|\right); \\ \left(|\psi(\mu,x)| \geqq \frac{|\delta|}{m\theta^2(m+1)(2\lambda+1)(1-\theta)^m}\right), \end{cases}$$

which, after simple computation, yields

$$\left|a_3 - \mu a_2^2\right| \leqq \begin{cases} \frac{|\delta||bx|}{m\theta^2(m+1)(2\lambda+1)(1-\theta)^m}; \\ \left(|\mu-1| \leqq \frac{\left|\left(b\delta m\theta^2(1-\theta)^m\varphi(\lambda,m,\theta) - pm^2\theta^2(\lambda+1)^2(1-\theta)^{2m}\right)bx^2 - qam^2\theta^2(\lambda+1)^2(1-\theta)^{2m}\right|}{b^2x^2|\delta|m\theta^2(m+1)(2\lambda+1)(1-\theta)^m}\right) \\ \frac{|\delta|^2|bx|^3|\mu-1|}{\left|\left(b\delta m\theta^2(1-\theta)^m\varphi(\lambda,m,\theta) - pm^2\theta^2(\lambda+1)^2(1-\theta)^{2m}\right)bx^2 - qam^2\theta^2(\lambda+1)^2(1-\theta)^{2m}\right|}; \\ \left(|\mu-1| \geqq \frac{\left|\left(b\delta m\theta^2(1-\theta)^m\varphi(\lambda,m,\theta) - pm^2\theta^2(\lambda+1)^2(1-\theta)^{2m}\right)bx^2 - qam^2\theta^2(\lambda+1)^2(1-\theta)^{2m}\right|}{b^2x^2|\delta|m\theta^2(m+1)(2\lambda+1)(1-\theta)^m}\right). \end{cases}$$

We have thus completed the proof of Theorem 6. □

3 Corollaries and Consequences

Our main results (Theorem 5 and Theorem 6) can be specialized to deduce a number of known or new results as their corollaries and consequences dealing with the initial Taylor-Maclaurin inequalities and the Fekete-Szegö inequalities. We choose to record here one example in which, by putting $\mu = 1$ in Theorem 6, we are led to the following corollary.

Corollary. *For $\delta \in \mathbb{C}^*$, $0 \leqq \lambda \leqq 1$, $m \geqq 1$, $0 \leqq \theta \leqq 1$ and $x \in \mathbb{R}$, let $f \in \mathcal{A}$ be in the class $\mathcal{H}_\Sigma(\delta, \lambda, m, \theta, x)$. Then*

$$\left|a_3 - a_2^2\right| \leqq \frac{|\delta|\,|bx|}{m\theta^2(m+1)(2\lambda+1)\,(1-\theta)^m}.$$

Remark 7. By taking some or all of the particular cases of the Horadam polynomial sequence $h_n(x)$ as shown in Remark 1 and using the same technique as in Section 2 above, we can derive analogous results for normalized analytic and bi-univalent functions in the class $\mathcal{H}_\Sigma(\delta, \lambda, m, \theta, x)$ associated with the Pascal distribution series. Furthermore, by suitably specializing the parameter λ, one can deduce the results for the subclasses $\mathcal{S}_\Sigma(\delta, m, \theta, x)$ and $\mathcal{K}_\Sigma(\delta, m, \theta, x)$ which are defined, respectively, in Example 3 and 4 and associated with the Pascal distribution series. The details involved may be left as an exercise for the interested reader.

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