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AN APPLICATION OF GENERALIZED DISTRIBUTION SERIES ON CERTAIN CLASSES OF UNIVALENT FUNCTIONS ASSOCIATED WITH CONIC DOMAINS

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Abstract. The purpose of the present paper is to obtain some necessary and sufficient conditions for generalized distribution series belonging to the classes $\mathcal{UCV}(\alpha,\beta)$, $\mathcal{S}_{\mathcal{P}}(\alpha,\beta)$, $\mathcal{UCV}(\beta)$, $C(\lambda)$, $S^*(\lambda)$ and obtain some inclusion relation between the classes $\Re^\tau(A,B)$ and $\mathcal{UCV}(\alpha,\beta)$, $\mathcal{UCV}(\beta)$, $C(\lambda)$. Finally, we attain some necessary and sufficient conditions of integral operator associated with the generalized distribution series for the classes $\mathcal{UCV}(\alpha,\beta)$, $\mathcal{UCV}(\beta)$ and $C(\lambda)$. Some particular cases of our main results are briefly indicated.

1 Introduction

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in C \text{ and } |z| < 1\}$. As usual, by $\mathcal{S}$ we shall represent the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$ and further, we denote $\mathcal{T}$ be the subclass of $\mathcal{S}$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n. \tag{1.2}$$

The convolution (or Hadamard product) of two series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is defined as the power series

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

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A function $f \in \mathcal{A}$ is said to be starlike of order α $(0 \leq \alpha < 1)$, if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad (z \in \mathbb{U}).$$

This function class is denoted by $\mathcal{S}^*(\alpha)$. We also write $\mathcal{S}^*(0) =: \mathcal{S}^*$, where $\mathcal{S}^*$ denotes the class of functions $f \in \mathcal{A}$ that $f(\mathbb{U})$ is starlike with respect to the origin.

A function $f \in \mathcal{A}$ is said to be convex of order α $(0 \leq \alpha < 1)$, if and only if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad (z \in \mathbb{U}).$$

This class is denoted by $\mathcal{K}(\alpha)$. Further, $\mathcal{K} = \mathcal{K}(0)$, the well-known standard class of convex functions.

It is an established fact that $f \in \mathcal{K}(\alpha) \Longleftrightarrow zf' \in \mathcal{S}^*(\alpha)$.

The classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ were initially introduced and studied by Robertson [31] and Silverman [32].

For some α $(0 \leq \alpha < 1)$, $\beta \geq 0$ and functions of the form (1.1), we let $\mathcal{S}_p(\alpha, \beta)$ be the subclass of $\mathcal{S}$ satisfying the analytic criteria

$$\Re\left(\frac{zf'(z)}{f(z)} - \alpha\right) > \beta\left|\frac{zf'(z)}{f(z)} - 1\right|, \ z \in \mathbb{U},$$

and also let $\mathcal{UCV}(\alpha, \beta)$ the subclass of $\mathcal{S}$ satisfying the analytic criteria

$$\Re\left(1 + \frac{zf''(z)}{f'(z)} - \alpha\right) > \beta\left|\frac{zf''(z)}{f'(z)}\right|, \ z \in \mathbb{U}.$$

The classes $\mathcal{S}_{\mathcal{P}}(\alpha, \beta)$ and $\mathcal{UCV}(\alpha, \beta)$ studied by Bharti *et al.* [7].

By specialzing the parameters in $\mathcal{S}_{\mathcal{P}}(\alpha, \beta)$ and $\mathcal{UCV}(\alpha, \beta)$, we obtain following known subclasses studied earlier by various researchers

1. $\mathcal{S}_{\mathcal{P}}(0, \beta) \equiv \mathcal{S}_{\mathcal{P}}(\beta)$ and $\mathcal{UCV}(\alpha, \beta) \equiv \mathcal{UCV}(\beta)$ were studied by Kanas and Wisniowaska [18], [17], (see also [19]).

2. $\mathcal{S}_{\mathcal{P}}(0, 1) \equiv \mathcal{S}_{\mathcal{P}}$ and $\mathcal{UCV}(0, 1) \equiv \mathcal{UCV}$ studied by Goodman [15], [16].

3. $\mathcal{S}_{\mathcal{P}}(\alpha, 0) \equiv \mathcal{S}^*(\alpha)$ and $\mathcal{UCV}(\alpha, 0) \equiv \mathcal{K}(\alpha)$ studied by Robertson [31] and Silverman [32].

In 1998, Ponnusamy and Rønning [22] introduced the class $S^*(\lambda)$, be the subclass of $\mathcal{T}$ consisting of functions which satisfy the condition

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \lambda, \quad (\lambda > 0, z \in \mathbb{U}). \tag{1.3}$$

Also, they introduce $C(\lambda)$ be the subclass of $\mathcal{T}$ consisting of functions which satisfy the condition

$$\left|\frac{zf''(z)}{f'(z)}\right| < \lambda, \quad (\lambda > 0, z \in \mathbb{U}). \tag{1.4}$$

By using (1.3) and (1.4) we have

$$f(z) \in C(\lambda) \Leftrightarrow zf'(z) \in S^*(\lambda).$$

A function $f \in \mathcal{A}$ is said to be in the class $f \in \Re^\tau(A, B)$ $(\tau \in \mathbb{C}\backslash\{0\}, -1 \leq B < A \leq 1)$, if it satisfies the inequality

$$\left|\frac{f'(z) - 1}{(A - B)\tau - B[f'(z) - 1]}\right| < 1, \quad (z \in \mathbb{U}).$$

The class $\Re^\tau(A, B)$ was introduced earlier by Dixit and Pal [10].

The applications of hypergeometric functions ([22], [34]), confluent hypergeometric functions [8], generalized hypergeometric functions [13], Wright function [30], Fox-Wright function [9], generalized Bessel functions ([5], [27]) are play an important role in Geometric Function Theory. In 2014, Porwal [25] (see also [2], [23]) introduced Poisson distribution series and obtain necessary and sufficient conditions for certain classes of univalent functions and co-relates probability density function with Geometric Function Theory. After the appearance of this paper several researchers introduced hypergeometric distribution series [1], Mittag-Leffler type Poisson distribution series [4], Pascal distribution series [11], hypergeometric distribution type series [28], confluent hypergeometric distribution series [29], Binomial distribution series [24], Borel distribution series [35] (see also [20]) and obtain some interesting properties of various classes of univalent functions. Recently Porwal [26] introduced generalized distribution series and obtain some necessary and sufficient conditions belonging to the certain classes of univalent functions. The study of generalized distribution series is of special interest because this distribution is a generalization of almost all the discrete probability distribution. For detailed study one may refer to [26]. Now, we recall the definition of generalized distribution. Let the series $\sum_{n=0}^{\infty} t_n$, where $t_n \geq 0, \forall n \in N$ is convergent and its sum is denoted by S, i.e.

$$S = \sum_{n=0}^{\infty} t_n. \tag{1.5}$$

Now, we introduce the generalized discrete probability distribution whose probability mass function is

$$p(n) = \frac{t_n}{S}, \quad n = 0, 1, 2, \ldots \tag{1.6}$$

Obviously $p(n)$ is a probability mass function because $p(n) \geq 0$ and $\sum_n p_n = 1$.

Now, we introduce the series

$$\phi(x) = \sum_{n=0}^{\infty} t_n x^n. \tag{1.7}$$

From (1.5) it is easy to see that the series given by (1.7) is convergent for $|x| < 1$ and for $x = 1$ it is also convergent.

Now, we introduce a power series whose coefficients are probabilities of the generalized distribution

$$K_\phi(z) = z + \sum_{n=2}^{\infty} \frac{t_{n-1}}{S} z^n. \tag{1.8}$$

Further, we define the function

$$TK_\phi(z) = z - \sum_{n=2}^{\infty} \frac{t_{n-1}}{S} z^n.$$

Next, we introduce the convolution operator $TK_\phi(f, z)$ for functions f of the form (1.2) as follows

$$TK_\phi(f, z) = K_\phi(z) * f(z) = z - \sum_{n=2}^{\infty} |a_n| \frac{t_{n-1}}{S} z^n. \tag{1.9}$$

In the present paper, motivated with the above metioned work, we obtain necessary and sufficient conditions for generalized distribution series belonging to the classes $\mathcal{UCV}(\alpha, \beta)$, $\mathcal{S_P}(\alpha, \beta)$, $\mathcal{UCV}(\beta)$, $C(\lambda)$, $S^*(\lambda)$ and obtain some inclusion relation between the classes $\Re^\tau(A, B)$ and $\mathcal{UCV}(\alpha, \beta)$, $\mathcal{UCV}(\beta)$, $C(\lambda)$. Finally, we obtain some necessary and sufficient condition of integral operator associated with the generalized distribution series for the classes $\mathcal{UCV}(\alpha, \beta)$, $\mathcal{UCV}(\beta)$ and $C(\lambda)$.

2 Preliminary Results

To establish our main results we shall require the following lemmas.

Lemma 1. *([7]) A function $f \in \mathcal{A}$ and of the form (1.1) belongs to the class $\mathcal{S_P}(\alpha, \beta)$ if*

$$\sum_{n=2}^{\infty} [n(1+\beta) - (\alpha+\beta)]|a_n| \le 1 - \alpha. \tag{2.1}$$

Lemma 2. *([7]) A function $f \in \mathcal{A}$ and of the form (1.1) belongs to the class $\mathcal{UCV}(\alpha, \beta)$ if*

$$\sum_{n=2}^{\infty} n[n(1+\beta) - (\alpha+\beta)]|a_n| \le 1 - \alpha.$$

Lemma 3. *([17]) Let $f \in \mathcal{A}$ and have the form (1.1). If for some β, $0 \leq \beta < \infty$, the inequality*

$$\sum_{n=2}^{\infty} n(n-1)|a_n| \leq \frac{1}{(\beta+2)},$$

holds, then $f \in \mathcal{UCV}(\beta)$. The number $\frac{1}{\beta+2}$ can not be increased.

Lemma 4. *([22]) A function $f(z)$ defined by (1.2) is in the class $S^*(\lambda)$, if and only if*

$$\sum_{n=2}^{\infty} [\lambda+n-1]\,|a_n| \leq \lambda.$$

Lemma 5. *([22]) A function $f(z)$ defined by (1.2) is in the class $C(\lambda)$, if and only if*

$$\sum_{n=2}^{\infty} n\,[\lambda+n-1]\,|a_n| \leq \lambda.$$

Lemma 6. *[10] A function $f \in \Re^{\tau}(A,B)$ is of form (1.1), then*

$$|a_n| \leq (A-B)\frac{|\tau|}{n}, \quad n \in \mathbb{N}\backslash\{1\}. \tag{2.2}$$

The bound given in (2.2) is sharp.

3 Main Results

In our first result, we obtain a sufficient condition for the distribution series $K_\phi(z)$ belonging to the class $\mathcal{UCV}(\alpha,\beta)$.

Theorem 7. *If $f \in \mathcal{A}$ is of the form (1.1) and the inequality*

$$(1+\beta)\phi''(1) + (2\beta+3-\alpha)\phi'(1) \leq (1-\alpha)\phi(0) \tag{3.1}$$

is satisfied, then $K_\phi(z)$ of the form (1.8) is in the class $\mathcal{UCV}(\alpha,\beta)$.

Proof. To prove that $K_\phi(z) \in \mathcal{UCV}(\alpha,\beta)$ from Lemma 2 it suffices to prove that

$$\sum_{n=2}^{\infty} n[n(1+\beta)-(\alpha+\beta)]\frac{t_{n-1}}{S} \leq 1-\alpha.$$

Now

$$\begin{aligned}
&\sum_{n=2}^{\infty} n[n(1+\beta)-(\alpha+\beta)]\frac{t_{n-1}}{S}\\
&=\frac{1}{S}\left[\sum_{n=2}^{\infty}(1+\beta)n(n-1)+(2\beta+3-\alpha)(n-1)+(1-\alpha)\right]t_{n-1}\\
&=\frac{1}{S}\sum_{n=1}^{\infty}[(1+\beta)n(n-1)t_n+n(2\beta+3-\alpha)t_n+(1-\alpha)t_n]\\
&=\frac{1}{S}\left[(1+\beta)\phi^{''}(1)+(2\beta+3-\alpha)\phi'(1)+(1-\alpha)\left(\phi(1)-\phi(0)\right)\right]\\
&\leq 1-\alpha.
\end{aligned}$$

This completes the proof of Theorem 7. □

Theorem 8. *If $f\in\mathcal{A}$ is of the form (1.1) and the inequality*

$$(1+\beta)\phi^{'}(1)\leq(1-\alpha)\phi(0). \tag{3.2}$$

is satisfied, then $K_\phi(z)$ of the form (1.8) is in the class $\mathcal{S}_{\mathcal{P}}(\alpha,\beta)$.

Proof. To prove that $K_\phi(z)\in\mathcal{S}_{\mathcal{P}}(\alpha,\beta)$ from Lemma 1 it suffices to prove that

$$\sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)]\frac{t_{n-1}}{S}\leq 1-\alpha.$$

Now

$$\begin{aligned}
&\sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)]\frac{t_{n-1}}{S}\\
&=\frac{1}{S}\left[\sum_{n=2}^{\infty}(1+\beta)(n-1)+(1-\alpha)\right]t_{n-1}\\
&=\frac{1}{S}\sum_{n=1}^{\infty}[(1+\beta)nt_n+(1-\alpha)t_n]\\
&=\frac{1}{S}\left[(1+\beta)\phi^{'}(1)+(1-\alpha)\left(\phi(1)-\phi(0)\right)\right]\\
&\leq 1-\alpha.
\end{aligned}$$

Thus the proof of Theorem 8 is established. □

Theorem 9. *If $f\in\mathcal{A}$ is of the form (1.1) and the inequality*

$$\phi^{'}(1)\leq\lambda\phi(0). \tag{3.3}$$

is satisfied, then $K_\phi(z)\in\mathcal{S}^*(\lambda)$.

Proof. To prove that $K_\phi(z) \in \mathcal{S}^*(\lambda)$ from Lemma 4 it suffices to prove that

$$\sum_{n=2}^{\infty}[n+\lambda-1]\frac{t_{n-1}}{S} \leq 1-\alpha.$$

Now

$$\begin{aligned}
&\sum_{n=2}^{\infty}[n+\lambda-1]\frac{t_{n-1}}{S}\\
&=\frac{1}{S}\left[\sum_{n=2}^{\infty}[(n-1)+\lambda]\right]t_{n-1}\\
&=\frac{1}{S}\sum_{n=1}^{\infty}[nt_n+\lambda t_n]\\
&=\frac{1}{S}\left[\phi^{'}(1)+\lambda\left(\phi(1)-\phi(0)\right)\right]\\
&\leq \lambda.
\end{aligned}$$

Thus the proof of Theorem 9 is established. □

Theorem 10. *If $f \in \mathcal{A}$ is of the form (1.1) and the inequality*

$$\phi^{''}(1)+(2+\lambda)\phi'(1) \leq \lambda\phi(0). \tag{3.4}$$

is satisfied then $K_\phi(z) \in C(\lambda)$.

Proof. The proof of this theorem is much akin to that of Theorem 9. Therefore we omit the details involved. □

Theorem 11. *If $f \in \mathcal{A}$ is of the form (1.1) and the inequality*

$$\phi^{''}(1)+2\phi'(1) \leq \frac{S}{2+\beta} \tag{3.5}$$

is satisfied, then $K_\phi(z) \in \mathcal{UCV}(\beta)$.

Proof. To prove that $K_\phi(z) \in \mathcal{S}_{\mathcal{P}}(\alpha,\beta)$ from Lemma 3 it suffices to prove that

$$\sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)]\frac{t_{n-1}}{S} \leq 1-\alpha.$$

Now

$$\begin{aligned}
&\sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)]\frac{t_{n-1}}{S} \\
&=\frac{1}{S}\left[\sum_{n=2}^{\infty}(1+\beta)(n-1)+(1-\alpha)\right]t_{n-1} \\
&=\frac{1}{S}\sum_{n=1}^{\infty}\left[(1+\beta)nt_n+(1-\alpha)t_n\right] \\
&=\frac{1}{S}\left[(1+\beta)\phi^{'}(1)+(1-\alpha)\left(\phi(1)-\phi(0)\right)\right] \\
&\leq 1-\alpha.
\end{aligned}$$

Thus the proof of Theorem 11 is established. □

Theorem 12. *If $f \in R^{\tau}(A,B)$ is of the form (1.2) and the operator $TK_{\phi}(f,z)$ defined by (1.9) is in the class $\mathcal{UCV}(\alpha,\beta)$, if and only if*

$$\frac{(A-B)|\tau|}{S}\left[(1+\beta)\phi^{'}(1)+(1-\alpha)\left(\phi(1)-\phi(0)\right)\right]\leq 1-\alpha. \tag{3.6}$$

Proof. To prove $TK_{\phi}(f,z)\in\mathcal{UCV}(\alpha,\beta)$, from Lemma 2, it suffices to prove that

$$P_1=\sum_{n=2}^{\infty}n\left[n-\lambda\alpha n-\alpha+\lambda\alpha\right]|a_n|\leq 1-\alpha.$$

Since $f\in R^{\tau}(A,B)$ then by using Lemma 6 we have

$$|a_n|\leq\frac{(A-B)|\tau|}{n}.$$

Hence

$$\begin{aligned}
P_1 \leq&\frac{(A-B)|\tau|}{S}\sum_{n=2}^{\infty}\left[n(1+\beta)-(\alpha+\beta)\right]t_{n-1} \\
=&\frac{(A-B)|\tau|}{S}\sum_{n=1}^{\infty}\left[(n+1)(1+\beta)-(\alpha+\beta)\right]t_n \\
=&\frac{(A-B)|\tau|}{S}\sum_{n=1}^{\infty}\left[n(1+\beta)+(1-\alpha)\right]t_n \\
=&\frac{(A-B)|\tau|}{S}\left[(1+\beta)\phi^{'}(1)+(1-\alpha)\left(\phi(1)-\phi(0)\right)\right] \\
&\leq 1-\alpha.
\end{aligned}$$

Thus the proof of Theorem 12 is established. □

Theorem 13. *If $f \in R^{\tau}(A,B)$ is of the form (1.2) and the operator $TK_{\phi}(f,z)$ defined by (1.9) is in the class $\mathcal{UCV}(\beta)$, if and only if*

$$\frac{(A-B)|\tau|}{S}\phi^{'}(1) \leq \frac{1}{\beta+2}. \tag{3.7}$$

Proof. The proof of above theorem is much akin to that of Theorem 12. Hence we omit the detail. □

Theorem 14. *If $f \in R^{\tau}(A,B)$ is of the form (1.2) and the operator $TK_{\phi}(f,z)$ defined by (1.9) is in the class $C(\lambda)$, if and only if*

$$\frac{(A-B)|\tau|}{S}\left[\phi^{'}(1)+\lambda\left(\phi(1)-\phi(0)\right)\right] \leq \lambda. \tag{3.8}$$

Proof. The proof of above theorem is much akin to that of Theorem 12. Hence we omit the detail. □

4 An Integral Operator

In this section, we introduce a integral operator $TG_{\phi}(z)$ as follows

$$TG_{\phi}(z) = \int_0^z \frac{TK_{\phi(t)}}{t}dt, \tag{4.1}$$

and obtain a necessary and sufficient condition for $TG_{\phi}(z)$ belonging to the classes $\mathcal{UCV}(\alpha,\beta)$, $\mathcal{UCV}(\beta)$ and $C(\lambda)$.

Theorem 15. *If $TK_{\phi}(z)$ defined by (1.9), then $TG_{\phi}(z)$ defined by (4.1) is in the class $\mathcal{UCV}(\alpha,\beta)$, if and only if (3.2) satisfies.*

Proof. Since

$$TG_{\phi}(z) = z - \sum_{n=2}^{\infty}\frac{t_{n-1}}{nS}z^n, \tag{4.2}$$

by Lemma 2, we have to prove that

$$\sum_{n=2}^{\infty} n\left[n(1+\beta)-(\alpha+\beta)\right]\frac{t_{n-1}}{nS} \leq 1-\alpha.$$

Now

$$\begin{aligned}
&\sum_{n=2}^{\infty} n\left[n(1+\beta)-(\alpha+\beta)\right]\frac{t_{n-1}}{nS}\\
=&\frac{1}{S}\sum_{n=2}^{\infty}\left[n(1+\beta)-(\alpha+\beta)\right]t_{n-1}\\
&=\frac{1}{S}\sum_{n=1}^{\infty}\left[(1+\beta)(n+1)-(\alpha+\beta)\right]t_{n}\\
&=\frac{1}{S}\sum_{n=1}^{\infty}\left[(1+\beta)n+(1-\alpha)\right]t_{n}\\
&=\frac{1}{S}\left[(1+\beta)\phi^{'}(1)+(1-\alpha)\left[\phi(1)-\phi(0)\right]\right]\\
&\leq 1-\alpha.
\end{aligned}$$

This completes the proof of Theorem 15. □

Theorem 16. *If $TK_{\phi}(z)$ defined by (1.9), then $TG_{\phi}(z)$ defined by (4.1) is in the class $C(\lambda)$, if and only if (3.3) satisfies.*

Proof. To prove that $TG_{\phi}(z)$ defined by (4.2) is in the class $\mathcal{C}(\lambda)$ from Lemma 5, it suffices to show that

$$\sum_{n=2}^{\infty} n[n+\lambda-1]\frac{t_{n-1}}{nS}\leq 1-\alpha.$$

Now

$$\begin{aligned}
&\sum_{n=2}^{\infty} n[n+\lambda-1]\frac{t_{n-1}}{nS}\\
&=\frac{1}{S}\left[\sum_{n=2}^{\infty}[(n-1)+\lambda]\right]t_{n-1}\\
&=\frac{1}{S}\sum_{n=1}^{\infty}[nt_n+\lambda t_n]\\
&=\frac{1}{S}\left[\phi^{'}(1)+\lambda\left(\phi(1)-\phi(0)\right)\right]\\
&\leq\lambda,\qquad (\text{by}(3.3)).
\end{aligned}$$

Thus the proof of Theorem 16 is established. □

Theorem 17. *If $TK_{\phi}(z)$ defined by (1.9), then $TG_{\phi}(z)$ defined by (4.1) is in the class $\mathcal{UCV}(\beta)$, if and only if*

$$\phi^{'}(1)\leq\frac{S}{2+\beta}.$$

Proof. The proof of above theorem is much akin to the proof of Theorem 16. Therefore we omit the details involved. □

Remark 18. *If we take* $t_n = \frac{m^n}{n!}$ *then the results of Theorem 7-17 reduce to the corresponding results for Poisson distribution series.*

Remark 19. *If we take* $t_n = \frac{(a)_n}{(c)_n}\frac{m^n}{n!}$ *then the results of Theorem 7-17 reduce to the corresponding results for confluent hypergeometric distribution series.*

Remark 20. *If we take* $t_n = \frac{(a)_n(b)_n}{(c)_n}\frac{m^n}{n!}$ *then the results of Theorem 7-17 reduce to the corresponding results for hypergeometric distribution type series.*

Remark 21. *If we take* $t_n = \frac{m^n}{\Gamma(\alpha n+\beta)}$ *then the results of Theorem 7-17 reduce to the corresponding results for Mittag-Leffler type Poisson distribution series.*

5 Scope of the Work and Conclusion

This paper deals with the necessary and sufficient conditions for generalized distribution series on various subclasses of univalent functions of conic domains. We also obtain some inclusion relation and studied integral operator associated with the generalized distribution series. Recently Hankel determinant [12], bounds for probabilities of the generalized distribution [21] have been investigated. Further properties of this distribution series like neighbourhood problems [3], Turan-type inequalities [6], Integral means inequalities and convolution properties [14], partial sums [33] may be investigated in future. We hope that this distribution series play a significant role in several branches of Mathematics, Science and Technology.

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