

A NOTE ON PROJECTIVE MODULES

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Abstract. This expository note delves into the theory of projective modules parallel to the one developed for injective modules by Matlis. Given a perfect ring R , we present a characterization of indecomposable projective R -modules and describe a one-to-one correspondence between the projective indecomposable R -modules and the simple R -modules.

1 Introduction

Throughout this note, R denotes an associative ring with identity. All modules are assumed to be left and unitary.

There is a well-known theorem in the representation theory of finite-dimensional algebras over a field which provides a 1-1 correspondence between the isomorphism classes of indecomposable projective modules over the algebra on the one hand, and the isomorphism classes of simple modules over the algebra on the other hand. More specifically, let k be a field and A a finite-dimensional k -algebra. Then every simple A -module is a quotient of some indecomposable projective A -module which is unique up to isomorphism. Conversely, for every indecomposable projective A -module P , there is a simple A -module, unique up to isomorphism, that is a quotient of P by some maximal submodule; see [Le]. For a generalization of this result to Artin algebras, see [ARS, Corollary 4.5], and to perfect rings, see [Sh, Proposition 5]. The purpose of this note is to present a novel descriptive proof of this theorem in the following more comprehensive form; see Theorems 12 and 15.

2020 Mathematics Subject Classification: 16L30; 16D40; 13C05.

Keywords: indecomposable module; perfect ring; projective module; sum-irreducible module.

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Theorem 1. *Let R be a perfect ring, and P a nonzero R -module. Then the following assertions are equivalent:*

- (i) P is an indecomposable projective R -module.
- (ii) P is a sum-irreducible projective R -module.
- (iii) P is the projective cover of its every nonzero quotient module.
- (iv) P is the projective cover of R/\mathfrak{m} for some maximal left ideal \mathfrak{m} of R .

Further if R is commutative, then the above assertions are equivalent to the following one:

- (v) $P \cong R_{\mathfrak{m}}$ for some maximal ideal \mathfrak{m} of R .

Theorem 2. *Let R be a perfect ring. There is a one-to-one correspondence between the indecomposable projective R -modules and the simple R -modules.*

2 Preliminaries

We begin with reminding the classical notion of a projective cover.

Definition 3. *Let R be a ring, and M an R -module. Then:*

- (i) *A submodule N of M is said to be superfluous, written as $N \subseteq_{sup} M$, if whenever $N + L = M$ for some submodule L of M , then we have $L = M$.*
- (ii) *By a projective cover of M , we mean a projective R -module P together with an epimorphism $\pi : P \rightarrow M$ such that $\ker \pi \subseteq_{sup} P$.*
- (ii) *R is said to be perfect if every R -module has a projective cover.*

Enochs and Jenda have defined the general notion of a cover as follows; see [EJ, Definition 5.1.1].

Definition 4. *Let R be a ring, M an R -module, and \mathcal{A} a class of R -modules. By an \mathcal{A} -cover of M , we mean an R -module $A \in \mathcal{A}$ together with an R -homomorphism $\varphi : A \rightarrow M$ that satisfy the following conditions:*

- (i) Any R -homomorphism $\psi : B \rightarrow M$ with $B \in \mathcal{A}$ factors through φ , i.e. there is an R -homomorphism $f : B \rightarrow A$ that makes the following diagram commutative:

$$\begin{array}{ccc} & B & \\ \swarrow f & \downarrow \psi & \\ A & \xrightarrow{\varphi} & M \end{array}$$

- (ii) Any R -homomorphism $f : A \rightarrow A$ that makes the diagram

$$\begin{array}{ccc} & A & \\ \swarrow f & \downarrow \varphi & \\ A & \xrightarrow{\varphi} & M \end{array}$$

commutative is an automorphism.

The classical definition of a projective cover amounts to the modern definition as recorded in the following result for the sake of bookkeeping.

Lemma 5. Let R be a ring, M an R -module, P a projective R -module, and $\pi : P \rightarrow M$ an R -homomorphism. Let $\mathcal{P}(R)$ denote the class of projective R -modules. Then the following assertions are equivalent:

- (i) $\pi : P \rightarrow M$ is a projective cover of M .
- (ii) $\pi : P \rightarrow M$ is a $\mathcal{P}(R)$ -cover of M .

Proof. See [Xu, Theorem 1.2.12]. □

In light of Lemma 5, the projective cover of a given R -module M is unique up to isomorphism if it exists, in the sense that if $\pi : P \rightarrow M$ and $\pi' : P' \rightarrow M$ are two projective covers of M , then there is an isomorphism $P' \xrightarrow{\cong} P$ that makes the following diagram commutative:

$$\begin{array}{ccc} & P' & \\ \swarrow \cong & \downarrow \pi' & \\ P & \xrightarrow{\pi} & M \end{array}$$

Accordingly, we denote a choice of a projective cover of M by $\pi_M : P_R(M) \rightarrow M$ whenever one exists.

Definition 6. *Let R be a ring. Then:*

- (i) *A nonzero R -module M is said to be indecomposable if whenever $M = N_1 \oplus N_2$ for some submodules N_1 and N_2 of M , then we have either $N_1 = 0$ or $N_2 = 0$.*
- (ii) *An R -module M is said to be sum-irreducible if whenever $M = N_1 + N_2$ for some submodules N_1 and N_2 of M , then we have either $N_1 = M$ or $N_2 = M$.*

By [AF, Theorem 28.4], one has the following lemma:

Lemma 7. *Let R be a ring, and $\mathfrak{J}(R)$ denote its Jacobson radical. Then the following assertions are equivalent:*

- (i) *R is perfect.*
- (ii) *Every descending chain of principal right ideals of R stabilizes.*
- (iii) *Every flat R -module is projective.*
- (iv) *$R/\mathfrak{J}(R)$ is semisimple and $\mathfrak{J}(R)$ is left T -nilpotent.*

Recall that a subset A of a ring R is said to be left T -nilpotent if for every sequence $a_1, a_2, \dots \in A$, there is a natural number n such that $a_1 a_2 \cdots a_n = 0$.

3 Main Results

In this section, we delve into the structure of indecomposable projective R -modules.

Lemma 8. *Let R be a perfect ring, and P a nonzero projective R -module. Then the following assertions are equivalent:*

- (i) *P is indecomposable.*

- (ii) Given any proper submodule N of P , there is an isomorphism $P \xrightarrow{\cong} P_R(P/N)$ that makes the following diagram commutative:

$$\begin{array}{ccc} & & P_R(P/N) \\ & \nearrow \cong & \downarrow \pi_{P/N} \\ P & \xrightarrow{\pi} & P/N \end{array}$$

where π is the natural epimorphism.

- (iii) P is sum-irreducible.

Proof. (i) \Rightarrow (ii): Let N be a proper submodule of P . By Lemma 5, the R -homomorphism $\pi_{P/N} : P_R(P/N) \rightarrow P/N$ is a $\mathcal{P}(R)$ -cover, so we can find an R -homomorphism $f : P \rightarrow P_R(P/N)$, and also using the projectivity of $P_R(P/N)$, we can find an R -homomorphism $g : P_R(P/N) \rightarrow P$ that make the following diagram commutative:

$$\begin{array}{ccccc} & & P_R(P/N) & & \\ & \nearrow g & \downarrow \pi_{P/N} & & \\ P & \xrightarrow{\pi} & P/N & \longrightarrow & 0 \end{array}$$

We thus have $\pi_{P/N}fg = \pi g = \pi_{P/N}$. Another use of Lemma 5 implies that fg is an automorphism, so f is surjective. The short exact sequence

$$0 \rightarrow \ker f \rightarrow P \xrightarrow{f} P_R(P/N) \rightarrow 0$$

splits, so $P \cong \ker f \oplus P_R(P/N)$. But $P_R(P/N) \neq 0$, so the hypothesis implies that $\ker f = 0$, whence $f : P \rightarrow P_R(P/N)$ is an isomorphism.

- (ii) \Rightarrow (iii): Suppose that N_1 and N_2 are two submodules of P such that $P = N_1 + N_2$. If N_1 is proper, then the hypothesis implies that there is an isomorphism $f : P \rightarrow P_R(P/N_1)$ that makes the following diagram commutative:

$$\begin{array}{ccc} & & P_R(P/N_1) \\ & \nearrow f & \downarrow \pi_{P/N_1} \\ P & \xrightarrow{\pi} & P/N_1 \end{array}$$

We thus obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & N_1 & \longrightarrow & P & \xrightarrow{\pi} & P/N_1 \longrightarrow 0 \\
 & & \downarrow \cong & & \downarrow f & & \parallel \\
 0 & \longrightarrow & \ker \pi_{P/N_1} & \longrightarrow & P_R(P/N_1) & \xrightarrow{\pi_{P/N_1}} & P/N_1 \longrightarrow 0
 \end{array}$$

Since $\ker \pi_{P/N_1} \subseteq_{\text{sup}} P_R(P/N_1)$, we deduces that $N_1 \subseteq_{\text{sup}} P$. It follows that $N_2 = P$, so P is sum-irreducible.

(iii) \Rightarrow (i): Suppose that $P = N_1 \oplus N_2$ for some submodules N_1 and N_2 of P . The hypothesis implies that either $N_1 = P$ or $N_2 = P$. It follows that either $N_2 = 0$ or $N_1 = 0$, so P is indecomposable. \square

Corollary 9. *Let R be a perfect ring, and M an R -module. Then M is sum-irreducible if and only if $P_R(M)$ is indecomposable.*

Proof. Let $\pi_M : P_R(M) \rightarrow M$ be a projective cover of M . Suppose that M is sum-irreducible, and let $P_R(M) = P_1 + P_2$ for some submodules P_1 and P_2 of $P_R(M)$. Then

$$M = \pi_M(P_R(M)) = \pi_M(P_1) + \pi_M(P_2).$$

Without loss of generality, we can conclude that $M = \pi_M(P_1)$. It follows that $P_R(M) = P_1 + \ker \pi_M$. But $\ker \pi_M \subseteq_{\text{sup}} P_R(M)$, so we infer that $P_R(M) = P_1$. It follows that $P_R(M)$ is sum-irreducible. Therefore, by Lemma 8, $P_R(M)$ is indecomposable.

Conversely, suppose that $P_R(M)$ is indecomposable. Hence by Lemma 8, $P_R(M)$ is sum-irreducible. Assume that $M = N_1 + N_2$. It is easy to see that

$$P_R(M) = \pi_M^{-1}(N_1) + \pi_M^{-1}(N_2) + \ker \pi_M.$$

But $\ker \pi_M \subseteq_{\text{sup}} P_R(M)$, so we infer that $P_R(M) = \pi_M^{-1}(N_1) + \pi_M^{-1}(N_2)$. Without loss of generality, we can conclude that $P_R(M) = \pi_M^{-1}(N_1)$. This shows that $M = N_1$, so M is sum-irreducible. \square

Corollary 10. *Let R be a perfect ring, and P a nonzero projective R -module. Then P is indecomposable if and only if $P \cong P_R(S)$ for some simple R -module S .*

Proof. Suppose that P is indecomposable. By [AF, Proposition 17.14], P has a maximal submodule N . Then by Lemma 8, $P \cong P_R(P/N)$. It is clear that $S := P/N$ is a simple R -module.

Conversely, suppose that $P \cong P_R(S)$ for some simple R -module S . Obviously, S is sum-irreducible. Now, Corollary 9 implies that $P_R(S)$ is indecomposable. \square

Lemma 11. *Let R be a commutative perfect ring. Then the following assertions hold:*

$$(i) \text{ Spec}(R) = \text{Max}(R).$$

$$(ii) \text{ Supp}_R(R_{\mathfrak{m}}) = \{\mathfrak{m}\} \text{ for every } \mathfrak{m} \in \text{Max}(R).$$

$$(iii) \text{ } R_{\mathfrak{m}} \text{ is an indecomposable } R\text{-module for every } \mathfrak{m} \in \text{Max}(R).$$

Proof. (i): Let $\mathfrak{p} \in \text{Spec}(R)$, and $0 \neq a + \mathfrak{p} \in R/\mathfrak{p}$. By Lemma 7, the descending chain $(a) \supseteq (a^2) \supseteq \cdots$ of principal ideals of R stabilizes, i.e. there is an integer $n \geq 1$ such that $(a^n) = (a^{n+1}) = \cdots$. In particular, $a^n = ra^{n+1}$ for some $r \in R$. It follows that $(1 - ra)a^n = 0$. As $a \notin \mathfrak{p}$, we get $1 - ra \in \mathfrak{p}$, which yields that $(r + \mathfrak{p})(a + \mathfrak{p}) = 1 + \mathfrak{p}$, i.e. $a + \mathfrak{p} \in (R/\mathfrak{p})^\times$. Hence R/\mathfrak{p} is a field, so $\mathfrak{p} \in \text{Max}(R)$.

(ii): Let $\mathfrak{m}, \mathfrak{n} \in \text{Max}(R)$ be such that $\mathfrak{m} \neq \mathfrak{n}$. Set

$$\mathfrak{J}_{\mathfrak{u}}(R) := \bigcap_{\mathfrak{v} \in \text{Max}(R) \setminus \{\mathfrak{u}\}} \mathfrak{v},$$

for any $\mathfrak{u} \in \text{Max}(R)$. By Lemma 7, the ring $R/\mathfrak{J}(R)$ is semisimple. This implies that R is semilocal, and so for any $i \geq 1$, there are elements $a_i \in \mathfrak{J}_{\mathfrak{m}}(R)^i \setminus \mathfrak{m}$ and $b_i \in \mathfrak{J}_{\mathfrak{n}}(R)^i \setminus \mathfrak{n}$. Set $c_i := a_i b_i$ for every $i \geq 1$. It is clear that $c_i \in \mathfrak{J}(R)$ for every $i \geq 1$. By Lemma 7, $\mathfrak{J}(R)$ is T-nilpotent. It follows that there is an integer $n \geq 1$ such that $c_1 c_2 \cdots c_n = 0$. Set $a := a_1 a_2 \cdots a_n$ and $b := b_1 b_2 \cdots b_n$. Then it is obvious that $a \notin \mathfrak{m}$, $b \notin \mathfrak{n}$, and $ab = 0$. This shows that $(R_{\mathfrak{m}})_{\mathfrak{n}} = 0$, so $\text{Supp}_R(R_{\mathfrak{m}}) = \{\mathfrak{m}\}$.

(iii): Suppose that $R_{\mathfrak{m}} = N \oplus N'$ for some R -submodules N and N' of $R_{\mathfrak{m}}$. Localizing at \mathfrak{m} , we get $R_{\mathfrak{m}} = N_{\mathfrak{m}} \oplus N'_{\mathfrak{m}}$. But since $R_{\mathfrak{m}}$ is a local ring, it follows that $R_{\mathfrak{m}}$ is an indecomposable $R_{\mathfrak{m}}$ -module. Hence, $N_{\mathfrak{m}} = 0$ or $N'_{\mathfrak{m}} = 0$. Say $N_{\mathfrak{m}} = 0$. But, (ii) implies that $\text{Supp}_R(N) \subseteq \{\mathfrak{m}\}$. Therefore, $N = 0$. \square

Theorem 12. *Let R be a perfect ring, and P a nonzero R -module. Then the following assertions are equivalent:*

- (i) P is an indecomposable projective R -module.
- (ii) P is a sum-irreducible projective R -module.
- (iii) P is the projective cover of its every nonzero quotient module.
- (iv) $P \cong P_R(R/\mathfrak{m})$ for some maximal left ideal \mathfrak{m} of R .

Further if R is commutative, then the above assertions are equivalent to the following one:

- (v) $P \cong R_{\mathfrak{m}}$ for some maximal ideal \mathfrak{m} of R .

Proof. Lemma 8 yields the equivalence of (i), (ii) and (iii).

(i) \Rightarrow (iv): Corollary 10 implies that $P \cong P_R(S)$ for some simple R -module S . But then $S \cong R/\mathfrak{m}$ for some maximal left ideal \mathfrak{m} of R .

(iv) \Rightarrow (v): Suppose that $P \cong P_R(R/\mathfrak{m})$ for some maximal ideal \mathfrak{m} of R . By Lemma 11, $R_{\mathfrak{m}}$ is an indecomposable R -module. Moreover, Lemma 7 yields that $R_{\mathfrak{m}}$ is a projective R -module. By Corollary 10, we have $R_{\mathfrak{m}} \cong P_R(R/\mathfrak{n})$ for some maximal ideal \mathfrak{n} of R . If $\mathfrak{n} \neq \mathfrak{m}$, then take any $a \in \mathfrak{n} \setminus \mathfrak{m}$. Therefore, the R -homomorphism $a1^{R_{\mathfrak{m}}} : R_{\mathfrak{m}} \rightarrow R_{\mathfrak{m}}$ is an isomorphism. The commutative diagram

$$\begin{array}{ccc}
 R_{\mathfrak{m}} & \xrightarrow{a} & R_{\mathfrak{m}} \\
 \downarrow \cong & & \downarrow \cong \\
 P_R(R/\mathfrak{n}) & \xrightarrow{a} & P_R(R/\mathfrak{n}) \\
 \downarrow \pi_{R/\mathfrak{n}} & & \downarrow \pi_{R/\mathfrak{n}} \\
 R/\mathfrak{n} & \xrightarrow{a} & R/\mathfrak{n}
 \end{array}$$

shows that the R -homomorphism $a1^{R/\mathfrak{n}} : R/\mathfrak{n} \rightarrow R/\mathfrak{n}$ is surjective, which is a contradiction. Hence, $\mathfrak{n} = \mathfrak{m}$.

(v) \Rightarrow (i): Follows from Lemmas 11 and 7. □

The following result may be considered as dual to Matlis Theorem which asserts that over a commutative noetherian ring R , every injective R -module decomposes uniquely into a direct sum of indecomposable injective R -modules; see [Ma].

Corollary 13. *Let R be a commutative artinian ring, and P a projective R -module. Then*

$$P \cong \prod_{\mathfrak{p} \in \text{Spec}(R)} P_R(R/\mathfrak{p})^{\pi_0(\mathfrak{p}, P)},$$

where $\pi_0(\mathfrak{p}, P)$ is the zeroth dual Bass number of P with respect to \mathfrak{p} , i.e.

$$\pi_0(\mathfrak{p}, P) = \text{rank}_{R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}} (R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} \text{Hom}_R(R_{\mathfrak{p}}, P)).$$

Proof. By [Xu, Theorem 1.2.13], R is perfect, and thus [Xu, Proposition 3.3.1] implies that P is a flat cotorsion R -module. It follows from [Xu, Theorem 4.1.15] that

$$P \cong \prod_{\mathfrak{p} \in \text{Spec}(R)} T_{\mathfrak{p}},$$

where $T_{\mathfrak{p}}$ is the $\mathfrak{p}R_{\mathfrak{p}}$ -adic completion of a free $R_{\mathfrak{p}}$ -module of rank $\pi_0(\mathfrak{p}, P)$. But $\mathfrak{J}(R_{\mathfrak{p}}) = \mathfrak{p}R_{\mathfrak{p}}$ is nilpotent, so every $R_{\mathfrak{p}}$ -module is $\mathfrak{p}R_{\mathfrak{p}}$ -adically complete. It follows that $T_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$ -module of rank $\pi_0(\mathfrak{p}, P)$. Now, the result follows from Theorem 12. \square

Lemma 14. *Let R be a perfect ring, and $f : M \rightarrow N$ an R -homomorphism. Then the following assertions hold:*

- (i) *There is an R -homomorphism $\tilde{f} : P_R(M) \rightarrow P_R(N)$ that makes the following diagram commutative:*

$$\begin{array}{ccc} P_R(M) & \xrightarrow{\tilde{f}} & P_R(N) \\ \downarrow \pi_M & & \downarrow \pi_N \\ M & \xrightarrow{f} & N \end{array}$$

- (ii) *If f is an epimorphism, then any R -homomorphism $g : P_R(M) \rightarrow P_R(N)$ that makes the diagram in (i) commutative is an epimorphism.*

- (iii) *If f is an isomorphism, then any R -homomorphism $g : P_R(M) \rightarrow P_R(N)$ that makes the diagram in (i) commutative is an isomorphism.*

Proof. (i): The existence of \tilde{f} follows readily from the projectivity of $P_R(M)$ in light of the following diagram:

$$\begin{array}{ccccc}
 & & P_R(M) & & \\
 & \swarrow \tilde{f} & \downarrow \pi_M & & \\
 & & M & & \\
 P_R(N) & \xrightarrow{\pi_N} & N & \longrightarrow & 0
 \end{array}$$

(ii): Suppose that f is an epimorphism and an R -homomorphism $g : P_R(M) \rightarrow P_R(N)$ makes the following diagram commutative:

$$\begin{array}{ccc}
 P_R(M) & \xrightarrow{g} & P_R(N) \\
 \downarrow \pi_M & & \downarrow \pi_N \\
 M & \xrightarrow{f} & N
 \end{array}$$

The diagram shows that $\pi_N g$ is surjective. Therefore, it can be seen by inspection that

$$\operatorname{im} g + \ker \pi_N = P_R(N).$$

But $\ker \pi_N \subseteq_{\text{sup}} P_R(N)$, so $\operatorname{im} g = P_R(N)$, i.e. g is surjective.

(iii): Suppose that f is an isomorphism and an R -homomorphism $g : P_R(M) \rightarrow P_R(N)$ makes the following diagram commutative:

$$\begin{array}{ccc}
 P_R(M) & \xrightarrow{g} & P_R(N) \\
 \downarrow \pi_M & & \downarrow \pi_N \\
 M & \xrightarrow{f} & N
 \end{array}$$

By (ii), g is surjective. The short exact sequence

$$0 \rightarrow \ker g \rightarrow P_R(M) \xrightarrow{g} P_R(N) \rightarrow 0$$

splits, so $P_R(M) = \ker g + \operatorname{im} g'$, where $g' : P_R(N) \rightarrow P_R(M)$ is an R -homomorphism such that $gg' = 1^{P_R(N)}$. It is clear that $\ker g \subseteq \ker \pi_M$, so $P_R(M) = \ker \pi_M + \operatorname{im} g'$.

But $\ker \pi_M \subseteq_{sup} P_R(M)$, so $P_R(M) = \operatorname{im} g'$. It follows that g' is an isomorphism, so g is an isomorphism. \square

The next result extends the main result of [Le] from finite-dimensional algebras over a field to perfect rings.

Theorem 15. *Let R be a perfect ring, and $\mathfrak{m}, \mathfrak{n}$ two maximal left ideals of R . Then $P_R(R/\mathfrak{m}) \cong P_R(R/\mathfrak{n})$ if and only if $\mathfrak{m} = \mathfrak{n}$. Hence, there is a one-to-one correspondence between the indecomposable projective R -modules and the simple R -modules.*

Proof. Let $\varphi : P_R(R/\mathfrak{m}) \rightarrow P_R(R/\mathfrak{n})$ be any isomorphism. Suppose to the contrary that $\mathfrak{m} \neq \mathfrak{n}$, and take any element $a \in \mathfrak{m} \setminus \mathfrak{n}$. It follows that the R -homomorphism $a1^{R/\mathfrak{n}} : R/\mathfrak{n} \rightarrow R/\mathfrak{n}$ is an isomorphism. In view of Lemma 14, the commutative diagram

$$\begin{array}{ccc} P_R(R/\mathfrak{n}) & \xrightarrow{a} & P_R(R/\mathfrak{n}) \\ \downarrow \pi_{R/\mathfrak{n}} & & \downarrow \pi_{R/\mathfrak{n}} \\ R/\mathfrak{n} & \xrightarrow{a} & R/\mathfrak{n} \end{array}$$

shows that the R -homomorphism $a1^{P_R(R/\mathfrak{n})} : P_R(R/\mathfrak{n}) \rightarrow P_R(R/\mathfrak{n})$ is an isomorphism. Therefore, the commutative diagram

$$\begin{array}{ccc} P_R(R/\mathfrak{m}) & \xrightarrow{a} & P_R(R/\mathfrak{m}) \\ \downarrow \varphi & & \downarrow \varphi \\ P_R(R/\mathfrak{n}) & \xrightarrow{a} & P_R(R/\mathfrak{n}) \end{array}$$

yields that the R -homomorphism $a1^{P_R(R/\mathfrak{m})} : P_R(R/\mathfrak{m}) \rightarrow P_R(R/\mathfrak{m})$ is an isomorphism. Now, the commutative diagram

$$\begin{array}{ccc} P_R(R/\mathfrak{m}) & \xrightarrow{a} & P_R(R/\mathfrak{m}) \\ \downarrow \pi_{R/\mathfrak{m}} & & \downarrow \pi_{R/\mathfrak{m}} \\ R/\mathfrak{m} & \xrightarrow{a} & R/\mathfrak{m} \end{array}$$

implies that the R -homomorphism $a1^{R/\mathfrak{m}} : R/\mathfrak{m} \rightarrow R/\mathfrak{m}$ is surjective. But this map is zero, so we arrive at a contradiction. Hence, $\mathfrak{m} = \mathfrak{n}$.

The converse is immediate. \square

References

- [AF] F.W. Anderson and K.R. Fuller, *Rings and categories of modules*, Second edition, Graduate Texts in Mathematics **13**, Springer-Verlag, New York, 1992. [MR1245487](#).
- [ARS] M. Auslander, I. Reiten and S.O. Smalø, *Representation theory of Artin algebras*, Cambridge Studies in Advanced Mathematics, **36**, Cambridge University Press, Cambridge, 1995. [MR1314422](#).
- [EJ] E.E. Enochs and O.M.G. Jenda, *Relative homological algebra, Volume 1*, Second edition, De Gruyter Exposition in Mathematics, **30**, Walter de Gruyter GmbH & Co. KG, Berlin, 2011.
- [Le] T. Leinster, *The bijection between projective indecomposable and simple modules*, Bull. Belg. Math. Soc. Simon Stevin, **22**(5), (2015), 725-735. [MR3435078](#). [Zbl 1334.16002](#).
- [Ma] E. Matlis, *Injective modules over Noetherian rings*, Pacific J. Math., **8**, (1958), 511-528. [MR0099360](#). [Zbl 0084.26601](#).
- [Sh] U. Shukla, *On the projective cover of a module and related results*, Pacific J. Math. **12** (1962), 709-717. [MR0146235](#). [Zbl 0109.02701](#).
- [Xu] J. Xu, *Flat covers of modules*, Lecture Notes in Mathematics **1634**, Springer-Verlag, Berlin, 1996. [MR1438789](#).

Received December 9, 2019; Accepted January 30, 2021

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Surveys in Mathematics and its Applications **16** (2021), 237 – 249

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Surveys in Mathematics and its Applications **16** (2021), 237 – 249
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