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ON A CLASS OF ALMOST PARACONTACT STRUCTURES ON T^2M

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Abstract. A new class of almost paracontact structures on second order tangent bundle built on a Riemannian space is defined and studied in this paper. In particular, characterizations for the integrability and normality of this class of almost paracontact structures are given. Also, compatible linear connections with this class of almost paracontact structures are introduced and certain characterization of them is obtained.

1 Introduction

The generalized Lagrange geometry of second order was defined and studied by R. Miron [6, 7] and represents the geometry of generalized Lagrangians modeled on the second order tangent bundle $(T^2M,\, p,\, M)$. These spaces are useful in the study of the geometry of higher-order Lagrangians [6, 7], for the prolongation of Riemannian, Finslerian and Lagrangian structures [6, 7], for the study of stationary curves [9], and for the development of a gauge theory having the second order tangent bundle as the geometrical model [3, 10, 13]. The term "homogeneity" has been discussed in Miron's papers [4, 5] where new geometrical models on Riemannian spaces and on Finslerian spaces are also introduced, respectively. In [13, 16] an extension of Miron's theory of homogeneity to the second order tangent bundle is presented.

The main goal of this note is to define and study a new class of almost paracontact structures on second order tangent bundle built on a Riemannian space. In particular, characterizations for the integrability and normality of this class of almost paracontact structures are given. Also, compatible linear connections with this class of almost paracontact structures are introduced and certain characterization of them is obtained.

Such geometrical structures are useful for the development of some gauge theories on TM or on higher order tangent spaces T^kM (for more details see for instance [2, 3, 6, 8, 9, 10, 12, 13, 14, 15, 16] and the references therein). We shall use different kind of metrics on T^2M, parallel to those introduced and studied in [1, 4, 5, 12], in order to obtain a "quasi–homogeneous" gauge theory on T^2M, cf. [11]. The

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obtained results within the present paper are strongly connected with some very interesting results in [17, 18].

2 Preliminaries

Consider $\mathfrak{R}^n = (M, \gamma)$ a Riemannian space generated by a real, differentiable, n–dimensional manifold M and by a Riemannian metric γ on M, given by the local components $(\gamma_{ij}(x))$, $x \in U \subset M$. It is possible to extend γ to $p^{-1}(U) \subset E = T^2M$ by

$$(\gamma_{ij} \circ p)(u) = \gamma_{ij}(x), \quad u \in p^{-1}(U), \quad p(u) = x. \tag{2.1}$$

In this case $\gamma_{ij} \circ p$ are the local components of a tensor field on E. Usually, we write these local components with γ_{ij} as well. Furthermore, with $\gamma^k_{ij}(x)$ we will denote the Christoffel symbols of the second species of the metric γ and with $r^k_{ijh}(x)$ we will denote the local components of the curvature tensor field of the metric γ. It is possible to introduce on E a nonlinear connection determined only by this metric, cf. [6]. Moreover, the coefficients of connection are determined by the following relations (see also [12])

$$N^{(0)i}_{(1)j}\left(x, y^{(1)}\right) = \gamma^i_{j0}, \tag{2.2}$$

$$N^{(0)i}_{(2)j}\left(x, y^{(1)}, y^{(2)}\right) = \frac{1}{2}\left(\frac{\partial \gamma^i_{j0}}{\partial x^p} y^{(1)p} + \gamma^i_{0m} \cdot \gamma^m_{j0}\right) + \gamma^i_{j\bar{0}}, \tag{2.3}$$

where the notation "0" stands for the contraction by $\left(y^{(1)}\right)$ and the notation "$\bar{0}$" stands for the contraction by $\left(y^{(2)}\right)$.

The nonlinear connection N assures the existence of a basis $\left(\delta_k, \delta^{(1)}_k, \delta^{(2)}_k\right)$ adapted to the tangent space T_uE. The vector fields of the adapted basis are defined with the help of the following relations

$$\delta_k = \frac{\partial}{\partial x^k} - N^i_{(1)k}\frac{\partial}{\partial y^{(1)i}} - N^i_{(2)k}\frac{\partial}{\partial y^{(2)i}}, \tag{2.4}$$

$$\delta^{(1)}_k = \frac{\partial}{\partial y^{(1)i}} - N^i_{(1)k}\frac{\partial}{\partial y^{(2)i}}, \quad \delta^{(2)}_k = \frac{\partial}{\partial y^{(2)k}}. \tag{2.5}$$

With respect to this basis the tangent space T_uE can be orthogonally decomposed as

$$T_uE = N_0(u) \oplus N_1(u) \oplus V_2(u), \quad u \in E,$$

where $N_0(u)$, $N_1(u)$ and $V_2(u)$ are linearly spanned by (δ_k), $(\delta^{(1)}_k)$ and $(\delta^{(2)}_k)$, respectively.

Furthermore, the dual basis to the basis (2.4)–(2.5) is given by the following locally defined 1–forms on T^2M: dx^i, $\delta y^{(1)i}$ and $\delta y^{(2)i}$, where

$$\begin{aligned} \delta y^{(1)i} &= dy^{(1)i} + M^{(0)i}_{(1)j} dx^j, \\ \delta y^{(2)i} &= dy^{(2)i} + M^{(0)i}_{(1)j} dy^{(1)j} + M^{(0)i}_{(2)j} dx^j. \end{aligned}$$

For further developments, we need the following result.

Theorem 1. *The Lie brackets of the vector fields of the adapted basis* $\left(\delta_k, \delta_k^{(1)}, \delta_k^{(2)}\right)$ *are given by:*

$$[\delta_j, \delta_k] = R^i_{(01)jk} \cdot \delta_i^{(1)} + R^i_{(02)jk} \cdot \delta_i^{(2)}, \tag{2.6}$$

$$\left[\delta_j, \delta_k^{(1)}\right] = B^i_{(11)jk} \cdot \delta_i^{(1)} + B^i_{(12)jk} \cdot \delta_i^{(2)}, \tag{2.7}$$

$$\left[\delta_j, \delta_k^{(2)}\right] = B^i_{(21)jk} \cdot \delta_i^{(1)} + B^i_{(22)jk} \cdot \delta_i^{(2)}, \tag{2.8}$$

$$\left[\delta_j^{(1)}, \delta_k^{(1)}\right] = R^i_{(12)jk} \cdot \delta_i^{(2)}, \quad \left[\delta_j^{(1)}, \delta_k^{(2)}\right] = B^i_{(21)jk} \cdot \delta_i^{(2)}, \tag{2.9}$$

where:

$$R^i_{(01)jk} = r^i_{0jk}, \quad R^i_{(12)jk} = 0, \tag{2.10}$$

$$R^i_{(02)jk} = \frac{1}{2} \cdot \frac{\partial r^i_{0jk}}{\partial x^p} \cdot y^{(1)p} + \frac{1}{2} \cdot \left(\gamma^i_{0m} \cdot r^m_{0jk} + \gamma^m_{0j} \cdot r^i_{0km} + \gamma^m_{0k} \cdot r^i_{0mj}\right) + r^i_{\bar{0}jk}, \tag{2.11}$$

$$B^i_{(11)jk} = B^i_{(22)jk} = \gamma^i_{jk}, \quad B^i_{(21)jk} = 0, \tag{2.12}$$

$$B^i_{(12)jk} = \frac{1}{2} \cdot r^i_{0jk} + \gamma^i_{0m} \cdot \gamma^m_{jk}. \tag{2.13}$$

3 β almost paracontact structures on T^2M

Assume that $\beta : \mathbb{R} \to (0, +\infty)$ is a smooth function. Define locally the following $\mathcal{F}(E)$ linear map $Q^\beta : \mathfrak{X}(E) \to \mathfrak{X}(E)$ by:

$$Q^\beta(\delta_i) = \frac{1}{\beta(F^2)} \cdot \delta_i^{(2)}, \ Q^\beta(\delta_i^{(1)}) = 0, \ Q^\beta(\delta_i^{(2)}) = \beta(F^2) \cdot \delta_i, \tag{3.1}$$

where $F^2 = \gamma_{ij} \cdot y^{(1)i} y^{(1)j}$.

The linear structure defined by (3.1) is called the β almost paracontact structure on T^2M. Some basic properties of this structure are listed within the next result.

Theorem 2. *The linear map* Q^β *has the following properties:*

1. Q^β *is globally defined on* $\widetilde{E}$*;*
2. Q^β *is a* $(1,1)$ *tensor field on* $\widetilde{E}$*;*
3. Q^β *is an almost paracontract structure on* $\widetilde{E}$*;*
4. $\left(Q^\beta\right)^3 - Q^\beta = 0$*;*
5. Q^β *depends only on the Riemannian metric* γ *and on a strictly positive smooth function* β*;*

6. $\ker\left(Q^{\beta}\right) = N_1$, $Im\left(Q^{\beta}\right) = N_0 + V_2$;

7. $rank\left(Q^{\beta}\right) = 2n$.

It is easily seen that Q^{β} can be represented as

$$Q^{\beta} = \frac{1}{\beta(F^2)} \cdot \delta_i^{(2)} \otimes dx^i + \beta(F^2) \cdot \delta_i \otimes \delta y^{(2)i}. \tag{3.2}$$

Definition 3. *The almost paracontact structure Q^{β} is said to be* normal *if the following relation holds true:*

$$\widetilde{N}_{Q^{\beta}}(X,Y) = N_{Q^{\beta}}(X,Y) + 2 \cdot \sum_{i=1}^{n} d(\delta y^{(1)i})(X,Y) \cdot \delta_i^{(1)} = 0, \tag{3.3}$$

for all X, $Y \in \mathfrak{X}(\widetilde{E})$, where $N_{Q^{\beta}}$ stands for the Nijenhuis tensor field associated to Q^{β}.

One obtains the following characterization for the normality of the β almost paracontact structure Q^{β}.

Theorem 4. *The almost paracontact structure Q^{β} is normal if and only if β is constant and $\gamma^i_{jk}(x) = 0$.*

The next result is an immediate consequence of a classical result due to T. Van Duc.

Theorem 5. *The almost paracontact structure Q^{β} is integrable if and only if $N_{Q^{\beta}} = 0$.*

Using Theorem 5, a direct computation using local coordinates leads to the following characterization for the integrability of the β almost paracontact structure.

Theorem 6. *The almost paracontact structure Q^{β} is integrable if and only if β is constant and $R^i_{(01)kh} = r^i_{jhk} = 0$.*

In other words, Theorem 6 says that the almost paracontact structure Q^{β} is integrable if and only if the underlying Riemannian space (M, γ) is flat.

4 Compatible connections with β paracontact structures

With respect to the adapted basis $\left(\delta_k, \delta_k^{(1)}, \delta_k^{(2)}\right)$, any linear connection D on E can be represented as follows

$$D_{\delta_k}\delta_j = L^{(H)i}_{jk} \cdot \delta_i + L^{(1)i}_{jk} \cdot \delta_i^{(1)} + L^{(2)i}_{jk} \cdot \delta_i^{(2)} \tag{4.1}$$

$$D_{\delta_k}\delta_j^{(1)} = L_{jk}^{(3)i}\cdot\delta_i + L_{jk}^{(v_1)i}\cdot\delta_i^{(1)} + L_{jk}^{(4)i}\cdot\delta_i^{(2)} \tag{4.2}$$

$$D_{\delta_k}\delta_j^{(2)} = L_{jk}^{(5)i}\cdot\delta_i + L_{jk}^{(6)i}\cdot\delta_i^{(1)} + L_{jk}^{(v_2)i}\cdot\delta_i^{(2)} \tag{4.3}$$

$$D_{\delta_k^{(1)}}\delta_j = F_{jk}^{(H)i}\cdot\delta_i + F_{jk}^{(1)i}\cdot\delta_i^{(1)} + F_{jk}^{(2)i}\cdot\delta_i^{(2)} \tag{4.4}$$

$$D_{\delta_k^{(1)}}\delta_j^{(1)} = F_{jk}^{(3)i}\cdot\delta_i + F_{jk}^{(v_1)i}\cdot\delta_i^{(1)} + F_{jk}^{(4)i}\cdot\delta_i^{(2)} \tag{4.5}$$

$$D_{\delta_k^{(1)}}\delta_j^{(2)} = F_{jk}^{(5)i}\cdot\delta_i + F_{jk}^{(6)i}\cdot\delta_i^{(1)} + F_{jk}^{(v_2)i}\cdot\delta_i^{(2)} \tag{4.6}$$

$$D_{\delta_k^{(2)}}\delta_j = C_{jk}^{(H)i}\cdot\delta_i + C_{jk}^{(1)i}\cdot\delta_i^{(1)} + C_{jk}^{(2)i}\cdot\delta_i^{(2)} \tag{4.7}$$

$$D_{\delta_k^{(2)}}\delta_j^{(1)} = C_{jk}^{(3)i}\cdot\delta_i + C_{jk}^{(v_1)i}\cdot\delta_i^{(1)} + C_{jk}^{(4)i}\cdot\delta_i^{(2)} \tag{4.8}$$

$$D_{\delta_k^{(2)}}\delta_j^{(2)} = C_{jk}^{(5)i}\cdot\delta_i + C_{jk}^{(6)i}\cdot\delta_i^{(1)} + C_{jk}^{(v_2)i}\cdot\delta_i^{(2)} \tag{4.9}$$

The set consisting of the functions $L_{jk}^{(H)i}$,..., $C_{jk}^{(v_2)i}$ represents the set of the coefficients of the linear connection D.

Definition 7. *A linear connection D on $E = T^2M$ is said to be compatible with the β paracontact structure Q^β if $D_X Q^\beta = 0$, for all $X \in \mathfrak{X}(E)$.*

Concerning the notion of compatible connection with the β almost paracontact structure Q^β, the following result can be proved.

Theorem 8. *A linear connection D on E is compatible with a β almost paracontact structure Q^β if and only if the coefficients of the linear connection satisfy the following relations:*

$$L_{jk}^{(H)i} = L_{jk}^{(V_2)i},\quad L_{jk}^{(5)i} = \beta^2\cdot L_{jk}^{(2)i},\quad L_{jk}^{(1)i} = L_{jk}^{(3)i} = L_{jk}^{(4)i} = L_{jk}^{(6)i} = 0,$$

$$F_{jk}^{(V_2)i} = F_{jk}^{(H)i} + 2\cdot\frac{\beta}{\beta'}\cdot y_k^{(1)}\cdot\delta_j^i,$$

$$F_{jk}^{(5)i} = \beta^2\cdot F_{jk}^{(2)i},\quad F_{jk}^{(1)i} = F_{jk}^{(3)i} = F_{jk}^{(4)i} = F_{jk}^{(6)i} = 0,$$

$$C_{jk}^{(V_2)i} = C_{jk}^{(H)i},\quad C_{jk}^{(5)i} = \beta^2\cdot C_{jk}^{(2)i},\quad C_{jk}^{(1)i} = C_{jk}^{(3)i} = C_{jk}^{(4)i} = C_{jk}^{(6)i} = 0.$$

The next result provides necessary and sufficient conditions for a d–linear connection to be compatible with an almost paracontact structure F^β.

Corollary 9. *A d–linear connection D on E is compatible with a β almost paracontact structure Q^β if and only if the coefficients of the linear connection satisfy the following relations:*

$$L_{jk}^{(H)i} = L_{jk}^{(V_2)i},\quad F_{jk}^{(H)i} = F_{jk}^{(V_2)i} + 2\cdot\frac{\beta}{\beta'}\cdot y_k^{(1)}\cdot\delta_j^i,\quad C_{jk}^{(H)i} = C_{jk}^{(V_2)i}.$$

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