ON A GENERALIZATION OF EULER’S CONSTANT

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Abstract. A one parameter generalization of Euler’s constant $\gamma$ from [Numer. Algorithms 46(2) (2007) 141–151] is investigated, and additional expressions for $\gamma$ are derived. Included are forms involving the Gregory coefficients and the Hurwitz Zeta function, and expressions of the later are shown to lead to an alternative proof of Lerch’s limit formula for the gamma function.

1 Introduction

Euler’s constant $\gamma$ is defined by the straightforward limit

$$\gamma = \lim_{n \to \infty} \left[ \sum_{k=1}^{n} \frac{1}{k} - \ln(n) \right] = 0.5772156649 \cdots,$$  

(1.1)

and is seen in a variety of diverse areas of mathematics while being featured prominently in both analysis and number theory. Its exact nature is still unknown – to this day no legitimate mathematical proof of its rationality or irrationality has been found. Given its ubiquity and mystery, it is not surprising that a variety of generalizations of $\gamma$ have been put forth.

In this paper the following generalization of $\gamma$ will be explored in detail:

$$\gamma(a) = \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k + a - 1} - \int_{1}^{n} \frac{1}{x + a - 1} \, dx \right),$$

(1.2)

where $a > 0$, and for which $\gamma(1) = \gamma$. This one parameter function $\gamma(a)$ from (1.2) was formulated by Sîntâmârian in [21], and can be related to some other well-known generalizations of $\gamma$. One of these is the generalized Stieltjes constants defined by

$$\gamma_n(a) = \lim_{m \to \infty} \left[ \sum_{k=0}^{m} \frac{(\ln(k + a))^n}{k + a} - \frac{(\ln(m + a))^{n+1}}{n + 1} \right].$$

2020 Mathematics Subject Classification: 30B10; 33B15; 11M35; 30E20; 11M06

Keywords: Euler’s Constant; Gamma Function; Hurwitz Zeta Function

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where \( a > 0 \) and \( n \) is a non-negative integer. The special case of \( a = 1 \) are the standard Stieltjes constants, which appear in the regular part of the Laurent expansion of the Hurwitz Zeta function given by

\[
\zeta(s,a) = \sum_{n=0}^{\infty} (n + a)^{-s}
\]

(1.3)

for \( \text{Re}(s) > 1 \) and \( \text{Re}(a) > 0 \). The Laurent expansion, which is valid under the same conditions, is

\[
\zeta(s,a) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n(a) (s-1)^n.
\]

In terms of the Stieltjes notation (1.2) becomes \( \gamma(a) = \gamma_0(a) + \ln(a) \) for \( a > 0 \).

Another prominent generalization of \( \gamma \) which can be related to \( \gamma(a) \) is the Euler-Lehmer constants [17] given by

\[
\gamma(a,q) = \lim_{n \to \infty} \sum_{0 < k \leq n}^{\text{mod } q} \left[ \frac{1}{k} - \frac{\ln(n)}{q} \right],
\]

(1.4)

where \( a \) and \( q \) are integers satisfying \( 0 < a \leq q \). In [19] Murty and Saradha demonstrated the usefulness of these constants with regard to the nature of \( \gamma \) by proving that at least one number in the following infinite list is algebraic:

\[
\gamma(a,q), \quad 1 \leq a < q, \quad q \geq 2.
\]

Using this notation our expression for \( \gamma(a) \) from (1.2) can be expressed by

\[
\gamma(a) = \gamma(a,1) + \ln(a)
\]

(1.5)

for positive integral \( a \).

Other generalizations include the \( q \)-analogue of the Euler constant discussed in [14] and denoted here by \( [\gamma]_q \) for \( q > 1 \). Its expression is

\[
[\gamma]_q = \sum_{n=1}^{\infty} \frac{1}{[n]_q} + \frac{(q-1) \ln(q-1)}{\ln(q)} - \frac{2}{(q-1)}.
\]

where \( [n]_q = (q^n - 1)/(q-1) \). In [14] it is shown that \( \lim_{q \to 1^+} [\gamma]_q = \gamma \), and that \( [\gamma]_2 \) is irrational. Additionally, there are the generalized Euler’s constants associated with the function \( y = 1/x^\alpha \) for \( \alpha > 0 \) which are given by

\[
\gamma_{\alpha} = \lim_{n \to \infty} \left[ \sum_{k=1}^{n} \frac{1}{k^\alpha} - \int_{1}^{n} \frac{1}{x^\alpha} dx \right],
\]

(1.6)

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and where $\gamma_1 = \gamma$. Because this generalization and the Stieltjes constants contain a summation along with a corresponding definite integral, they are sometimes addressed in the literature in the context of the Euler-Maclaurin summation formula [2],[6].

In contrast to some of these other generalizations, $\gamma(a)$, as it is given in (1.2), has not been examined in as great detail in the literature related to Euler’s constant. In the following analysis a variety of expressions for $\gamma(a)$ will be derived and shown to be consistent with analogous results for Euler’s constant as the parameter $a$ approaches 1. These expressions include generalizations of a few of the most well-known expressions for $\gamma$, and in many cases, references explaining other known methods for proving a given result will be provided. The various expressions for $\gamma(a)$ given here reveal its relationship to the Gregory coefficients and the Hurwitz zeta function. Several identities involving $\gamma(a)$ are brought together from the literature through its close connection to the digamma function $\psi$, and other straightforward results are derived. Additionally, asymptotic formulas for (1.2) are derived for both small and large values of the parameter $a$, and other derived results are used to evaluate a class of definite integrals. This work will also motivate and lead to other expressions for Euler’s constant, as well as to another avenue to prove a limit formula attributed to Lerch given in [14] and [18].

2 A Classic Definite Integral

A definite integral expression for $\gamma(a)$ can be derived using a modification of a method that starts from the definition of $\gamma$ in (1.1) and leads to the following identity which has been attributed to Euler [15]:

$$\gamma = \int_0^1 \left( \frac{1}{z} + \frac{1}{\ln(1-z)} \right) dz.$$  \hspace{1cm} (2.1)

One way to arrive at (2.1) is through the Frullani integral [12, p. 406] given by

$$\ln \left( \frac{b}{a} \right) = \int_0^\infty \frac{e^{-at} - e^{-bt}}{t} dt,$$  \hspace{1cm} (2.2)

where $a$ and $b$ are arbitrary positive real numbers. This expression can be obtained by separately evaluating each of the following integrals:

$$\int_a^b \int_0^\infty e^{-tx} dx dt = \int_0^\infty \int_a^b e^{-tx} dt dx.$$

Leaving $a$ unchanged and letting $b = n + a - 1$ in the Frullani integral expression leads to

$$\ln \left( \frac{n + a - 1}{a} \right) = \int_0^\infty \frac{e^{-at} - e^{-(n+a-1)t}}{t} dt.$$  \hspace{1cm} (2.3)
Define $H_{a,n} = \sum_{k=1}^{n} (k + a - 1)^{-1}$. Then

$$H_{n,a} = \sum_{k=0}^{n-1} \left( \int_{0}^{\infty} e^{-(k+a)t} dt \right) = \int_{0}^{\infty} \left( e^{-at} \sum_{k=0}^{n-1} e^{-kt} \right) dt = \int_{0}^{\infty} e^{-at} \frac{(1 - e^{-nt})}{1 - e^{-t}} dt.$$  

Subtracting (2.3) from this expression for $H_{n,a}$, and using the definition in (1.2) yields

$$\gamma(a) = \lim_{n \to \infty} \int_{0}^{\infty} \left[ \frac{e^{-at} (1 - e^{-nt})}{1 - e^{-t}} - \frac{(e^{-at} - e^{-(n+a-1)t})}{t} \right] dt.$$  

Simplifying and moving the limit into the integral leads to

$$\gamma(a) = \int_{0}^{\infty} e^{-at} \left( \frac{1}{1 - e^{-t}} - \frac{1}{t} \right) dt = \int_{0}^{1} (1 - x)^{a-1} \left( \frac{1}{x} + \frac{1}{\ln(1 - x)} \right) dx, \quad (2.4)$$  

where the second integral in (2.4) arises from the substitution $x = 1 - e^{-t}$, and as expected, the case of $a = 1$ leads back to (2.1). It will be seen that the first integral in (2.4) can also be obtained from (5.6) along with Binet’s formula for the digamma function in [4, Vol. 1, Sect. 1.7, Eq. 22].

Another integral expression that can be reduced to the rightmost equation of (2.4) is

$$\gamma(a) = \int_{0}^{1} \int_{0}^{1} \frac{t^{a-1} (1 - tx)}{1 - t} dtdx.$$  

This equivalence can be seen by starting with this double integral, switching the order of integration, and completing only the integration with respect to $x$. Incidentally, the case of $a = 1$ yields the concise result

$$\gamma = \int_{0}^{1} H_{x} dx$$  

where $H_{x} = \int_{0}^{1} (1 - t^x)/(1 - t)dt$ is Euler’s generalization of the harmonic numbers for real $x$. This function is so defined because when $n$ is a positive integer

$$H_{n} = \sum_{k=1}^{n} \frac{1}{k} = \int_{0}^{1} \left( \sum_{k=0}^{n-1} t^{k} \right) dt = \int_{0}^{1} \frac{1 - t^{n}}{1 - t} dt.$$  

### 3 Another Expression for $\gamma$

Two methods for obtaining integral expressions for the harmonic numbers $H_{n}$ have been used in the previous work. A key component in these procedures is the use of elementary definite integral expressions for $1/k$ such as

$$\frac{1}{k} = \int_{0}^{\infty} e^{-kt} dt = \int_{0}^{1} t^{k-1} dt.$$  

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In a similar manner, by integrating functions of the form 1/x^{k+1}, another expression for \( H_n \) can be found. This method will be used in the proof of the following theorem.

**Theorem 1.** Euler’s constant can be obtained from the following limit:

\[
\gamma = \lim_{n \to \infty} \int_1^n \frac{2 - x - x^{-n}}{x(x - 1)} \, dx. \tag{3.1}
\]

**Proof.** Express the harmonic numbers \( H_n \) in the form

\[
H_n = \sum_{k=1}^{n} \left( \int_1^n \frac{1}{x^{k+1}} \, dx \right) + E_n,
\]

where \( E_n = \sum_{k=1}^{n} \frac{1}{(kn)^k} \). From here, interchange the integral and the summation, simplify the resulting geometric series, and use (1.1) to obtain

\[
H_n - \ln(n) = \int_1^n \left( \frac{1 - x^{-n}}{x(x - 1)} - \frac{1}{x} \right) \, dx + E_n. \tag{3.2}
\]

Because \( 0 < E_n < \sum_{k=1}^{\infty} \frac{1}{(kn)^k} = \ln(n/(n - 1)) \), the \( \lim_{n \to \infty} E_n = 0 \). Equation (3.1) follows after simplifying (3.2) and appealing to (1.1).

This unusual expression for Euler’s constant is not found among the standard lists of expressions for \( \gamma \) in references such as [7], [8], [11], and [25].

A \( \gamma(a) \) analog of (3.1) can be obtained by carrying out the same steps of the above process except that the integration is performed on terms of the form 1/x^{(k+a)}.

After summing and utilizing (1.2) the result is

\[
\gamma(a) = \lim_{n \to \infty} \int_1^n \frac{(1 - x^{-n})(x + a - 1) - x^a(x - 1)}{x^a(x - 1)(x + a - 1)} \, dx.
\]

### 4 Gregory Coefficients

In addition to forms involving definite integrals, there are several series expansions for Euler’s constant that can be generalized to expansions for \( \gamma(a) \). In this section one particular series for \( \gamma \) is derived from first principles, and this result will be referred to again in the next section where an analogous form for \( \gamma(a) \) is obtained from a result in the literature for the digamma function \( \psi \).

A series for \( \gamma(a) \) consisting of only positive terms can be obtained by expanding the integrand of (2.1) into a Maclaurin series and dividing to obtain

\[
f(z) = \frac{\ln(1 - z) + z}{z \ln(1 - z)} = \sum_{n=0}^{\infty} \frac{z^n}{n(n + 2)} = \sum_{n=0}^{\infty} a_n z^n.
\]
Upon clearing fractions

\[
\left(1 + \frac{z^2}{2} + \frac{z^4}{3} + \ldots\right) \left(\sum_{n=0}^{\infty} a_n z^n\right) = \frac{1}{2} + \frac{z}{3} + \frac{z^2}{4} + \ldots,
\]

and after multiplying and equating powers, the coefficients \(a_n\) can be shown to satisfy

\[
a_n = \frac{1}{(n + 2)} - \sum_{k=0}^{n-1} \frac{a_k}{n + 1 - k},
\]

where the first four values of the sequence \(\{a_n\}\) are

\[
a_0 = \frac{1}{2}, \ a_1 = \frac{1}{12}, \ a_2 = \frac{1}{24}, \ a_3 = \frac{19}{720}, \ a_4 = \frac{3}{160}.
\]

These are the absolute values of the Gregory coefficients which are also referred to as Bernoulli numbers of the second kind [13, p. 285]. (This sequence was first introduced by James Gregory (1638–1675) in the context of numerical integration.) If the Gregory coefficients are denoted by \(G_n\), then \(a_n = |G_{n+1}|\) and \(G_{n+1} = (-1)^n a_n\). Integrating \(f(z)\) over the interval \([0, 1]\) yields the Fontana-Mascheroni series

\[
\gamma = \frac{1}{2} + \frac{1}{24} + \frac{1}{72} + \frac{19}{2880} + \frac{3}{800} + \cdots = \sum_{n=1}^{\infty} \frac{|G_n|}{n}.
\]

(4.1)

Scott in [20] provides an alternative derivation of this formula by expanding \(f(z)\) after rewriting it in the following form:

\[
f(z) = \frac{1}{z} - \int_0^1 \frac{(1-z)^t}{z} dt = \frac{1}{z} \left[ 1 - \int_0^1 \left(1 - tz + t(t - 1) \frac{z^2}{2} - \cdots\right) dt \right].
\]

(4.2)

Equating like powers of \(z\) in this expression with those of the series \(\sum_{n=0}^{\infty} a_n z^n\) demonstrates that the coefficient \(a_n\) can be expressed by the definite integral

\[
a_n = \frac{1}{(n + 1)!} \int_0^1 t (1 - t) (2 - t) \cdots (n - t) dt.
\]

5 The Connection Between \(\gamma(a)\) and \(\psi(a)\)

Some useful expressions for \(\gamma(a)\) can be established through its relationship to the gamma and digamma functions \(\Gamma\) and \(\psi\). To proceed in this direction, we state for reference the following well-known definite integral form of \(\Gamma(z)\) valid for \(\text{Re}(z) > 0\)

\[
\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt.
\]

(5.1)
The digamma function $\psi$ can be defined by

$$\psi(z) = \frac{d}{dz} \ln \Gamma (z) = \frac{\Gamma'(z)}{\Gamma(z)}. \quad (5.2)$$

With these definitions in place, we can proceed to a straightforward expression which casts $\gamma(a)$ within the framework of the gamma function. Differentiating (5.1) with respect to $z$ and applying the definition in (5.2) yields

$$\psi(z) = 1\times \frac{\Gamma(z)}{\Gamma(z)} \int_0^\infty t^{(z-1)} \ln(t) e^{-t} dt. \quad (5.3)$$

Then from (5.6)

$$\gamma(a) = \ln(a) - 1\times \frac{\Gamma(a)}{\Gamma(a)} \int_0^\infty t^{(a-1)} \ln(t) e^{-t} dt. \quad (5.4)$$

Another relationship between $\gamma(a)$ and the gamma function is given in the following theorem:

**Theorem 2.** For $\text{Re}(z) \neq -1, -2, -3, \ldots$

$$\Gamma(z + 1) = a^ze^{-\gamma(a)z} \cdot \lim_{n \to \infty} n^z \cdot \prod_{k=1}^n \left( \frac{k}{z+k} \right).$$

**Proof.** The function $\Gamma(z + 1)$ can be expressed, as in [26, p. 231], by

$$\Gamma(z + 1) = \lim_{n \to \infty} n^z \cdot \prod_{k=1}^n \left( \frac{k}{z+k} \right).$$

Since $\lim_{n \to \infty} \ln((n+a-1)/n) = 0$, definition (1.2) is equivalent to

$$\gamma(a) = \lim_{n \to \infty} [H_{a,n} - \ln(n/a)]. \quad (5.5)$$

Since $\prod_{k=1}^n \exp(-z/(k+a-1)) = e^{-zH_{a,n}}$

$$\Gamma(z + 1) = \lim_{n \to \infty} e^{z\ln(a)} \cdot e^{-z(H_{a,n}-\ln(n)+\ln(a))} \prod_{k=1}^n \left( \frac{k}{z+k} \right) \exp \left( \frac{z}{k + a - 1} \right),$$

and after evaluating the limit in the exponential function, the theorem follows. \(\square\)

This theorem is a generalization of the Weierstrass definition of the gamma function

$$\Gamma(z + 1) = e^{-\gamma z} \prod_{k=1}^{\infty} \left( 1 + \frac{z}{k} \right)^{-1} e^{z/n},$$

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which can be found in [26, p. 230] and [1, p. 255, Eq. 6.1.3]. Taking the natural logarithm, differentiating the result in Theorem 1, and using (5.2) yields

\[ \psi(z + 1) = \ln(a) - \gamma(a) + \sum_{k=1}^{\infty} \left( \frac{1}{k + a - 1} - \frac{1}{z + k} \right), \]

and upon substituting \( z = a - 1 \) into the above we obtain

\[ \gamma(a) = \ln(a) - \psi(a). \quad (5.6) \]

Letting \( a = 1 \) in (5.6) brings us to the well-known result \( \gamma = -\psi(1) \). Another way to arrive at (5.6) is to start with the following expression for \( \psi(a) \) found in [4, Vol. 1, Sec. 1.7, Eq. 2]:

\[ \psi(a) = \lim_{n \to \infty} \left[ \ln(n) - \sum_{k=0}^{n-1} \frac{1}{k + a} \right]. \quad (5.7) \]

Then express the definition for \( \gamma(a) \) in (1.2) with the integral portion evaluated. That is,

\[ \gamma(a) = \lim_{n \to \infty} \left[ \sum_{k=0}^{n-1} \frac{1}{k + a} - \ln(n + a - 1) \right] + \ln(a). \]

Adding and subtracting an \( \ln(n) \), and doing the same with an additional series term \( (n + a)^{-1} \) yields

\[ \gamma(a) = \ln(a) - \lim_{n \to \infty} \left[ \left( \ln(n) - \sum_{k=0}^{n-1} \frac{1}{k + a} \right) + \frac{1}{n + a} + \ln \left( 1 + \frac{a - 1}{n} \right) \right]. \quad (5.8) \]

The last two terms within the square brackets vanish in the limit, and upon comparing the resulting expression to (5.7) the identity in (5.6) follows.

Equation (5.6) reveals significant differences between \( \psi(a) \) and \( \gamma(a) \). The function \( \gamma(a) \) is decreasing on \((0, \infty)\), and its range, like its domain, is the positive reals. In contrast \( \psi(a) \) is increasing on the same domain; its range is \((-\infty, \infty)\); and it crosses the horizontal axis at approximately 1.4616321. Furthermore, the integrand and summand in the expression for \( \gamma(a) \) from (1.2) share a convenient symmetry which is not found in the formula for \( \psi(a) \) given in (5.7) – as the integral portion there is given by \( \ln(n) = \int_{1}^{n} (1/t) dt \).

A result similar to (5.6) has been obtained in terms of the Euler-Lehmer coefficients in Theorem 7 of [17]. This theorem states that \( -q \gamma(a, q) = \psi(a/q) + \ln(q) \), which along with (1.5) can be combined to bring us to (5.6) for the case of \( q = 1 \) and positive integral values of \( a \).

Many other expressions involving \( \gamma(a) \) can be obtained by utilizing (5.6) in conjunction with established formulas for \( \psi \) found in references such as [1],[4],[9],

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A few examples are listed below.

\[ \gamma(1/2) = \gamma + \ln(2) \]
\[ \gamma(1/3) = \gamma + \ln\sqrt{3} + \pi/(2\sqrt{3}) \quad (5.9) \]
\[ \gamma(n) = \gamma - H_{n-1} + \ln(n), \ n \in \mathbb{N} \]
\[ \gamma(a + 1) = \gamma(a) + \ln\left(1 + \frac{1}{a}\right) - \frac{1}{a} \quad (5.10) \]
\[ \gamma'(1) = 1 - \frac{\pi^2}{6} \]
\[ \frac{1}{2x} \leq \gamma(x) \leq \frac{1}{x}, \ x \in \mathbb{R}^+ \quad (5.11) \]

Gauss’s digamma theorem \cite[Eq. 8.3636]{9} can be used to obtain additional \( \gamma(a) \) values at rational \( a \) along with those given above for \( a = 1/2 \) and \( a = 1/3 \). Equations 6.3.2, 6.3.5, and 6.4.2 of \cite{1} can be used to produce the remaining equalities listed above, and the inequality \( 1/(2x) \leq \gamma(x) \leq 1/x \), follows from Eq. 2.2 in \cite{3}.

In a similar manner, asymptotic formulas series can also be obtained. For example, by combining (5.6) with the rational zeta series for \( \psi(x+1) \) found in 8.363 of \cite{9} along with the digamma recurrence relation of 8.365 in \cite{9} leads to

\[ \gamma(x) = \gamma + \frac{1}{x} + \ln(x) + \sum_{n=2}^{\infty} \zeta(n)(-x)^{n-1}, \]

and therefore

\[ \gamma(x) \sim \gamma + 1/x + \ln(x) - x \left(\frac{\pi^2}{6}\right), \quad x \to 0. \]

Differentiating the Stirling series as it is expressed in Eq. 6.1.40 of \cite{1}, and using the definition in (5.2) along with (5.6) yields

\[ \gamma(x) \sim \frac{1}{2x} + \sum_{n=1}^{N} \frac{B_{2n}}{2nx^{2n}} + O\left(x^{-2N-1}\right), \quad x \to \infty \]

where the \( B_{2n} \) are Bernoulli numbers.

Another series for \( \gamma(a) \), which can be obtained from (5.6), is the following generalization of the Gregory coefficient series from (4.1):

\[ \gamma(a) = \sum_{n=1}^{\infty} \left( \frac{G_n}{n} \cdot \prod_{k=1}^{n} \frac{k}{a+k-1} \right). \]

This result clearly reduces to \( \gamma \) in the case of \( a = 1 \), and was obtained using (5.6) along with the formula for \( \psi \) found in Eq. 93 of \cite{5}.

6 \( \gamma(a) \) in Terms of the Hurwitz Zeta Function

The Hurwitz Zeta function \( \zeta(s,a) \) from (1.3) has been used to obtain an expressions for \( \gamma(a) \) in [16], which will be discussed in this section. Additionally, a form for \( \gamma(a) \)
in terms of a series of Hurwitz Zeta function values is also derived. All of the results presented below are analogous to the well-known Riemann Zeta function expressions for \( \gamma \). The first of such forms for \( \gamma(a) \) is given in the following theorem:

**Theorem 3.** Şintămărian’s expression for \( \gamma(a) \) in (1.2) can be obtained from the following formula:

\[
\gamma(a) = \frac{1}{a} - \sum_{n=2}^{\infty} \frac{1}{n} \left( \zeta(n, a) - \frac{1}{a^n} \right).
\]

(6.1)

**Proof.** By expressing (1.2) in the form

\[
\gamma(a) = \lim_{n \to \infty} \left[ \frac{1}{a} + \sum_{k=1}^{n-1} \left( \frac{1}{a+k} + \ln \left(1 - \frac{1}{a+k}\right) \right) \right] = \lim_{n \to \infty} \gamma_{n-1}(a),
\]

(6.2)

and expanding the logarithm we obtain

\[
\frac{1}{a} - \gamma_{n-1}(a) = \sum_{k=1}^{n-1} \sum_{m=2}^{\infty} \frac{1}{m(a+k)^m}.
\]

The double series consists of positive terms and is convergent; therefore, summations can be interchanged, and the above result expressed in terms of \( \zeta(m, a) \) is as follows:

\[
\frac{1}{a} - \gamma_{n-1}(a) = \sum_{m=2}^{\infty} \frac{1}{m} \left( \zeta(m, a) - \frac{1}{a^m} \right) - \sum_{m=2}^{\infty} \sum_{k=n}^{\infty} \frac{1}{m(a+k)^m}.
\]

(6.3)

The inside sum of the last term is less than \( \int_{a+n-1}^{\infty} t^{-m} dt \). Therefore

\[
\sum_{m=2}^{\infty} \frac{1}{m} \sum_{k=n}^{\infty} \frac{1}{(a+k)^m} < \sum_{m=2}^{\infty} \frac{1}{m(m-1)(a+n-1)^{m-1}} < \frac{1}{2(a+n-2)}
\]

for \( n \geq 2 \), where the last inequality was obtained by first fixing \( m \) to a value of 2 in the \( m(m-1) \) factor, and afterwards summing the geometric series. Then (6.1) follows after applying the limit as \( n \to \infty \) to the expression in (6.3).

Lampert in [16] provides the following formula for \( \gamma(a) \) involving the Hurwitz Zeta function:

\[
\gamma(a) = \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n, a).
\]

This alternating series expression is proved in detail in [19] directly from Şintămărian’s definition of \( \gamma(a) \) in reference [21].
An Alternative Proof of Lerch’s Limit Formula

Forms involving limits of the Hurwitz zeta functions can also be used to obtain expressions for $\gamma(a)$ which can motivate different paths to some results in classical analysis. One way this can be done is through the following expression which was obtained by adding the parameter $x$ where $x \geq 1$ within the definition from (1.2):

$$G(x, n; a) = \sum_{k=1}^{n} \frac{1}{(k + a - 1)^x} - \int_{1}^{n} \frac{du}{(u + a - 1)^x}.$$

Evaluating this integral and applying a limit on $n$ yields

$$\lim_{n \to \infty} G(x, n; a) = \zeta(x, a) - a^{1-x}(x-1)^{-1}, \quad (6.4)$$

and this convergence is uniform for any $x > 1$. Whereas from (1.2)

$$\gamma(a) = \lim_{n \to \infty} \left[ \lim_{x \to 1^+} G(x, n; a) \right]$$

because for each $n$, the $\lim_{x \to 1^+} G(x, n; a) = \gamma_n(a)$ . (Recall that $\gamma_n(a)$ is defined in (6.2).) Therefore, conditions for the the Moore-Osgood Theorem [24, p. 140] are met, and consequentially it is permissible to switch the order of the limits. Thus

$$\gamma(a) = \lim_{x \to 1^+} \left[ \zeta(x, a) - \frac{1}{a^{1-x}(x-1)} \right].$$

This result is an analog of “Lerch’s limit formula” from [18] (and cited in [14]) which can be written as

$$-\psi(a) = \lim_{x \to 1^+} \left[ \zeta(x, a) - \frac{1}{x-1} \right].$$

The connection between these two limit formulas can be clarified further by (5.6) along with the fact that $\lim_{x \to 1^+} (1 - a^{1-x})/(x-1) = \ln(a)$ . In either case, setting $a = 1$ yields the more familiar result

$$\gamma = \lim_{x \to 1^+} \left[ \zeta(x) - \frac{1}{x-1} \right].$$

(See e.g., [26, Sec. 13.21, p. 265].)

There are other classical expressions for $\gamma$ that consist of a limiting difference between two terms, each of which increase without bound under a common limit. Examples include the definition of $\gamma$ in (1.1) and the expression

$$\gamma = \lim_{x \to 0^+} \left[ \frac{1}{x} - \Gamma(x) \right],$$

which is given in [11, p. 109]. Through a direct application of the previous work, another expression for $\gamma$ along these lines can be found, and which is spelled out in the next theorem.
Theorem 4. Euler’s constant is given by

$$\gamma = \lim_{x \to 1^+} \left[ -\ln \left( 1 - \frac{1}{x} \right) - \sum_{n=2}^{\infty} \frac{\zeta(n, x)}{n} \right]. \tag{6.5}$$

Proof. If \( a > 1 \) in Theorem 3, then the sum of the \( 1/a^n \) terms in (6.1) converges to \( -\ln (1 - 1/a) \). Therefore

$$\gamma(a) = \frac{1}{a} - \sum_{n=2}^{\infty} \frac{1}{n} \left( \zeta(n, a) - \frac{1}{a^n} \right) = -\ln \left( 1 - \frac{1}{a} \right) - \sum_{n=2}^{\infty} \frac{\zeta(n, a)}{n}.$$

Replacing \( a \) with \( x \), and taking the limit as \( x \) approaches 1 completes the proof. \( \square \)

This expression for Euler’s constant is not listed among the standard collections of formulas for \( \gamma \) found in the references below.

Further confirmation of the validity of (6.5) can be found from the following integral form of Hurwitz Zeta function [26, Ch. 13, p. 260] which is valid for \( n > 1 \) and \( x > 0 \):

$$\zeta(n, x) = \frac{1}{\Gamma(n)} \int_0^\infty \frac{t^{n-1}}{e^{xt} (1 - e^{-t})} dt.$$

Using \( n! \Gamma(n) = n! \) along with the above integral, the series term in (6.5) can be expressed as a definite integral. That is,

$$\sum_{n=2}^{\infty} \frac{\zeta(n, x)}{n} = \int_0^\infty \frac{1}{te^{xt} (1 - e^{-t})} \sum_{n=2}^{\infty} \left( \frac{t^n}{n!} \right) dt = \int_0^\infty \frac{e^t - t - 1}{te^{xt} (1 - e^{-t})} dt. \tag{6.6}$$

Next write the natural logarithm term in (6.5) in terms of a Frullani integral as follows:

$$-\ln \left( 1 - \frac{1}{x} \right) = \ln \left( \frac{x}{x-1} \right) = \int_0^\infty \frac{e^{-xt} - e^{-(t-1)}(x-1)}{t} dt. \tag{6.7}$$

After subtracting the integral in (6.6) from the result in (6.7) it can be seen that the expression within the limit of (6.5) is equivalent to \( \gamma(x) \) as expressed by the first integral in (2.4). Therefore (6.5) follows from the fact that \( \lim_{x \to 1^+} \gamma(x) = \gamma \).

The sequence \( \left\{ \gamma_{n-1}(a) \right\}_{n=2}^{\infty} \) as defined in (6.2) is monotonically increasing and converges pointwise to \( \gamma(a) \). And since each function \( \gamma_{n-1}(a) \) is differentiable for \( a \in (0, \infty) \), the \( n \)th derivative of \( \gamma(a) \) can be obtained via term by term differentiation of (6.2)

$$\gamma^{(n)}(a) = (-1)^n n! \left[ \zeta(n+1, a) - \frac{1}{na^n} \right]. \tag{6.8}$$

(This result also follows from a fundamental expression for the polygamma function (cf. Eq. 6.4.2 on p. 260 of [1].) An incidental application of the above work is
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the evaluation of a family of definite integrals. That is, equating (6.8) with the $n$th derivative of (2.4) with respect to $a$ leads to the following identity:

$$
\int_0^1 x^{a-1}(\ln x)^n \left( \frac{1}{1-x} + \frac{1}{\ln x} \right) dx = (-1)^n n! \left[ \zeta(n+1, a) - \frac{1}{na^n} \right].
$$

In the appendix below, additional expressions for $\gamma(a)$ are provided along with some brief notes explaining how they can be derived.

References


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Appendix: Other Expressions for $\gamma(a)$

In this appendix several additional expressions for $\gamma(a)$ are given and explained. The first such expression can be obtained by applying the substitution $x = e^{-t}$ to the integral in (5.4) to yield

$$\gamma(a) = \ln(a) - \frac{1}{\Gamma(a)} \int_0^1 \left( \ln \left( \frac{1}{x} \right) \right)^{a-1} \ln \left( \ln \left( \frac{1}{x} \right) \right) dx.$$ 

Another expression is

$$\gamma(a) = \frac{1}{a} - \int_1^\infty \frac{(x - \lfloor x \rfloor) dx}{(x + a - 1)^2},$$

which can be verified by evaluating the integral over intervals of the form $[n, n + 1]$, summing over natural numbers $n$ and, and then appealing to the definition in (1.2).

Using (5.6) along with known expressions for the digamma function yields

$$\gamma(a) = \int_0^\infty \left( \frac{1}{(t + 1)^a} - e^{-at} \right) \frac{dt}{t} = \frac{1}{2a} + 2 \int_0^\infty \frac{t}{(t^2 + a^2)(e^{2\pi t} - 1)} dt.$$ 

Additionally, we have the following expression for $\gamma(a) - \ln(a)$ given by

$$\frac{1}{\Gamma(a)} \int_0^1 x^{a-2} (1 - e^{-x}) (1 + (a - 1) \ln x) dx = \frac{1}{\Gamma(a)} \int_0^1 \left( 1 + (1 - a) \ln x \right) e^{-1/x} dx,$$

which can be derived using a procedure analogous to one used in [11, p. 107] for deriving the corresponding $\gamma$ expression. Then applying a Maclaurin series expansion of $(1 - e^{-x})$ to the first integral term of the above expression yields

$$\gamma(a) = \ln(a) + \frac{1}{\Gamma(a)} \left[ \sum_{n=0}^{\infty} \frac{(-1)^n}{n! (a + n)^2} - \int_1^\infty \frac{(1 + (a - 1) \ln x)}{x^{2-a} e^x} dx \right].$$

Finally, the double integral

$$\gamma(a) = - \int_0^1 \int_0^1 \frac{(1 - x)(xy)^{a-1}}{\ln(xy)} \frac{1 - xy}{(1 - xy)} dy dx.$$ (6.9)

can be obtained using a modified form of a procedure outlined in [22], which will be explained in detail below.

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A path connecting (1.2) to (6.9) can be established through the expression

\[ I(a) = \sum_{n=0}^{\infty} \left( \int_0^1 \int_0^1 \int_{n+1}^{\infty} (1-x)(xy)^{t+a-1} \, dt \, dy \, dx \right), \]

by showing that the quantity \( I(a) \) is equivalent to \( \gamma(a) \). This can be accomplished by first switching the order of integration in \( I(a) \) and evaluating the \( x \) and \( y \) portions as follows:

\[ \left( \int_0^1 y^{t+a-1} \, dy \right) \left( \int_0^1 (x^{t+a-1} - x^{t+a}) \, dx \right) = \left( \frac{1}{t+a} \right) \left( \frac{1}{t+a} - \frac{1}{t+a+1} \right). \]

After distributing and expanding, the resulting integral in the variable \( t \) becomes

\[ \int_n^\infty \left[ \frac{1}{(t+a)^2} - \frac{1}{t+a} + \frac{1}{t+a+1} \right] \, dt = \frac{1}{n+a} + \ln \left( \frac{n+a}{n+a+1} \right). \]

Then applying a limit to the summation in the definition of \( I(a) \) yields

\[ I(a) = \lim_{N \to \infty} \sum_{n=0}^{N} \left[ \frac{1}{n+a} + \ln \left( \frac{n+a}{n+a+1} \right) \right] = \lim_{N \to \infty} \left[ H_{a,N} - \ln \left( \frac{N+a-1}{a} \right) \right], \]

where the expression on the right is \( \gamma(a) \). To complete the task of verifying (6.9), it is necessary to show that \( I(a) \) is also equivalent to the right hand side of (6.9). Integrating within \( I(a) \) with respect to \( t \) only, moving the summation into the integral, and using the formula for the sum of a geometric series with ratio \( xy \) yields

\[ \sum_{n=0}^{\infty} \left( \int_n^\infty (1-x)(xy)^{t+a-1} \, dt \right) = \frac{x-1}{\ln(xy)} \sum_{n=0}^{\infty} (xy)^{n+a-1} = \frac{(x-1)(xy)^{a-1}}{(1-xy) \ln(xy)}, \]

which is the integrand of (6.9). Applying the double integral in \( x \) and \( y \) to both sides of the above completes the identity.

Equation (6.9) can also be obtained through the digamma expression in Corollary 4.1 of [10], which was derived from the Lerch transcendent.

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