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EXISTENCE RESULTS FOR A FUNCTIONAL INTEGRO-DIFFERENTIAL INCLUSIONS WITH RIEMANN-STIELTJES INTEGRAL OR INFINITE-POINT BOUNDARY CONDITIONS

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Abstract. In this article, we establish the existence of solutions for initial value problem of fractional-order differential inclusion with nonlocal infinite-point or Riemann–Stieltjes integral boundary conditions. The continuous dependence of the solution on the set of selections and some other functions will be proved.

1 Introduction

In this paper, we study the existence of solutions for functional integro-differential inclusion of the form

$$\frac{dx}{dt} \in F_1(t, I^\alpha f_2(t, x(\varphi(t)))), \quad \alpha \in (0,1), \ t \in (0,T] \tag{1.1}$$

equipped with Riemann–Stieltjes integro boundary conditions

$$x(0) + \int_0^T x(s)\, dh(s) = x_\circ, \quad h : [0,T] \to \mathbb{R} \ \text{ is nondecreasing function} \tag{1.2}$$

or the infinite-point boundary conditions given by with the nonlocal condition

$$x(0) + \sum_{k=1}^{\infty} a_k\, x(\tau_k) = x_\circ, \quad a_k > 0, \tau_k \in (0,T]. \tag{1.3}$$

Where $F_1 : [0,T] \times \mathbb{R}^+ \to P(\mathbb{R})$ is a set-valued mapping and $P(\mathbb{R})$ denote the family of nonempty subsets of $\mathbb{R}$ under a set of several suitable assumptions on the function F_1. Our study is based on the selections of the set-valued function F_1 by reformulating the functional integral inclusion into a coupled system.

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We first find the continuous solution of the problem (1.1) with the m-point BCs given by

$$x(0) \ + \ \sum_{k=1}^{m} a_k \ x(\tau_k) = x_\circ, \quad a_k \ > 0, \tau_k \in [0, T] \tag{1.4}$$

and then, by using the properties of the Riemann sum for continuous functions, we investigate the solutions of the BVP with the Riemann-Stieltjes integral given by (1.1) and (1.2) as well as the BVP with infinite points given by (1.1) and (1.3). Here establish two approaches, the first approach present the existence of at least one continuous solution, where the set-valued map F_1 has Lipschitz selection and the function f_2 satisfies growth condition and the second approach presents the sufficient condition for the uniqueness of the solution where the set-valued map F_1 has Lipschitz selection and f_2 is Lipschitz function. Our approach is based on Schauder fixed point theorem, many authors use fixed point theorems to prove the existence and the uniqueness of the solution to nonlinear fractional differential equations; (see[2, 3, 10, 11, 12, 13]).

We arrange the rest of the paper as follows. Sect.2, we recall some preliminary facts that will be used in the subsequent part of the paper. Sect.3, contains our main result for the problem (1.1)– (1.4). Motivated by the developments result, we consider the BVP given by (1.1)-(1.2) and by (1.1)-(1.3). In each case, we determine sufficient conditions on existence result for the problem (1.1) under the Riemann-Stieltjes functional integral BC (1.2) and under infinite-point BC (1.3), While the continuous dependence and the uniqueness of solutions were discussed in Sect.4. Finally, Sect.5, examples are given to show the applications of our results. Conclusion is mentioned in the last Sect.6.

2 Preliminaries

Assume that E is a Banach space with the norm $\|.\|_E$. For an interval $I = [0, T]$, and denote by $C = C(I)$ the space consisting of all continuous functions defined on I and taking values in the space E. This space will be furnished with the sup-norm

$$\|x\|_C = \sup_{t \in I} |x(t)|.$$

Let X be the class of all ordered pair $u = (x, y), \ \ x, y \in C(I)$ and define the Banach space $X = C(I) \times C(I)$ with the norm

$$\|(x, y)\|_X = \|x\|_C + \|y\|_C.$$

Definition 1. *The Riemann-Liouville of fractional integral of the function $f \in L^1(I)$ of order $\alpha \in \mathbb{R}^+$ is defined by (see [17, 18, 19, 20])*

$$I_a^\alpha \ f(t) = \int_a^t \ \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \ f(s) \ ds$$

and when $a = 0$*, we have* $I^{\alpha}\ f(t) = I_0^{\alpha}\ f(t)$.

Definition 2. The Liouville-Caputo fractional derivative of $f(t)$ of order $\alpha \in (0,1]$ is defined as follows (see [18, 19])

$$D^{\alpha} f(t) = I^{1-\alpha}\frac{df(t)}{dt} = \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}\frac{df(s)}{ds}\,ds.$$

For further properties of fractional calculus operator (see [4, 18, 19, 20])

Definition 3. *Let* X *and* Y *be two nonempty sets, a set-valued (multivalued) map* $F : X \to Y$ *is a function that associates to any element* $x \in X$ *a subset* $F(x)$ *of* Y*, called the (image) valued of* F *at* x*.*

Definition 4. *Let* F *be a strict set-valued map (we say* F *is strict if the domain of* F *is* X *itself),* f *is called a selection of* F *if* $f(x) \in F(x)$*, for every* $x \in X$*, we denote by* $S_F = \{f : f(x) \in F(x), x \in X\}$ *the set of all selection of* F *(for the properties of the selection of* F *see([6, 8, 14])).*

Definition 5. *A set-valued map* F *from* $I \times E$ *to family of all nonempty closed subsets of* E *is called Lipschitzian if there exists* $L > 0$ *such that for all* $t_1, t_2 \in I$ *and all* $x_1, x_2 \in E$*, we have*

$$H(F(t, x_1), F(s, x_2)) \le L(\|t - s\| + \|x_1 - x_2\|) \tag{2.1}$$

where $H(A, B)$ *is the Hausdorff metric distance between the two subsets* $A, B \in I \times E$ *(for properties of the Hausdorff distance see ([1])).*
Denote $S_F = Lip(I, E)$ *be the set of all Lipschitz selections of the set-valued function* F *with values in the Banach space* E*.*
Let $E = \mathbb{R}^n$*.The following Theorem [[1], Sect.9, Chap. 1, Th. 1] assume the existence of Lipschitzian selection.*

Theorem 6. *Let* M *be a metric space and* F *be Lipschitzian set-valued function from* M *into the nonempty compact convex subsets of* R^n*. Assume, moreover, that for some* $\lambda > 0$*,* $F(x) \subset \lambda B$ *for all* $x \in M$ *where* B *is the unit ball on* $\mathbb{R}^n$*. Then there exists a constant* c *and a single-valued function* $f : M \to \mathbb{R}^n$*,* $f(x) \in F(x)$ *for* $x \in M$*; this function is Lipschitzian with constant* c*.*

Theorem 7. *[7] "Schauder fixed point theorem"*
Let Q *be a convex subset of a Banach space* X*,* $T : Q \to Q$ *be a compact, continuous map. Then* T *has at least one fixed point in* Q*.*

3 Existence of at least one continuous solution to (1.1) with the m-Point BCs (1.4)

Definition 8. *A function* x *is called a solution of problem (1.1) with the m-Point BCs (1.4) if* $x \in C(I)$ *and satisfies (1.1) and (1.4).*

Consider the following assumptions:

(i) The set-valued map $F_1 : I \times \mathbb{R} \rightarrow 2^{\mathbb{R}}$ is Lipschitzian set-valued map with a nonempty compact convex subset of $2^{\mathbb{R}^+}$, with a Lipschitz constant $k > 0$.

Remark From this assumption and Theorem 6, we can deduce the set of Lipschitz selections of F_1 not empty and there exists $f_1 \in F_1$ such that

$$|f_1(t,x) - f_1(t,y)| \leq k|x-y|.$$

(ii) $\varphi : I \rightarrow I$ is a continuous function.

(iii) $f_2 : I \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies Caratheodory condition, i.e., f_2 is measurable in t for any $x \in \mathbb{R}$ and continuous in x for almost all $t \in I$. There exist a function $a(t)$ that is measurable bounded and there is a positive constant $b > 0$, such that

$$|f_2(t,x)| \leq \ a(t) + \ b|x|, \forall \ \ t \ \in I \ and \ x \in \mathbb{R}.$$

(iv) The following condition holds,
$[a(|x_\circ| + \sum_{k=1}^{m} |a_k|) + 1]kT < 1$, $\frac{b \ T^\alpha}{\Gamma(\alpha+1)} < 1$, and $I_c^\gamma a(.) \leq M \ \ \forall \gamma \ \leq \alpha, \ c \geq 0.$

Lemma 9. *The boundary single-valued problem given by*

$$\frac{dx}{dt} \ = \ f_1(t, I^\alpha f_2(t, x(\varphi(t)))), \quad \alpha \in (0,1), \ t \in I \tag{3.1}$$

with the non-local condition (1.4), is equivalent to the following integral equation

$$x(t) = a\left(x_\circ - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f_1(s, I^\alpha f_2(s, x(\varphi(s))))ds\right) + \int_0^t f_1(s, I^\alpha f_2(s, x(\varphi(s))))ds \tag{3.2}$$

where $a = (1 + \sum_{k=1}^{m} \ a_k)^{-1}$.

Proof. We start by looking at the problem (3.1) with m-point BCs in (1.4). Integrating both sides of (3.1),we get

$$x(t) = \ x(0) \ + \ If_1(t, I^\alpha f_2(t, x(\varphi(t)))). \tag{3.3}$$

Substituting for the value of $x(0)$ from (1.4), we get

$$x(t) = \ x_\circ \ - \ \sum_{k=1}^{m} \ a_k \ x(\tau_k) \ + \ \int_0^t f_1(s, I^\alpha f_2(s, x(\varphi(s)))) \ ds. \tag{3.4}$$

Indeed, upon setting $t = \tau_k \in [0,T]$ in Equation (3.4), we get

$$x(\tau_k) = \ x_\circ \ - \ \sum_{k=1}^{m} \ a_k \ x(\tau_k) \ + \int_0^{\tau_k} f_1(s, I^\alpha f_2(s, x(\varphi(s)))) \ ds. \tag{3.5}$$

So, we have

$$x(\tau_k) = x(t) + \int_0^{\tau_k} f_1(s, I^\alpha f_2(s, x(\varphi(s))))ds - \int_0^t f_1(s, I^\alpha f_2(s, x(\varphi(s))))ds. \quad (3.6)$$

Substituting (3.6) in (3.4)

$$\begin{aligned} x(t) &= x_\circ - \sum_{k=1}^m a_k \left(x(t) + \int_0^{\tau_k} f_1(s, I^\alpha f_2(s, x(\varphi(s))))\, ds\right. \\ &- \left.\int_0^t f_1(s, I^\alpha f_2(s, x(\varphi(s))))\, ds\right) + \int_0^t f_1(s, I^\alpha f_2(s, x(\varphi(s))))\, ds. \end{aligned}$$

Consequently, we can get

$$\begin{aligned} (1 + \sum_{k=1}^m a_k)\, x(t) &= x_\circ - \sum_{k=1}^m a_k \int_0^{\tau_k} f_1(s, I^\alpha f_2(s, x(\varphi(s))))\, ds \\ &+ (1 + \sum_{k=1}^m a_k) \int_0^t f_1(s, I^\alpha f_2(s, x(\varphi(s))))\, ds. \end{aligned}$$

Letting $a = (1 + \sum_{k=1}^m a_k)^{-1}$, we deduce that the non-local problem (3.1)-(1.4) transformed to the integral equation

$$x(t) = a\left(x_\circ - \sum_{k=1}^m a_k \int_0^{\tau_k} f_1(s, I^\alpha f_2(s, x(\varphi(s))))ds\right) + \int_0^t f_1(s, I^\alpha f_2(s, x(\varphi(s))))ds.$$

Finally, in order to complete the proof of the above Lemma, we show that Equation (3.2) satisfies problem (3.1) together with the m-point BCs in (1.4). In fact, upon differentiating (3.2) with respect to t, we obtain

$$\frac{dx}{dt} = f_1(t, I^\alpha f_2(t, x(\varphi(t)))).$$

Again, from (3.2), we have

$$x(\tau_k) = a\left(x_\circ - \sum_{k=1}^m a_k \int_0^{\tau_k} f_1(s, I^\alpha f_2(s, x(\varphi(s))))ds\right) + \int_0^{\tau_k} f_1(s, I^\alpha f_2(s, x(\varphi(s))))ds, \quad (3.7)$$

$$\begin{aligned} (1 + \sum_{k=1}^m a_k)x(\tau_k) &= x_\circ - \sum_{k=1}^m a_k \int_0^{\tau_k} f_1(s, I^\alpha f_2(s, x(\varphi(s))))\, ds \\ &+ (1 + \sum_{k=1}^m a_k) \int_0^{\tau_k} f_1(s, I^\alpha f_2(s, x(\varphi(s))))ds \\ x(\tau_k) + \sum_{k=1}^m a_k\, x(\tau_k) &= x_\circ + \int_0^{\tau_k} f_1(s, I^\alpha f_2(s, x(\varphi(s))))ds, \end{aligned}$$

then

$$\sum_{k=1}^{m} a_k\ x(\tau_k) \quad = \quad x_\circ - x(\tau_k) + \int_0^{\tau_k} f_1(s, I^\alpha f_2(s, x(\varphi(s))))ds. \tag{3.8}$$

From (3.2) we have

$$x(0) = a\left(x_\circ \ - \ \sum_{k=1}^{m} a_k \int_0^{\tau_k} f_1(s, I^\alpha f_2(s, x(\varphi(s))))ds\right).$$

Substitute the value of $x(0)$ in (3.7)

$$x(\tau_k) = x(0) \ + \ \int_0^{\tau_k} f_1(s, I^\alpha f_2(s, x(\varphi(s))))ds$$

and

$$x(0) = x(\tau_k) - \int_0^{\tau_k} f_1(s, I^\alpha f_2(s, x(\varphi(s))))ds. \tag{3.9}$$

Adding (3.8) and (3.9), we get m-point BC (1.4)

$$x(0) \ + \ \sum_{k=1}^{\infty} a_k\ x(\tau_k) = x_\circ.$$

This complete the proof. □

Definition 10. *A function $x \in C(I,R)$ is called a solution of the nonlocal problem (1.1)-(1.4) if there exists a function $f_1 \in L^1(I,R)$ with $f_1(t,x(t) \in F_1(t,x(t)$, a.e. on I such that*

$$\frac{dx}{dt} = f_1(t, I^\alpha f_2(t, x(\varphi(t)))), \qquad a.e.\ on\ I$$

with the nonlocal condition $x(0) \ + \ \sum_{k=1}^{m} \ a_k\ x(\tau_k) = x_\circ$.

It is clear that from Theorem 6 and assumption (i), the set of Lipschitz selection of F_1 is non empty. So, the solution of the single-valued integral equation (3.2) where $f_1 \in S_{F_1}$, is a solution to the inclusion (1.1) with $x(0) \ + \ \sum_{k=1}^{m} \ a_k\ x(\tau_k) = x_\circ$.

It must be noted that f_1 satisfied the Lipschitz selection

$$|f_1(t,x) - f_1(t,y)| \le k|x-y|.$$

Now, let

$$y(t) = I^\alpha f_2(t, x(\varphi(t))), \qquad t \in I, \tag{3.10}$$

then the nonlinear functional integral equation (3.2) can be written in the form

$$x(t) = a\left(x_\circ - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f_1(s, y(s))\, ds \right) + \int_0^t f_1(s, y(s))\, ds, \quad t \in I. \tag{3.11}$$

Hence, the functional integral equation (3.2) is equivalent to the coupled system (3.10) and (3.11).

Now, we study the existence of a continuous solution of the functional integral equation (3.2), which is a solution of the functional integral inclusion (1.1) with nonlocal condition (1.4) , by getting the continuous solution of the coupled system (3.10) and (3.11).

Definition 11. *By a solution of the coupled system (3.11), (3.10) we mean the functions $x, y \in C(I)$ satisfying (3.11), (3.10).*

Remark 12. *From the Lipschitz condition of f_1, we have*

$$|f_1(t,x)| - |f_1(t,0)| \le |f_1(t,x) - f_1(t,0)| \le k|x|$$

i.e.,

$$|f_1(t,x)| \le k|x| + \sup_{t\in[0,T]} |f_1(t,0)| \le k|x| + f_1^*$$

where

$$f_1^* = \sup_{t\in[0,T]} |f_1(t,0)|.$$

Now for the existence of at least one solution, $u = (x, y)$, $x, y \in C(I)$ of the coupled system (3.10), (3.11) we have the following theorem.

Theorem 13. *Let the assumptions $(i) - (iv)$ be satisfied. Then there exists at least one continuous solution $u = (x, y)$, $x, y \in C(I)$ of the coupled system (3.10), (3.11).*

Proof. Let the set Q_r be defined as

$$Q_r = \{u = (x, y) \in R^2, \|u\| \le r\}$$

where $r = r_1 + r_2 = \frac{a|x_\circ| + [a\sum_{k=1}^m |a_k| + 1] f_1^* T}{1 - [a\sum_{k=1}^m |a_k| + 1] kT} + (1 - \frac{b\, T^\alpha}{\Gamma(\alpha+1)})^{-1} \frac{M T^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}$. It is clear that the set Q_r is nonempty, bounded, closed and convex.

Next, let us denote by A the operator defined on the space $C(I)$ by

$$Au(t) = A(x, y)(t) = (A_1 y(t), A_2 x(t)),$$

$$A_1 y(t) = a(x_\circ - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f_1(s, y(s))\, ds) + \int_0^t f_1(s, y(s)))\, ds, \quad t \in I$$

and

$$A_2x(t) \;=\; \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_2(s, x(\varphi(s)))\; ds \qquad t \in I$$

where for $u = (x, y) \in Q_r$, and from Remark (12) we have

$$\begin{aligned}
|A_1y(t)| &= |a(\; x_\circ - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f_1(s, y(s))\; ds\;) \;+\; \int_0^t f_1(s, y(s)))\; ds| \\
&\leq a\;|x_\circ| + a \sum_{k=1}^{m} a_k \int_0^{\tau_k} |f_1(s, y(s))|\; ds \;+\; \int_0^t |f_1(s, y(s)))|\; ds \\
&\leq a\;|x_\circ| + [a \sum_{k=1}^{m} |a_k|\;\; + 1](k|y| + f_1^*)T,
\end{aligned}$$

then

$$\begin{aligned}
\|A_1y\| &\leq \quad a\;|x_\circ| + [a \sum_{k=1}^{m} |a_k|\;\; + 1](k|y| + f_1^*)T = r_1, \\
r_1 &= \frac{a|x_\circ| + [a\sum_{k=1}^{m} |a_k|\;\; + 1]f_1^*T}{1 - [a\sum_{k=1}^{m} |a_k|\;\; + 1]kT}.
\end{aligned}$$

Also

$$\begin{aligned}
|A_2x(t)| &= |\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_2(s, x(\varphi(s)))ds| \\
&\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f_2(s, x(\varphi(s)))|\; ds \\
&\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} [a(s) + b\;|x(\varphi(s))|]ds.
\end{aligned}$$

Taking supremum over $t \in I$,

$$\begin{aligned}
\|A_2x\| &\leq \int_0^t a(s)\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds + \int_0^t b\;|x(\varphi(s))|\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds \\
&\leq I^\alpha a(t) + br_2 \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds \\
&\leq I^{\alpha-\gamma} I^\gamma a(t) + br_2 I^\alpha(t) \\
&\leq M \int_0^t \frac{(t-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}ds + br_2 \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds \\
&\leq \frac{M\; t^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} + br_2\; \frac{t^\alpha}{\Gamma(\alpha+1)} \\
&\leq \frac{M\; T^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} + \frac{br_2\; T^\alpha}{\Gamma(\alpha+1)} = r_2, \quad r_2 = (1 - \frac{b\;T^\alpha}{\Gamma(\alpha+1)})^{-1} \frac{M\; T^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}.
\end{aligned}$$

Now

$$\begin{aligned}\|Au\|_X &= \|A_1y\|_C+\|A_2x\|_C \ \leq\ r_1+r_2\\ &\leq \frac{a|x_\circ|+[a\sum_{k=1}^{m}\ |a_k|\ +1]f_1^*T}{1-[a\sum_{k=1}^{m}\ |a_k|\ +1]kT}+(1-\frac{b\,T^\alpha}{\Gamma(\alpha+1)})^{-1}\frac{M\ T^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}=r.\end{aligned}$$

Then $AQ_r\subset Q_r$ and the class $\{Au\}$, $u\in Q_r$ is uniformly bounded.

Now, for $u=(x,y)\in Q_r$, for all $\epsilon>0$, $\delta>0$ and for each $t_1,t_2\in[0,T]$, $t_1<t_2$ such that $|t_2-t_1|<\delta$, we have

$$\begin{aligned}|A_1y(t_2)-A_1y(t_1)| &= |a(\ x_\circ\ -\ \sum_{k=1}^{m}\ a_k\ \int_0^{\tau_k}f_1(s,y(s))ds)\ +\ \int_0^{t_2}f_1(s,y(s))ds\\ &-\ a(\ x_\circ\ -\ \sum_{k=1}^{m}\ a_k\ \int_0^{\tau_k}f_1(s,y(s))ds)\ +\ \int_0^{t_1}f_1(s,y(s))ds|\\ &\leq \int_{t_1}^{t_2}|f_1(s,y(s))|ds\\ &\leq (k|y|+f_1^*)\int_{t_1}^{t_2}ds\\ &\leq (kr_1+f_1^*)(t_2-t_1),\end{aligned}$$

and

$$\begin{aligned}&|A_2x(t_2)-A_2x(t_1)|\\ \leq\ & |\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}f_2(s,x(\varphi(s)))ds-\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}f_2(s,x(\varphi(s)))ds|\\ \leq\ & |\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}f_2(s,x(\varphi(s)))ds-\int_0^{t_1}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}f_2(s,x(\varphi(s)))ds|\\ +\ & |\int_0^{t_1}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}f_2(s,x(\varphi(s)))ds-\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}f_2(s,x(\varphi(s)))ds|\\ \leq\ & |\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}f_2(s,x(\varphi(s)))ds|+|\int_0^{t_1}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}f_2(s,x(\varphi(s)))ds|\\ -\ & |\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}f_2(s,x(\varphi(s)))ds|\\ \leq\ & \int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}|f_2(s,x(\varphi(s)))|ds+\int_0^{t_1}\frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}|f_2(s,x(\varphi(s)))|ds\\ \leq\ & \int_{t_1}^{t_2}[a+b|x(\varphi(s))|]\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}ds+\int_0^{t_1}[a+b|x(\varphi(s))|]\frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}ds\\ \leq\ & (a+br_2)\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}ds+(a+br_2)\int_0^{t_1}\frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}ds\end{aligned}$$

$$\leq (a+br_2)\frac{(t_2-t_1)^\alpha}{\Gamma(\alpha+1)} + (a+br_2)\left(\frac{-(t_2-t_1)^\alpha}{\Gamma(\alpha+1)} + \frac{t_2^\alpha}{\Gamma(\alpha+1)} - \frac{t_1^\alpha}{\Gamma(\alpha+1)}\right)$$
$$\leq (a+br_2)\frac{(t_2^\alpha - t_1^\alpha)}{\Gamma(\alpha+1)}.$$

For the operator A and $u \in Q_r$, we have

$$\begin{aligned} Au(t_2) - Au(t_1) &= A(x,y)(t_2) - A(x,y)(t_1) \\ &= (A_2x(t_2), A_1y(t_2)) - (A_2x(t_1), A_1y(t_1)) \\ &= (A_2x(t_2) - A_2x(t_1), A_1y(t_2) - A_1y(t_1)), \end{aligned}$$

then

$$\begin{aligned} |Au(t_2) - Au(t_1)|_X &= |A(x,y)(t_2) - A(x,y)(t_1)|_X, \\ &= |A_1y(t_2) - A_1y(t_2)|_C + |A_2x(t_2) - A_2x(t_1)|_C \\ &= (kr_1 + f_1^*)(t_2 - t_1) \ + \ (a+br_2)\,\frac{(t_2^\alpha - t_1^\alpha)}{\Gamma(\alpha+1)}. \end{aligned}$$

This means that the class of functions $\{Au\}$ is equi-continuous on Qr. Then by the Arzela-Ascoli Theorem [5], the operator A is compact. It remains to prove the continuity of $A: Qr \to Qr$. Let $u_n = (x_n, y_n)$ be a sequence in Qr with $x_n \to x$,and $y_n \to y$ and since $f_2(t, x(t))$ is continuous in $C(I) \times R$, then $f_2(t, x_n(t))$ converges to $f_2(t, x(t))$, thus $f_2(t, x_n(\varphi(t)))$ converges to $f_2(t, x(\varphi(t)))$, using assumptions (iii)-(iv) and applying Lebesgue Dominated Convergence Theorem, we get

$$\lim_{n\to\infty} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\ f_2(s, x_n(\varphi(s)))\ ds \ = \ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\ f_2(s, x(\varphi(s)))\ ds,$$

then

$$\begin{aligned} \lim_{n\to\infty} A_2x_n(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \lim_{n\to\infty} f_2(s, x_n(\varphi(s)))ds \\ &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_2(s, x(\varphi(s)))ds = A_2x(t) \\ \lim_{n\to\infty} A_1y_n(t) &= a(x_\circ - \sum_{k=1}^m a_k \int_0^{\tau_k} \lim_{n\to\infty} f_1(s, y_n(s))ds) + \int_0^t \lim_{n\to\infty} f_1(s, y_n(s))ds \\ &= a(x_\circ - \sum_{k=1}^m a_k \int_0^{\tau_k} f_1(s, y(s))\ ds) + \int_0^t f_1(s, y(s))ds = A_1y(t). \end{aligned}$$

So,

$$\begin{aligned} \lim_{n\to\infty} Au_n(t) &= \lim_{n\to\infty} (A_1y_n(t), A_2x_n(t)) \\ &= (\lim_{n\to\infty} A_1y_n(t) \lim_{n\to\infty} A_2x_n(t)) = (A_1y(t), A_2x(t)) \ = \ Au(t). \end{aligned}$$

Then $Au_n \to Au$ as $n \to 1$. This mean that the operator A is continuous.
Since all conditions of the Schauder fixed-point theorem [7] hold then A has a fixed point $u \in Q_r$, and then the system (3.11), (3.10) has at least one continuous solutions $u = (x, y) \in Q_r$, $x; y \in C(I)$.

Consequently, the functional integral equation (1.1) has at least one solution $x \in C(I)$.

Conversely, differentiating (3.2), we obtain

$$\frac{dx}{dt} = \frac{d}{dt}\left\{ a\left(x_\circ - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f_1(s, y(s))ds \right) + \int_0^t f_1(s, y(s))ds \right\},$$
$$y(t) = I^\alpha f_2(t, x(\varphi(t))).$$

Also, from the integral equation (3.10)–(3.11), we obtain

$$\begin{aligned} x(\tau_k) &= a\left(x_\circ - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f_1(s, y(s))\ ds \right) + \int_0^{\tau_k} f_1(s, y(s))\ ds, \\ x(0) &= a\left(x_\circ - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f_1(s, y(s))\ ds \right), \\ y(t) &= I^\alpha f_2(t, x(\varphi(t))) \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} \sum_{k=1}^{m} a_k x(\tau_k) = a \sum_{k=1}^{m} a_k \left(x_\circ - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f_1(s, y(s))ds \right) + \sum_{k=1}^{m} a_k \int_0^{\tau_k} f_1(s, y(s))ds, \\ y(t) = I^\alpha f_2(t, x(\varphi(t))). \end{aligned} \tag{3.13}$$

From (3.12) and (3.13), we have

$$\begin{aligned} x(0) + \sum_{k=1}^{m} a_k\ x(\tau_k) = a(1 + \sum_{k=1}^{m} a_k) \left(x_\circ - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f_1(s, y(s))ds \right) \\ + \sum_{k=1}^{m} a_k \int_0^{\tau_k} f_1(s, y(s))ds. \end{aligned}$$

Then

$$x(0) + \sum_{k=1}^{m} a_k\ x(\tau_k) = x_\circ.$$

Hence, there exist at least one solution $x \in C(I)$ of the nonlocal problem of functional differential inclusion (1.1)-(1.4). □

3.1 Riemann-Stieltjes Integral BCs (1.2)

Let $x \in C(I)$ be the solution of the non-local problem of (1.1) − (1.4). Let $a_k = h(t_k) - h(t_{k-1})$, h is nondecreasing function, $\tau_k \in (t_{k-1}, t_k)$, $0 = t_0 < t_1 < t_2 \cdots < T$. Then, the nonlocal condition (1.4) will be in the form

$$x(0) + \sum_{k=1}^{m} x(\tau_k)\ (h(t_j) - h(t_{j-1}))\ = x_\circ.$$

From the continuity of the solution of the nonlocal problem (1.1) − (1.4), we obtain from [18] as $m \to \infty$

$$\lim_{m\to\infty} \sum_{k=1}^{m} x(\tau_k)\ (h(t_j) - h(t_{j-1}))\ = \int_0^T x(s)\ dh(s),$$

that is, the non-local conditions (1.4) is transformed to Riemann-Steltjes integral condition as $m \to \infty$

$$x(0) + \lim_{m\to\infty} \sum_{k=1}^{m} x(\tau_k)\ (h(t_j) - h(t_{j-1}))\ = x(0) + \int_0^T x(s)\ dh(s) = x_\circ.$$

Now, we can state the following theorem.

Theorem 14. *Let the assumptions (i)–(iv) of Theorem 13 be satisfied and $h : [0,T] \to [0,T]$ is an increasing function, then there exists a solution $x \in C(I)$ of the non-local problem of (1.1) together with the Riemann-Stieltjes functional integral condition (1.2) which is represented by*

$$\begin{aligned} x(t) &= (1 + h(T) - h(0))^{-1} x_\circ - (1 + h(T) \\ &- h(0))^{-1} \int_0^T \int_0^t f_1(s, I^\alpha f_2(s, x(\varphi(s))))dsdh(s) + \int_0^t f_1(s, I^\alpha f_2(s, x(\varphi(s))))ds. \end{aligned} \quad (3.14)$$

Proof. As $m \to \infty$, the solution of the the non-local problem (1.1) − (1.4) will be

$$\begin{aligned} & x(t) \\ = & \lim_{m\to\infty} \frac{1}{(1+\sum_{k=1}^{m} a_k)} \big(x_\circ - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f_1(s, I^\alpha f_2(s, x(\varphi(s))))ds\big) \\ + & \int_0^t f_1(s, I^\alpha f_2(s, x(\varphi(s))))ds \\ = & \frac{1}{(1+h(T)-h(0))} \big(x_\circ - \lim_{m\to\infty} \sum_{k=1}^{m} (h(t_j)-h(t_{j-1})) \int_0^{\tau_k} f_1(s, I^\alpha f_2(s, x(\varphi(s))))ds\big) \\ + & \int_0^t f_1(s, I^\alpha f_2(s, x(\varphi(s))))\ ds \\ = & \frac{1}{(1+h(T)-h(0))} \big(x_\circ - \int_0^T \int_0^t f_1(s, I^\alpha f_2(s, x(\varphi(s))))dsdh\big) \\ + & \int_0^t f_1(s, I^\alpha f_2(s, x(\varphi(s))))\ ds. \end{aligned}$$

□

Therefore, the solution $x \in C(I)$ of the first-order nonlinear differential Equation (1.1) with the Riemann-Stieltjes integral condition (1.2) is given by (3.14).
Hence, there exist at least one solution $x \in C(I)$ of the nonlocal problem of functional differential inclusion (1.1)-(1.2).

3.2 Infinite-Point Boundary Condition (1.3)

Assume $x \in C(I)$ be the solution of the non-local problem given by (1.1) and (1.3). Then, we can state the following theorem.

Theorem 15. *Let the assumptions (i)–(iv) of Theorem 13 be satisfied and let* $S_m^{-1} = 1+\sum_{k=1}^{m} a_k$ *be convergent sequence, Then the non-local problem of (1.1)-(1.3) given by the following integral equation*

$$x(t) = S_m x_\circ - S_m \sum_{k=1}^{m} a_k \int_0^{\tau_k} f_1(s, I^\alpha f_2(s, x(\varphi(s))))ds + \int_0^t f_1(s, I^\alpha f_2(s, x(\varphi(s))))ds \tag{3.15}$$

has at least one solution $x \in C(I)$.

Proof. Let $x \in C(I)$ be a solution of the infinite point BVP (1.1) and (1.3) given by

(3.2).

$$x_m(t) = \frac{1}{(1+\sum_{k=1}^{m} \ a_k)}\left(x_\circ \ - \ \sum_{k=1}^{m} a_k \ \int_0^{\tau_k} f_1(s, I^\alpha f_2(s, x(\varphi(s))))ds\right) + \int_0^t f_1(s, I^\alpha f_2(s, x_m(\varphi(s))))ds. \tag{3.16}$$

Take the limit to (3.16), as $m \to \infty$, we have

$$\begin{aligned} &\lim_{m\to\infty} x_m(t) \\ = &\lim_{m\to\infty}\left[\frac{1}{(1+\sum_{k=1}^{m} \ a_k)}\left(x_\circ \ - \ \sum_{k=1}^{m} \ a_k \ \int_0^{\tau_k} f_1(s, I^\alpha f_2(s, x(\varphi(s)))) \ ds\right)\right. \\ + &\left.\int_0^t f_1(s, I^\alpha f_2(s, x_m(\varphi(s)))) \ ds\right] \\ = &\lim_{m\to\infty}\frac{1}{(1+\sum_{k=1}^{m} \ a_k)}\left[x_\circ \ - \ \sum_{k=1}^{m} \ a_k \ \int_0^{\tau_k} f_1(s, I^\alpha f_2(s, x(\varphi(s)))) \ ds\right] \\ + &\lim_{m\to\infty}\int_0^t f_1(s, I^\alpha f_2(s, x_m(\varphi(s)))) \ ds. \end{aligned} \tag{3.17}$$

Now $|a_k x(\tau_k)| \leq |a_k| \|x\|$, therefore by comparison test $\sum_{k=1}^{m} \ a_k x(\tau_k)$is convergent. Also

$$\begin{aligned} &\left|\int_0^{\tau_k} f_1(s, I^\alpha f_2(s, x(\varphi(s)))) \ ds\right| \\ \leq &\int_0^{\tau_k}(k \ |I^\alpha f_2(s, x(\varphi(s)))| \ + f_1^*)ds \\ \leq & \ k \ \int_0^{\tau_k}[I^\alpha a(s) + b\int_0^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)}|x(\varphi(\theta))|d\theta \ ]ds + k \ \int_0^{\tau_k} f_1^* ds \\ \leq & \ k \ \int_0^{\tau_k}[I^{\alpha-\gamma}I^\gamma a(s) + br_2\int_0^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)}d\theta \ ]ds + k \ f_1^* \ T \\ \leq & \ k \ M \ \int_0^{\tau_k}\int_0^s \frac{(s-\theta)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}d\theta \ ds + k \ br_2\int_0^{\tau_k}\int_0^s \frac{(s-\theta)^{\alpha-1}}{\Gamma(\alpha)}d\theta \ ds \\ + & \ k \ f_1^* \ T \\ \leq & \ \frac{k \ M \ T^{\alpha-\gamma+1}}{\Gamma(\alpha-\gamma+1)} + \frac{k \ br_2 \ T^{\alpha+1}}{\Gamma(\alpha+1)} + k \ f_1^* \ T \leq N, \end{aligned}$$

then

$$|a_k| \ |\int_0^{\tau_k} f_1(s, I^\alpha f_2(s, x(\varphi(s)))) \ ds| \leq |a_k| \ N,$$

and by the comparison test $\sum_{k=1}^{m} a_k \int_0^{\tau_k} f_1(s, I^{\alpha} f_2(s, x(\varphi(s))))\ ds$ is convergent. Using assumptions $(i)-(iii)$ and applying Lebesgue Dominated convergence Theorem [15], from (3.17) we obtain (3.15). Furthermore, from (3.15), we have

$$\begin{aligned}(1+\sum_{k=1}^{m} a_k)x(\tau_k) &= S_m^{-1}S_m x_\circ - S_m^{-1}S_m \sum_{k=1}^{m} a_k \int_0^{\tau_k} f_1(s, I^{\alpha} f_2(s, x(\varphi(s))))ds \\ &+ (1+\sum_{k=1}^{m} a_k) \int_0^{\tau_k} f_1(s, I^{\alpha} f_2(s, x(\varphi(s))))ds \\ x(\tau_k)+\sum_{k=1}^{m} a_k\ x(\tau_k) &= x_\circ - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f_1(s, I^{\alpha} f_2(s, x(\varphi(s))))\ ds \\ &+ \int_0^{\tau_k} f_1(s, I^{\alpha} f_2(s, x(\varphi(s))))ds \\ &+ \sum_{k=1}^{m} a_k \int_0^{\tau_k} f_1(s, I^{\alpha} f_2(s, x(\varphi(s))))ds\end{aligned}$$

$$\sum_{k=1}^{m} a_k\ x(\tau_k) = x_\circ - x(\tau_k) + \int_0^{\tau_k} f_1(s, I^{\alpha} f_2(s, x(\varphi(s))))ds. \tag{3.18}$$

From (3.2), we have

$$x(0) = a\left(x_\circ - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f_1(s, I^{\alpha} f_2(s, x(\varphi(s))))ds\right)$$

and

$$x(\tau_k) = a\left(x_\circ - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f_1(s, I^{\alpha} f_2(s, x(\varphi(s))))ds\right) + \int_0^{\tau_k} f_1(s, I^{\alpha} f_2(s, x(\varphi(s))))ds,$$

So

$$x(0) = x(\tau_k) - \int_0^{\tau_k} f_1(s, I^{\alpha} f_2(s, x(\varphi(s))))ds.$$

Back to (3.18) we get infinite-point BC (1.3)

$$x(0)\ +\ \sum_{k=1}^{\infty} a_k\ x(\tau_k) = x_\circ.$$

Hence, there exist at least one solution $x \in C(I)$ of the nonlocal problem of functional differential inclusion (1.1)-(1.3). This complete the proof. □

4 Existence of unique solutions to (1.1) with the m-Point BCs (1.4)

In this section, we give the sufficient condition for the uniqueness result for non-local problem (1.1)-(1.4). Let us assume the following assumption

$(iii)^*$ Let $f_2 : I \times R \to R$, be a continuous function satisfying the Lipschitz condition, such that $|f_2(t,x) - f_2(t,y)| \leq c\ |x - y|$.

Theorem 16. *Let the assumptions of Theorem 13 be satisfied with replace condition (iii) by $(iii)^*$, if $1 - \frac{(\ a\ \sum_{k=1}^{m}\ a_k+1)\ T^{\alpha+1}\ k\ c}{\Gamma(\alpha+1)} < 1$. Then the solution $x \in C(I)$ of non-local problem (1.1)-(1.4) is unique.*

Proof. Let $x_1(t)$ and $x_2(t)$ be two solutions of the functional integral equation (3.2), then

$$\begin{aligned}
& x_1(t) - x_2(t) \\
= &\ a\left(x_\circ - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f_1(s, I^\alpha f_2(s, x_1(\varphi(s))))ds\right) + \int_0^t f_1(s, I^\alpha f_2(s, x_1(\varphi(s))))ds \\
- &\ a\left(x_\circ - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f_1(s, I^\alpha f_2(s, x_2(\varphi(s))))ds\right) - \int_0^t f_1(s, I^\alpha f_2(s, x_2(\varphi(s))))ds \\
& |x_1(t) - x_2(t)| \\
\leq &\ a\left|\sum_{k=1}^{m} a_k\ (\int_0^{\tau_k} f_1(s, I^\alpha f_2(s, x_2(\varphi(s))))ds - \int_0^{\tau_k} f_1(s, I^\alpha f_2(s, x_1(\varphi(s))))\ ds)\right| \\
+ &\ |\int_0^t\ [f_1(s, I^\alpha f_2(s, x_1(\varphi(s))))\ - f_1(s, I^\alpha f_2(s, x_2(\varphi(s))))\ ]\ ds| \\
\leq &\ a\sum_{k=1}^{m} a_k\ (\int_0^{\tau_k} |f_1(s, I^\alpha f_2(s, x_2(\varphi(s)))) - f_1(s, I^\alpha f_2(s, x_1(\varphi(s))))|ds) \\
+ &\int_0^t\ |f_1(s, I^\alpha f_2(s, x_1(\varphi(s))))\ - f_1(s, I^\alpha f_2(s, x_2(\varphi(s))))|\ ds.
\end{aligned}$$

Using Lipschitz condition for f_1, we obtain

$$\begin{aligned}
|x_1(t) - x_2(t)| \leq &\ a\sum_{k=1}^{m} a_k\ k\int_0^{\tau_k} |I^\alpha f_2(s, x_1(\varphi(s))) - I^\alpha f_2(s, x_2(\varphi(s)))|\ ds \\
+ &\ k\int_0^t |I^\alpha f_2(s, x_1(\varphi(s))) - I^\alpha f_2(s, x_2(\varphi(s)))|\ ds
\end{aligned}$$

$$\leq \quad a \sum_{k=1}^{m} a_k \; k \; (\int_0^{\tau_k} \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} |f_2(\tau, x_1(\varphi(\tau))) - f_2(\tau, x_2(\varphi(\tau)))| d\tau \; ds$$
$$+ \quad k \int_0^t \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} |f_2(\tau, x_1(\varphi(\tau))) - f_2(\tau, x_2(\varphi(\tau)))| d\tau \; ds.$$

Using Lipschitz condition for f_2, we obtain

$$\begin{aligned}
|x_1(t) - x_2(t)| \quad &\leq \quad a \sum_{k=1}^{m} a_k \; k \; c \int_0^{\tau_k} \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} |x_1(\varphi(\tau)) - x_2(\varphi(\tau))\;| d\tau \; ds \\
&+ \quad k \; c \int_0^t \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} |x_1(\varphi(\tau)) - x_2(\varphi(\tau))\;| d\tau \; ds \\
&\leq \quad a \sum_{k=1}^{m} a_k \; k \; c \, \|x_1 - x_2\| \int_0^{\tau_k} \int_0^s \frac{(s-\tau\;)^{\alpha-1}}{\Gamma(\alpha)} \; d\tau \quad ds \\
&+ \quad k \; c \, \|x_1 - x_2\| \int_0^t \int_0^s \frac{(s-\tau\;)^{\alpha-1}}{\Gamma(\alpha)} \; d\tau \quad ds \\
\|x_1 - x_2\| \quad &\leq \quad \frac{(\; a \; \sum_{k=1}^m \; a_k + 1) \; T^{\alpha+1} \; k \; c}{\Gamma(\alpha+1)} \|x_1 - x_2\|.
\end{aligned}$$

Hence

$$\left(1 - \frac{(\; a \; \sum_{k=1}^m \; a_k + 1) \; T^{\alpha+1} \; k \; c}{\Gamma(\alpha+1)}\right) \|x_1 - x_2\| \quad \leq \quad 0.$$

Since $\frac{(\; a \; \sum_{k=1}^m \; a_k+1) \; T^{\alpha+1} \; k \; c}{\Gamma(\alpha+1)} < 1$, then $x_1(t) = x_2(t)$ and the solution of the integral equation (3.2) is unique, and consequence this prove the existence of unique solutions for non-local problem (1.1)-(1.4). This completes the proof. □

Corollary 17. *let the assumptions of Theorem 16 be satisfied. Then the solution of the non-local problem* (1.1)-(1.4) *depends continuously on the* S_{F_1} *the set of all Lipschitzian selections of* F_1*.*

Proof. Let $f_1(t, x(t))$ and $f_1^*(t, x(t))$ be two different Lipschitzian selections of $F_1(t, x(t))$, such that

$$|f_1(t, x(t)) - f_1^*(t, x(t)| < \epsilon, \quad \epsilon > 0, \quad t \in I,$$

then for the two corresponding solution $x_{f_1}(t)$ and $x_{f_1^*}(t)$ of (3.2) we have.

$$\begin{aligned}
|x(t) - x^*(t)| \quad &= \quad |a \sum_{k=1}^{m} a_k \int_0^{\tau_k} f_1(s, I^\alpha f_2(s, x(\varphi(s))) - f_1^*(t, I^\alpha f_2(s, x^*(\varphi(s)))ds \\
&+ \quad \int_0^t f_1(s, I^\alpha f_2(s, x(\varphi(s))) - f_1^*(t, I^\alpha f_2(s, x^*(\varphi(s))) \; ds \; | \\
&\leq \quad a \sum_{k=1}^{m} a_k \int_0^{\tau_k} |f_1(s, I^\alpha f_2(s, x(\varphi(s))) - f_1^*(t, I^\alpha f_2(s, x^*(\varphi(s)))| ds
\end{aligned}$$

$$
\begin{aligned}
&+ \int_0^t |f_1(s, I^\alpha f_2(s, x^*(\varphi(s))) - f_1^*(s, I^\alpha f_2(s, x^*(\varphi(s)))|ds \\
&\leq a \sum_{k=1}^m a_k \int_0^{\tau_k} |f_1(s, I^\alpha f_2(s, x(\varphi(s))) - f_1(s, I^\alpha f_2(s, x^*(\varphi(s)))| \, ds \\
&+ a \sum_{k=1}^m a_k \int_0^{\tau_k} |f_1(s, I^\alpha f_1(s, x^*(\varphi(s))) - f_1^*(s, I^\alpha f_2(s, x^*(\varphi(s)))| \, ds \\
&+ \int_0^t |f_1(s, I^\alpha f_2(s, x(\varphi(s))) - f_1(s, I^\alpha f_2(s, x^*(\varphi(s)))| \, ds \\
&+ \int_0^t |f_1(s, I^\alpha f_1(s, x^*(\varphi(s))) - f_1^*(s, I^\alpha f_2(s, x^*(\varphi(s)))| \, ds \\
&\leq a \sum_{k=1}^m a_k \int_0^{\tau_k} (|f_1(s, I^\alpha f_2(s, x(\varphi(s))) - f_1(s, I^\alpha f_2(s, x^*(\varphi(s)))| + \delta) \, ds \\
&+ \int_0^t |f_1(s, I^\alpha f_2(s, x(\varphi(s))) - f_1(s, I^\alpha f_2(s, x^*(\varphi(s)))| \, ds + \int_0^t \delta \, ds \\
&\leq a \sum_{k=1}^m a_k \, k \, (\int_0^{\tau_k} |I^\alpha f_2(t, x(\varphi(t))) - I^\alpha f_2(t, x^*(\varphi(t)))| \, ds + \delta \, T) \\
&+ k \int_0^t |I^\alpha f_2(t, x(\varphi(t))) - I^\alpha f_2(t, x^*(\varphi(t)))| \, ds + \delta \, T \\
&\leq a \sum_{k=1}^m a_k k (\int_0^{\tau_k} \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} |f_2(\tau, x(\varphi(\tau))) - f_2(\tau, x^*(\varphi(\tau)))| d\tau \, ds + T\delta) \\
&+ k \int_0^t \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} |f_2(\tau, x(\varphi(\tau))) - f_2(\tau, x^*(\varphi(\tau)))| \, d\tau \, ds + T \, \delta \\
&\leq a \sum_{k=1}^m a_k \, k \, c \, (\int_0^{\tau_k} \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} |x(\varphi(s)) - x^*(\varphi(s))| \, d\tau \, ds + T \, \delta) \\
&+ k \, c \int_0^t \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} |x(\varphi(s)) - x^*(\varphi(s))| \, d\tau \, ds + T \, \delta \\
&\leq \|x - x^*\| (a \sum_{k=1}^m a_k \, k \, c \int_0^{\tau_k} \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} \, d\tau \, ds \\
&+ k \, c \int_0^t \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} \, d\tau \, ds) + (a \sum_{k=1}^m a_k \, k \, c + 1) \, T \, \delta
\end{aligned}
$$

$$\|x_{f_1} - x_{f_1^*}\| \leq \frac{(a\sum_{k=1}^{m} a_k + 1)kcT^{\alpha+1}}{\Gamma(\alpha+1)}\|x_{f_1} - x_{f_1^*}\| + (a\sum_{k=1}^{m} a_k kc + 1)T\delta$$

$$\|x_{f_1} - x_{f_1^*}\| \leq \left(1 - \frac{(a \sum_{k=1}^{m} a_k + 1)kcT^{\alpha+1}}{\Gamma(\alpha+1)}\right)^{-1} (a\sum_{k=1}^{m} a_k\ kc + 1)\ T\ \delta = \epsilon.$$

Hence,

$$\|x_{f_1} - x_{f_1^*}\| \leq \epsilon.$$

which proves the continuous dependence of the solution on the set S_{F_1} of all Lipschitzian selection of F_1. This completes the proof. □

Corollary 18. *let the assumptions of Theorem 16 be satisfied. Then the solution of the non-local problem (1.1)-(1.4) depends continuously on the Lipschitz function f_2.*

Proof. Let $f_2(t, x(t))$ and $f_2^*(t, x(t))$ be two different Lipschitz functions such that

$$|f_2(t, x(t)) - f_2^*(t, x(t)| < \delta, \quad \delta > 0, \quad t \in I,$$

then for the two corresponding solutions x and x^* of (3.2), we have

$$\begin{aligned}
&|x(t) - x^*(t)| \\
\leq\ & a \sum_{k=1}^{m} a_k \int_0^{\tau_k} |f_1(s, I^\alpha f_2(s, x_2(\varphi(s)))) - f_1(s, I^\alpha f_2^*(s, x^*(\varphi(s))))|\, ds \\
+\ & \int_0^t |f_1(s, I^\alpha f_2(s, x_1(\varphi(s)))) - f_1(s, I^\alpha f_2^*(s, x^*(\varphi(s))))|\, ds \\
&|x(t) - x^*(t)| \\
\leq\ & a \sum_{k=1}^{m} a_k\ k \int_0^{\tau_k} |I^\alpha f_2(s, x(\varphi(s))) - I^\alpha f_2^*(s, x^*(\varphi(s)))|\, ds \\
+\ & k \int_0^t |I^\alpha f_2(s, x^*(\varphi(s))) - I^\alpha f_2^*(s, x^*(\varphi(s)))|\, ds \\
\leq\ & a \sum_{k=1}^{m} a_k\ k \int_0^{\tau_k} |I^\alpha f_2(s, x(\varphi(s))) - I^\alpha f_2(s, x^*(\varphi(s)))|\, ds \\
+\ & a \sum_{k=1}^{m} a_k\ k \int_0^{\tau_k} |I^\alpha f_2(s, x^*(\varphi(s))) - I^\alpha f_2^*(s, x^*(\varphi(s)))|\, ds \\
+\ & k \int_0^t |I^\alpha f_2(s, x(\varphi(s))) - I^\alpha f_2(s, x^*(\varphi(s)))|\, ds \\
+\ & k \int_0^t |I^\alpha f_2(s, x^*(\varphi(s))) - I^\alpha f_2^*(s, x^*(\varphi(s)))|\, ds \\
\leq\ & a \sum_{k=1}^{m} a_k \left[k \int_0^{\tau_k} \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} |f_2(\tau, x(\varphi(\tau))) - f_2(s, x^*(\varphi(s)))|\, d\tau\, ds\right.
\end{aligned}$$

$$
\begin{aligned}
&+ \int_0^{\tau_k}\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}|f_2(\tau,x^*(\varphi(\tau)))-f_2^*(\tau,x^*(\varphi(\tau)))|\,d\tau\,ds\Big]\\
&+ k\Big[\int_0^t\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}|f_2(\tau,x(\varphi(\tau)))-f_2(s,x^*(\varphi(s)))|\,d\tau\,ds\\
&+ \int_0^t\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}|f_2(\tau,x^*(\varphi(\tau)))-f_2^*(\tau,x^*(\varphi(\tau)))|\,d\tau\,ds\Big]\\
&\le a\sum_{k=1}^m a_k\ k\ c\int_0^{\tau_k}\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}\big[|x(\varphi(\tau))-x^*(\varphi(\tau))|+k\ \delta\big]\,d\tau\,ds\\
&+ k\ c\int_0^t\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}\big[|x(\varphi(\tau))-x^*(\varphi(\tau))|+k\ \delta\big]\,d\tau\,ds\\
&\le a\sum_{k=1}^m a_k[k\ c\ \|x-x^*\|+k\ \delta]\int_0^{\tau_k}\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}\,d\tau\,ds\\
&+ [k\ c\ \|x-x^*\|+k\ \delta]\int_0^t\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}\,d\tau\,ds\\
&\le a\sum_{k=1}^m a_k\ k\ [\ c\ \|x-x^*\|+\ \delta]\frac{T^{\alpha+1}}{\Gamma(\alpha+1)}+k\ [\ c\ \|x-x^*\|+\ \delta]\frac{T^{\alpha+1}}{\Gamma(\alpha+1)}.
\end{aligned}
$$

Taking supremum over $t\in I$

$$
\begin{aligned}
\|x-x^*\| &\le \frac{(a\sum_{k=1}^m a_k+1)kcT^{\alpha+1}}{\Gamma(\alpha+1)}\|x-x^*\|+\frac{(a\sum_{k=1}^m a_k+1)kcT^{\alpha+1}}{\Gamma(\alpha+1)}\\
\|x-x^*\| &\le (1-\frac{(a\sum_{k=1}^m a_k+1)kcT^{\alpha+1}}{\Gamma(\alpha+1)})^{-1}\frac{(a\sum_{k=1}^m a_k+1)kcT^{\alpha+1}}{\Gamma(\alpha+1)}=\epsilon.
\end{aligned}
$$

Hence,

$$\|x-x^*\|\le\epsilon.$$

which proves the continuous dependence of the solution on the Lipschitz function f_2. This completes the proof. □

Corollary 19. *let assumptions of Theorem 16 be satisfied. Then the solution of the non-local problem (1.1)-(1.4) depends continuously on initial data $x_\circ$.*

Proof. The solution of the integral inclusion (3.2) depends continuously on initial data $x_\circ$, if

$$\forall\ \epsilon>0,\quad \exists\ \delta(\epsilon)\quad \text{s.t}\quad |x_\circ-x_\circ^*|<\delta\quad\Rightarrow\quad \|x-x^*\|<\epsilon,$$

then for the two corresponding solutions $x(t)$ and $x^*(t)$ of the integral equation (3.2) we have

$$
\begin{aligned}
& |x(t) - x^*(t)| \\
= & \left|a\left(x_\circ - \sum_{k=1}^{m} a_k \int_0^{\tau_k} |f_1(s, I^\alpha f_2(s, x_2(\varphi(s)))ds\right) + \int_0^t f_1(s, I^\alpha f_2(s, x(\varphi(s))))ds\right. \\
- & \left. a\left(x_\circ^* - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f_1(s, I^\alpha f_2(s, x^*(\varphi(s)))ds\right) - \int_0^t f_1(s, I^\alpha f_2(s, x^*(\varphi(s))))ds\right| \\
\leq & a|x_\circ - x_\circ^*| + a\sum_{k=1}^{m} a_k \int_0^{\tau_k} |f_1(s, I^\alpha f_2(s, x(\varphi(s))) - f_1(s, I^\alpha f_2(s, x^*(\varphi(s)))|ds \\
+ & \int_0^t |f_1(s, I^\alpha f_2(s, x(\varphi(s))) - f_1(s, I^\alpha f_2(s, x^*(\varphi(s)))|ds \\
\leq & a\ \delta\ +\ a \sum_{k=1}^{m} a_k\ k \int_0^{\tau_k} |I^\alpha f_2(s, x(\varphi(s))) - I^\alpha f_2(s, x^*(\varphi(s)))|\ ds \\
+ & k \int_0^t |I^\alpha f_2(s, x(\varphi(s))) - I^\alpha f_2(s, x^*(\varphi(s)))|\ ds \\
\leq & a\delta + a\sum_{k=1}^{m} a_k k \int_0^{\tau_k} \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} |f_2(\tau, x(\varphi(\tau))) - f_2(\tau, x^*(\varphi(\tau)))|d\tau ds \\
+ & k \int_0^t \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} |f_2(\tau, x(\varphi(\tau))) - f_2(\tau, x^*(\varphi(\tau)))|\ d\tau\ ds \\
\leq & a\ \delta\ +\ a \sum_{k=1}^{m} a_k\ k\ c \int_0^{\tau_k} \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} |x(\varphi(\tau)) - x^*(\varphi(\tau))|\ d\tau\ ds \\
+ & k\ c \int_0^t \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} |x(\varphi(\tau)) - x^*(\varphi(\tau))|\ d\tau\ ds \\
\leq & a\ \delta\ +\ a \sum_{k=1}^{m} a_k\ k\ c\ \|x - x^*\| \int_0^{\tau_k} \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}\ d\tau\ ds \\
+ & k\ c\ \|x - x^*\| \int_0^t \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}\ d\tau\ ds.
\end{aligned}
$$

Taking supermum over $t \in I$

$$
\begin{aligned}
\|x - x^*\| & \leq a\ \delta\ +\ \frac{[a\ \sum_{k=1}^{m}\ a_k + 1]\ k\ c\ T^{\alpha+1}}{\Gamma(\alpha+1)} \|x - x^*\| \\
\|x - x^*\| & \leq a\delta\ (1 - \frac{[a\ \sum_{k=1}^{m}\ a_k + 1]\ k\ c\ T^{\alpha+1}}{\Gamma(\alpha+1)})^{-1} = \epsilon.
\end{aligned}
$$

Hence,

$$\|x - x^*\| \leq \epsilon.$$

This mean that the solution of the integral equation (3.2) depends continuously on $x_\circ$. Then the solution of the non-local problem (1.1)-(1.4) depends continuously on initial data $x_\circ$. This completes the proof. □

5 Illustrative examples

In this section, we offer some examples to illustrate our results.
Example 1. Consider the following nonlinear integro-differential inclusion:

$$\frac{dx}{dt} \in F_1(t, I^{\alpha} f_2(t, x(\varphi(t))), \quad t \in [0,1], \;\; \alpha \in (0,1) \tag{5.1}$$

with infinite point boundary condition

$$x(0) + \sum_{k=1}^{\infty} \frac{1}{k^2}\, x\big(\frac{k-1}{k}\big) = x_\circ. \tag{5.2}$$

For illustrating Theorem 13, we choose $F_1 : [0,1] \times \mathbb{R} \to 2^{\mathbb{R}^+}$ in (5.3) as follows:

$$F_1(t, I^{\frac{1}{4}} f_2(t, x(t)) = \Big[0, t^3 + t + 1 + \int_0^t \frac{(t-s)^{-\frac{3}{4}}}{2\,\Gamma(\frac{1}{4})}\big(\cos(x(s)+1) + \frac{x(s)}{e^s}\big)\, ds\Big],$$

set

$$f_2(t, x(t)) = \frac{1}{2}\big(\cos(x(s)+1) + \frac{x(s)}{e^s}\big).$$

Define the continuous map $f_1 : [0,1] \times \mathbb{R} \to \mathbb{R}$, notice that for $f_1 \in S_{F_1}$, then we have

$$|f_1(t, I^{\frac{1}{4}} f_2(t, x(\varphi(t)))) - f_1(t, I^{\frac{1}{4}} f_2(t, y(\varphi(t))))| \le \frac{1+e}{2\;e\;\Gamma(\frac{3}{4})}|x-y|,$$

and

$$\big|f_2(t, x(t))\big| \le \frac{1}{2}|\cos(x(t)+1)| + \frac{|x(t)|}{2e}.$$

Thus conditions (i) and (iii) are satisfied with $k = \frac{1+e}{2\;e\;\Gamma(\frac{3}{4})} \approx 0.558129 < 1$, $a(t) = \frac{1}{2}\cos(x(t)+1) \in L^1[0,1]$, $b = \frac{1}{2e}$ and the series $\sum_{k=1}^{\infty} \frac{1}{k^4}$ is convergent. Also, $[a(|x_\circ| + \sum_{k=1}^{m} |a_k|) + 1]kT \approx 0.6136 < 1$ and $\frac{b\;T^{\alpha}}{\Gamma(\alpha+1)} \approx 0.2029 < 1$. It follows from Theorem 13 that the given nonlocal problem (5.3)-(5.4) has at least one continuous solution.

Example 2. Consider the following nonlinear integro-differential inclusion:

$$\frac{dx}{dt} \in F_1(t, I^{\alpha} f_2(t, x(\varphi(t))), \quad t \in [0,1], \;\; \alpha \in (0,1) \tag{5.3}$$

with infinite point boundary condition

$$x(0) + \sum_{k=1}^{\infty} \frac{1}{k^4}\, x\big(\frac{k^2+k-1}{k^2+k}\big) = x_\circ. \tag{5.4}$$

For illustrating Theorem 13, we choose $F_1 : [0,1] \times \mathbb{R} \to 2^{\mathbb{R}^+}$ in (5.3) as follows

$$F_1(t, I^{\frac{1}{2}} f_2(t, x(t)) = \left[0, \frac{e^{-t}}{e^t+5} + \int_0^t \frac{\sqrt{(t-s)}}{2\sqrt{\pi}e^{s+1}} \left(\frac{2+|\sin x(s)|}{1+|\sin x(s)|}\right) ds\right].$$

Set

$$f_2(t, x(t)) = \frac{1}{2e^{t+1}} \left(\frac{2+|\sin x(t)|}{1+|\sin x(s)|}\right).$$

Define the continuous map $f_1 : [0,1] \times \mathbb{R} \to \mathbb{R}$, notice that for $f_1 \in S_{F_1}$, then we have

$$|f_1(t, I^{\frac{1}{2}} f_2(t, x(\varphi(t)))) - f_1(t, I^{1/2} f_2(t, y(\varphi(t))))| \le \frac{1}{2e^2\sqrt{\pi}} |x-y|.$$

and

$$\big|f_2(t, x(t)) - f_2(t, y(t))\big| \le \frac{1}{2e^2} |x-y|.$$

Thus conditions (i)-$(iii)^*$ are satisfied with $k_1 = \frac{1}{2e^2\sqrt{\pi}}$, and $k_2 = \frac{1}{2e^2}$ and the series $\sum_{k=1}^{\infty} \frac{1}{k^3}$ is convergent. Also $\frac{(\ a\ \sum_{k=1}^{m}\ a_k+1)\ T^\alpha\ \ k\ \ c}{\Gamma(\alpha+1)} = \frac{1}{4e^4\pi} < 1$. It follows from Theorem 13 that the given nonlocal problem (5.3)-(5.4) has a unique continuous solution.

6 Conclusion

In our current research, we considered a continuous solution for a class of first-degree non-linear differential equations with integrated boundary conditions (BCs) or with infinite-point BCs. We have proven it, if we can get continuous solutions for BVPs with m-point BCs, It is easy to get solutions to this problems with integrated BCs or with infinite-point BCs. Note that the fractional differential inclusions (1.1) includes the ordinary derivative $\frac{dx}{dt}$ of order 1 on its left-hand side. In the expected future, we suggest that investigate the possibility of extending our results to other higher-order derivatives such as

$$\frac{d^2x}{dt^2}, \frac{d^3x}{dt^3}, \frac{d^4x}{dt^4}, \ \ldots.$$

It occurs on the left side of the fractional differential equation (1.1), which includes the Riemann-Liouville fractional integration with integrated BCs and / or infinite point BCs.

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