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# ON UNIFIED SUBCLASS OF COMPLEX ORDER CONNECTED WITH $q-$CONFLUENT HYPERGEOMETRIC DISTRIBUTION

Sheza M. El-Deeb

**Abstract**. In this paper, we apply the concept of $q-$confluent hypergeometric distribution to introduce and study a unified subclass of univalent functions of complex order $\mathcal{B}_{\zeta,\gamma}^{\lambda,q}(\Upsilon;b,c,m)$ consisting of all analytic functions, and some known consequences of the results are also derived for this class.

## 1 Introduction

In [16] Srivastava presented and motivated about brief expository overview of the classical $q$-analysis versus the so-called $(p,q)$-analysis with an obviously redundant additional parameter $p$. We also briefly consider several other families of such extensivelyand widely-investigated linear convolution operators as (for example) the Dziok–Srivastava, Srivastava–Wright and Srivastava–Attiya linear convolution operators, together with their extended and generalized versions. The theory of $(p,q)$-analysis has important role in many areas of mathematics and physics. Our usages here of the $q$-calculus and the fractional qcalculus in geometric function theory of complex analysis are believed to encourage and motivate significant further developments on these and other related topics (see Srivastava and Karlsson [18, pp. 350–351], Srivastava [14, 15]). Our main objective in this survey-cum-expository article is based chiefly upon the fact that the recent and future usages of the classical $q$-calculus and the fractional $q$-calculus in geometric function theory of complex analysis have the potential to encourage and motivate significant further researches on many of these and other related subjects. Jackson [5, 6] was the first that gave some application of $q$-calculus and introduced the $q$-analogue of derivative and integral operator (see also [1]). Let $\mathcal{A}$ denote the subclass of functions of the form

$$f(z) := z + \sum_{k=2}^{\infty} a_k z^k,\ z \in \Delta, \tag{1.1}$$

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and, let the function $\Omega \in \mathcal{A}$ is given by

$$\Omega(z) := z + \sum_{k=2}^{\infty} \psi_k z^k \; z \in \Delta. \tag{1.2}$$

The *Hadamard (or convolution) product* of $f$ and $\Omega$ is defined by

$$(f * \Omega)(z) := z + \sum_{k=2}^{\infty} a_k \psi_k z^k, \; z \in \Delta.$$

**Definition 1.** *For $f, g \in \mathcal{A}$, we say that $f$* is subordinate to $g$, *written $f(z) \prec g(z)$, if there exists a* Schwarz function $w$, *which is analytic in $\Delta$, with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \Delta$, such that $f(z) = g(w(z))$, $z \in \Delta$. Furthermore, if the function $g$ is univalent in $\Delta$, then we have the following equivalence (see [2, 8]):*

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\Delta) \subset g(\Delta).$$

Let $\Upsilon(z)$ be analytic function with positive real part of $\Upsilon$ with $\Upsilon(0) = 1$, $\Upsilon'(0) > 0$, which maps the unit disk $\Delta$ onto a region starlike with respect to 1, which is symmetric with respect to the real axis. Let $S^*(\Upsilon)$ be the class of functions $f \in \mathcal{A}$ such that

$$\frac{z\, f'(z)}{f(z)} \prec \Upsilon(z),$$

and $C(\Upsilon)$ be the class of functions $f \in \mathcal{A}$ for which

$$1 + \frac{z\, f''(z)}{f'(z)} \prec \Upsilon(z).$$

Ma and Minda [7] and Ravichandran et al. [12] defined the classes $S^*_\zeta(\Upsilon)$ and $C_\zeta(\Upsilon)$ of complex order $\zeta$ ($\zeta \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$) as follows

$$S^*_\zeta(\Upsilon) = \left\{ f \in \mathcal{A} : \; 1 + \frac{1}{\zeta}\left( \frac{z\, f'(z)}{f(z)} - 1 \right) \prec \Upsilon(z) \right\}, \tag{1.3}$$

and

$$C_\zeta(\Upsilon) = \left\{ f \in \mathcal{A} : \; 1 + \frac{1}{\zeta} \frac{z\, f''(z)}{f'(z)} \prec \Upsilon(z) \right\}. \tag{1.4}$$

From (1.3) and (1.4), we obtain

$$f \in C_\zeta(\Upsilon) \;\Leftrightarrow\; zf'(z) \in S^*_\zeta(\Upsilon).$$

The confluent hypergeometric function of the first kind is given by the power series

$$\begin{aligned} F(b;c;z) &= 1+\frac{b}{c}z+\frac{b\ (b+1)}{c\ (c+1)}\frac{z^2}{2!}+.......... \\ &= \sum_{k=0}^{\infty}\frac{(b)_k}{(c)_k\ (1)_k}z^k, \quad \left(b\in\mathbb{C},\ c\in\mathbb{C}\setminus\{0,-1,-2,...\}\right), \end{aligned}$$

where $(b)_k$ is the Pochhammer symbol defined in terms of the Gamma function by

$$(b)_k=\frac{\Gamma(b+k)}{\Gamma(b)}=\begin{cases} 1, & \text{if } \ k=0, \\ b(b+1)\dots(b+k-1), & \text{if } \ k\in\mathbb{N}=\{1,2,...\}, \end{cases}$$

is convergent for all finite values of $z$ (see [11]). It can be written otherwise

$$F(b;c;m)=\sum_{k=0}^{\infty}\frac{(b)_k}{(c)_k\ (1)_k}m^k, \quad \left(b\in\mathbb{C},\ c\in\mathbb{C}\setminus\{0,-1,-2,...\}\right),$$

is convergent for $b,c,m>0$. Very recently, Porwal and Kumar [10] introduced the confluent hypergeometric distribution (CHD) whose probability mass function is

$$P(k)=\frac{(b)_k}{(c)_k\ k!F(b;c;m)}m^k, \ \left(b,c,m>0,\ k=0,1,2,...\right).$$

Porwal [9] introduced a series $\mathcal{I}(b;c;m;z)$ whose coefficients are probabilities of confluent hypergeometric distribution

$$\mathcal{I}(b;c;m;z)=z+\sum_{k=2}^{\infty}\frac{(b)_{k-1}\,m^{k-1}}{(c)_{k-1}\ (k-1)!F(b;c;m)}z^k, \ \left(b,c,m>0\right), \tag{1.5}$$

and defined a linear operator $\Omega(b;c;m)f:\mathcal{A}\to\mathcal{A}$ as follows

$$\begin{aligned} \Omega(b;c;m)f(z) &= \mathcal{I}(b;c;m;z)*f(z) \\ &= z+\sum_{k=2}^{\infty}\frac{(b)_{k-1}\,m^{k-1}}{(c)_{k-1}\ (k-1)!F(b;c;m)}a_k\ z^k, \ \left(b,c,m>0\right). \end{aligned}$$

Srivastava [16] (see also [17]) made use of various operators of $q$-calculus and fractional $q$-calculus and recalling the definition and notations. The $q$-shifted factorial is defined for $\lambda,q\in\mathbb{C}$ and $n\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$ as follows

$$(\lambda;q)_k=\begin{cases} 1 & k=0, \\ (1-\lambda)(1-\lambda q)\dots\left(1-\lambda q^{k-1}\right) & k\in\mathbb{N}. \end{cases}$$

By using the $q$-gamma function $\Gamma_q(z)$, we get

$$\left(q^{\lambda};q\right)_k=\frac{(1-q)^k\ \Gamma_q(\lambda+k)}{\Gamma_q(\lambda)}, \qquad \left(k\in\mathbb{N}_0\right),$$

where (see [4])

$$\Gamma_q(z) = (1-q)^{1-z}\,\frac{(q;q)_\infty}{(q^z;q)_\infty}, \qquad (|q|<1)\,.$$

Also, we note that

$$(\lambda;q)_\infty = \prod_{k=0}^{\infty}\left(1-\lambda q^k\right), \qquad (|q|<1)\,,$$

and, the $q$-gamma function $\Gamma_q(z)$ is known

$$\Gamma_q(z+1) = [z]_q\ \Gamma_q(z),$$

where $[k]_q$ denotes the basic $q$-number defined as follows

$$[k]_q := \begin{cases} \frac{1-q^k}{1-q}, & k\in\mathbb{C}, \\ 1+\sum\limits_{j=1}^{k-1} q^j, & k\in\mathbb{N}. \end{cases} \tag{1.6}$$

Using the definition formula (1.6) we have the next two products:
(i) For any non negative integer $k$, the *$q$-shifted factorial* is given by

$$[k]_q! := \begin{cases} 1, & \text{if} \quad k=0, \\ \prod\limits_{n=1}^{k}[n]_q, & \text{if} \quad k\in\mathbb{N}. \end{cases}$$

(ii) For any positive number $r$, the *$q$-generalized Pochhammer symbol* is defined by

$$[r]_{q,k} := \begin{cases} 1, & \text{if} \quad k=0, \\ \prod\limits_{n=r}^{r+k-1}[n]_q, & \text{if} \quad k\in\mathbb{N}. \end{cases}$$

It is known in terms of the classical (Euler's) gamma function $\Gamma(z)$, that

$$\Gamma_q(z)\to\Gamma(z) \qquad \text{as } q\to 1^-.$$

Also, we observe that

$$\lim_{q\to 1^-}\left\{\frac{(q^\lambda;q)_k}{(1-q)^k}\right\} = (\lambda)_k\,.$$

For $0<q<1$, the *$q$-derivative operator* [6] for $\mathcal{I}(b;c;m;z)$ is defined by

$$\begin{aligned} D_q\left(\Omega(b;c;m)f(z)\right) \quad : \quad &= \frac{\Omega(b;c;m)f(z)-\Omega(b;c;m)f(qz)}{z(1-q)} \\ &= 1+\sum_{k=2}^{\infty}[k]_q\frac{(b)_{k-1}\,m^{k-1}}{(c)_{k-1}\ (k-1)!F(b;c;m)}a_k\ z^{k-1},\ (b,c,m>0,\ z\in\Delta)\,, \end{aligned}$$

where

$$[k]_q := \frac{1-q^k}{1-q} = 1 + \sum_{j=1}^{k-1} q^j, \qquad [0,q] := 0. \tag{1.7}$$

For $\lambda > -1$ and $0 < q < 1$, we defined the linear operator $\mathcal{I}^{\lambda,q}(b;c;m)f : \mathcal{A} \to \mathcal{A}$ by

$$\mathcal{I}^{\lambda,q}(b;c;m)f(z) * \mathcal{N}^{m}_{q,\lambda+1}(z) = z\, D_q\left(\Omega(b;c;m)f(z)\right),\ z \in \Delta,$$

where the function $\mathcal{N}^{m}_{q,\lambda+1}$ is given by

$$\mathcal{N}^{m}_{q,\lambda+1}(z) := z + \sum_{k=2}^{\infty} \frac{[\lambda+1]_{q,k-1}}{[k-1]_q!} z^k,\ z \in \Delta.$$

A simple computation shows that

$$\mathcal{I}^{\lambda,q}(b;c;m)f(z) := z + \sum_{k=2}^{\infty} \frac{(b)_{k-1} m^{k-1} [k]_q!}{(c)_{k-1}\ (k-1)! F(b;c;m)[\lambda+1]_{q,k-1}} a_k\, z^k\ \ (b,c,m>0,\ \lambda>-1,\ 0<q<1,\ z\in\Delta). \tag{1.8}$$

From the definition relation (1.8), we can easily verify that the next relations hold for all $f \in \mathcal{A}$:

(i) $[\lambda+1]_q\, \mathcal{I}^{\lambda,q}(b;c;m)f(z) = [\lambda]_q \mathcal{I}^{\lambda+1,q}(b;c;m)f(z) + q^\lambda\, z\, D_q\left(\mathcal{I}^{\lambda+1,q}(b;c;m)f(z)\right),\ z \in \Delta;$ (1.9)

(ii) $\mathcal{M}^{\lambda}(b;c;m)f(z) := \lim_{q\to 1^-} \mathcal{I}^{\lambda,q}(b;c;m)f(z) = z + \sum_{k=2}^{\infty} \frac{k(b)_{k-1} m^{k-1}}{(c)_{k-1}\ F(b;c;m)(\lambda+1)_{k-1}} a_k z^k,\ z \in \Delta.$ (1.10)

**Remark 2.** *Putting $b = c$ in the operator $\mathcal{I}^{\lambda,q}(b;c;m)$, we obtain the $q$-analogue of Poisson operator $I_q^{\lambda,m}$ defined by El-Deeb et al. [3] as follows*

$$I_q^{\lambda,m} f(z) := z + \sum_{k=2}^{\infty} \frac{m^{k-1}}{(k-1)!} e^{-m} \cdot \frac{[k]_q!}{[\lambda+1]_{q,m-1}} a_k\, z^k,\ z \in \Delta. \tag{1.11}$$

*By using the operator $\mathcal{I}^{\lambda,q}(b;c;m)$, we defined a subclass $\mathcal{B}^{\lambda,q}_{\zeta,\gamma}(\Upsilon;b,c,m)$ of the class $\mathcal{A}$ as follows:*

**Definition 3.** *Let $\Upsilon(z)$ be analytic function with positive real part of $\Upsilon$ with $\Upsilon(0) = 1,\ \Upsilon'(0) > 0$, which maps $\Delta$ onto a region starlike with respect to 1, then the class $\mathcal{B}^{\lambda,q}_{\zeta,\gamma}(\Upsilon;b,c,m)$ consists of all analytic functions $f \in \mathcal{A}$ satisfies*

$$\frac{1}{\zeta}\left[(1-\gamma)\frac{z\left(\mathcal{I}^{\lambda,q}(b;c;m)f(z)\right)'}{\mathcal{I}^{\lambda,q}(b;c;m)f(z)} + \gamma\frac{z\left(\mathcal{I}^{\lambda+1,q}(b;c;m)f(z)\right)'}{\mathcal{I}^{\lambda+1,q}(b;c;m)f(z)} - (1-\zeta)\right] \prec \Upsilon(z). \tag{1.12}$$

**Remark 4.** *Taking different particular cases for the coefficients $b$, $c$, $q$ we obtain the next special cases for the class $\mathcal{B}_{\zeta,\gamma}^{\lambda,q}(\Upsilon; b, c, m)$:*
*(i) Putting $q \to 1^-$, we obtain that $\lim\limits_{q\to 1^-} \mathcal{B}_{\zeta,\gamma}^{\lambda,q}(\Upsilon; b, c, m) =: \mathcal{P}_{\zeta,\gamma}^{\lambda}(\Upsilon; b, c, m)$, where $\mathcal{P}_{\zeta,\gamma}^{\lambda}(\Upsilon; b, c, m)$ represents the functions $f \in \mathcal{A}$ that satisfies (1.12) for $\mathcal{I}^{\lambda,q}(b; c; m)$ replaced with $\mathcal{M}^{\lambda}(b; c; m)$;*
*(ii) Putting $b = c$, we obtain the class $\mathcal{W}_{\zeta,\gamma}^{\lambda,q}(\Upsilon; m)$, that represents the functions $f \in \mathcal{A}$ satisfies (1.12) for $\mathcal{I}^{\lambda,q}(b; c; m)$ replaced with $I_q^{\lambda,m}$;*
*(iii) Putting $\gamma = 0$, we obtain the class $\mathcal{S}_{\zeta}^{\lambda,q}(\Upsilon; b, c, m)$, that represents the functions $f \in \mathcal{A}$ satisfies*

$$\frac{1}{\zeta}\left[\frac{z\left(\mathcal{I}^{\lambda,q}(b; c; m)f(z)\right)'}{\mathcal{I}^{\lambda,q}(b; c; m)f(z)} - (1-\zeta)\right] \prec \Upsilon(z);$$

*(iv) Putting $\gamma = 1$, we obtain the class $\mathcal{C}_{\zeta}^{\lambda,q}(\Upsilon; b, c, m)$, that represents the functions $f \in \mathcal{A}$ satisfies*

$$\frac{1}{\zeta}\left[\frac{z\left(\mathcal{I}^{\lambda+1,q}(b; c; m)f(z)\right)'}{\mathcal{I}^{\lambda+1,q}(b; c; m)f(z)} - (1-\zeta)\right] \prec \Upsilon(z).$$

## 2 Preliminaries

In order to prove our results, we shall need the following lemmas.

**Lemma 5.** *[13] Let $\Upsilon$ be a convex function defined on $\Delta$, $\Upsilon(0) = 1$. Define $\mathcal{F}(z)$ by*

$$\mathcal{F}(z) = z\, e^{\left(\int_0^z \frac{\Upsilon(t)-1}{t}dt\right)}, \tag{2.1}$$

*and let*

$$p(z) = 1 + p_1 z + p_2 z^2 + \dots$$

*be analytic in $\Delta$, then*

$$1 + \frac{z\, q'(z)}{q(z)} \prec \Upsilon(z),$$

*if and only if for all $|s| \leq 1$ and $|t| \leq 1$, we have*

$$\frac{p(tz)}{p(sz)} \prec \frac{s\mathcal{F}(tz)}{t\mathcal{F}(sz)}.$$

**Lemma 6.** *[8] Let $q(z)$ be univalent in $\Delta$ and let $\Upsilon(z)$ be analytic in domain containing $q'(z)$ if $\frac{z\, q'(z)}{q(z)}$ is starlike, then*

$$z\, p'(z)\Upsilon(p(z)) \prec z\, q'(z)\Upsilon(q(z)),$$

*then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.*

# 3 Main Results

Unless otherwise mentioned, we shall assume in the reminder of this paper that $\zeta \in \mathbb{C}^*$, $\gamma \geq 0$, $b, c, m > 0$, $\lambda > -1$, $0 < q < 1$, the powers are understood as principle values.

**Theorem 7.** *Let $\Upsilon$ and $\mathcal{F}$ be as Lemma 5, the function $f \in \mathcal{B}^{\lambda,q}_{\zeta,\gamma}(\Upsilon; b, c, m)$ if and only if $|s| \leq 1$ and $|t| \leq 1$, we have*

$$\frac{s}{t}\left[\left(\frac{\mathcal{I}^{\lambda,q}(b;c;m)f(tz)}{\mathcal{I}^{\lambda,q}(b;c;m)f(sz)}\right)^{1-\gamma}\left(\frac{\mathcal{I}^{\lambda+1,q}(b;c;m)f(tz)}{\mathcal{I}^{\lambda+1,q}(b;c;m)f(sz)}\right)^{\gamma}\right]^{\frac{1}{\zeta}} \prec \frac{s\mathcal{F}(tz)}{t\mathcal{F}(sz)}. \tag{3.1}$$

*Proof.* Let

$$p(z) = \left[\frac{\mathcal{I}^{\lambda,q}(b;c;m)f(z)}{z}\left(\frac{\mathcal{I}^{\lambda+1,q}(b;c;m)f(z)}{\mathcal{I}^{\lambda,q}(b;c;m)f(z)}\right)^{\gamma}\right]^{\frac{1}{\zeta}}. \tag{3.2}$$

Taking logarithm derivative, we obtain

$$\frac{z\,p'(z)}{p(z)} = \frac{1}{\zeta}\left[(1-\gamma)\frac{z\left(\mathcal{I}^{\lambda,q}(b;c;m)f(z)\right)'}{\mathcal{I}^{\lambda,q}(b;c;m)f(z)} + \gamma\frac{z\left(\mathcal{I}^{\lambda+1,q}(b;c;m)f(z)\right)'}{\mathcal{I}^{\lambda+1,q}(b;c;m)f(z)} - 1\right], \tag{3.3}$$

then

$$1+\frac{z\,p'(z)}{p(z)} = \frac{1}{\zeta}\left[(1-\gamma)\frac{z\left(\mathcal{I}^{\lambda,q}(b;c;m)f(z)\right)'}{\mathcal{I}^{\lambda,q}(b;c;m)f(z)} + \gamma\frac{z\left(\mathcal{I}^{\lambda+1,q}(b;c;m)f(z)\right)'}{\mathcal{I}^{\lambda+1,q}(b;c;m)f(z)} - (1-\zeta)\right], \tag{3.4}$$

since $f \in \mathcal{B}^{\lambda,q}_{\zeta,\gamma}(\Upsilon; b, c, m)$, then we have

$$1 + \frac{z\,p'(z)}{p(z)} \prec \Upsilon(z)$$

The assertion (3.1) of Theorem 7 now follows by an application of Lemma 5. □

Taking $q \to 1^-$ in Theorem 7, we get the following corollary.

**Corollary 8.** *Let $\Upsilon$ and $\mathcal{F}$ be as Lemma 5, the function $f \in \mathcal{P}^{\lambda}_{\zeta,\gamma}(\Upsilon; b, c, m)$ if and only if $|s| \leq 1$ and $|t| \leq 1$, we have*

$$\frac{s}{t}\left[\left(\frac{\mathcal{M}^{\lambda}(b;c;m)f(tz)}{\mathcal{M}^{\lambda}(b;c;m)f(sz)}\right)^{1-\gamma}\left(\frac{\mathcal{M}^{\lambda+1}(b;c;m)f(tz)}{\mathcal{M}^{\lambda+1}(b;c;m)f(sz)}\right)^{\gamma}\right]^{\frac{1}{\zeta}} \prec \frac{s\mathcal{F}(tz)}{t\mathcal{F}(sz)}.$$

Taking $b = c$ in Theorem 7, we get the following corollary.

**Corollary 9.** *Let $\Upsilon$ and $\mathcal{F}$ be as Lemma 5, the function $f \in \mathcal{W}_{\zeta,\gamma}^{\lambda,q}(\Upsilon; m)$ if and only if $|s| \leq 1$ and $|t| \leq 1$, we have*

$$\frac{s}{t}\left[\left(\frac{I_q^{\lambda,m} f(tz)}{I_q^{\lambda,m} f(sz)}\right)^{1-\gamma}\left(\frac{I_q^{\lambda+1,m} f(tz)}{I_q^{\lambda+1,m} f(sz)}\right)^{\gamma}\right]^{\frac{1}{\zeta}} \prec \frac{s\mathcal{F}(tz)}{t\mathcal{F}(sz)}.$$

Taking $\gamma = 0$ in Theorem 7, we get the following example.

**Example 10.** *Let $\Upsilon$ and $\mathcal{F}$ be as Lemma 5, the function $f \in \mathcal{S}_{\zeta}^{\lambda,q}(\Upsilon; b, c, m)$ if and only if $|s| \leq 1$ and $|t| \leq 1$, we have*

$$\frac{s}{t}\left(\frac{\mathcal{I}^{\lambda,q}(b; c; m) f(tz)}{\mathcal{I}^{\lambda,q}(b; c; m) f(sz)}\right)^{\frac{1}{\zeta}} \prec \frac{s\mathcal{F}(tz)}{t\mathcal{F}(sz)}.$$

*Taking $\gamma = 1$ in Theorem 7, we get the following example.*

**Example 11.** *Let $\Upsilon$ and $\mathcal{F}$ be as Lemma 5, the function $f \in \mathcal{C}_{\zeta}^{\lambda,q}(\Upsilon; b, c, m)$ if and only if $|s| \leq 1$ and $|t| \leq 1$, we have*

$$\frac{s}{t}\left(\frac{\mathcal{I}^{\lambda+1,q}(b; c; m) f(tz)}{\mathcal{I}^{\lambda+1,q}(b; c; m) f(sz)}\right)^{\frac{1}{\zeta}} \prec \frac{s\mathcal{F}(tz)}{t\mathcal{F}(sz)}.$$

**Theorem 12.** *Let $\Upsilon$ be starlike with respect to 1 and $\mathcal{F}$ given by (2.1) be starlike. If the function $f \in \mathcal{B}_{\zeta,\gamma}^{\lambda,q}(\Upsilon; b, c, m)$, then we have*

$$\frac{\mathcal{I}^{\lambda,q}(b; c; m) f(z)}{z}\left(\frac{\mathcal{I}^{\lambda+1,q}(b; c; m) f(z)}{\mathcal{I}^{\lambda,q}(b; c; m) f(z)}\right)^{\gamma} \prec \left(\frac{\mathcal{F}(z)}{z}\right)^{\zeta}. \tag{3.5}$$

*Proof.* Let $p(z)$ be given by (3.2) and

$$q(z) = \frac{\mathcal{F}(z)}{z} \quad (z \in \Delta). \tag{3.6}$$

Taking logarithm derivative, we obtain

$$\frac{z\, q^{'}(z)}{q(z)} = \frac{z\, \mathcal{F}^{'}(z)}{\mathcal{F}(z)} - 1 = \Upsilon(z) - 1, \tag{3.7}$$

since $f \in \mathcal{B}_{\zeta,\gamma}^{\lambda,q}(\Upsilon; b, c, m)$, from (3.7) and (3.4), we have

$$\frac{z\, p^{'}(z)}{p(z)} \prec \frac{z\, q^{'}(z)}{q(z)}, \tag{3.8}$$

The assertion (3.5) of Theorem 12 now follows by an application of Lemma 6. □

Taking $q \to 1^-$ in Theorem 12, we get the following corollary.

**Corollary 13.** *Let $\Upsilon$ be starlike with respect to 1 and $\mathcal{F}$ given by (2.1) be starlike. If the function $f \in \mathcal{P}^{\lambda}_{\zeta,\gamma}(\Upsilon; b, c, m)$, then we have*

$$\frac{\mathcal{M}^{\lambda}(b;c;m)f(z)}{z}\left(\frac{\mathcal{M}^{\lambda+1}(b;c;m)f(z)}{\mathcal{M}^{\lambda}(b;c;m)f(z)}\right)^{\gamma} \prec \left(\frac{\mathcal{F}(z)}{z}\right)^{\zeta}.$$

Taking $b = c$ in Theorem 12, we get the following corollary.

**Corollary 14.** *Let $\Upsilon$ be starlike with respect to 1 and $\mathcal{F}$ given by (2.1) be starlike. If the function $f \in \mathcal{W}^{\lambda,q}_{\zeta,\gamma}(\Upsilon; m)$, then we have*

$$\frac{I_q^{\lambda,m}f(z)}{z}\left(\frac{I_q^{\lambda+1,m}f(z)}{I_q^{\lambda,m}f(z)}\right)^{\gamma} \prec \left(\frac{\mathcal{F}(z)}{z}\right)^{\zeta}.$$

Taking $\Upsilon(z) = \frac{1+Az}{1+Bz}$ $(-1 \le B < A \le 1)$ and $\gamma = 0$ in Theorem 12, we get the following example.

**Example 15.** *Let $\Upsilon$ be starlike with respect to 1 and $\mathcal{F}$ given by (2.1) be starlike. If the function $f \in \mathcal{S}^{\lambda,q}_{\zeta}(\frac{1+Az}{1+Bz}; b, c, m)$, then we have*

$$\frac{\mathcal{I}^{\lambda,q}(b;c;m)f(z)}{z} \prec (1+Bz)^{\frac{\zeta(A-B)}{B}}.$$

Taking $\Upsilon(z) = \frac{1+Az}{1+Bz}$ and $\gamma = 1$ in Theorem 12, we get the following example.

**Example 16.** *If the function $f \in \mathcal{C}^{\lambda,q}_{\zeta}(\frac{1+Az}{1+Bz}; b, c, m)$, then we have*

$$\frac{\mathcal{I}^{\lambda,q}(b;c;m)f(z)}{z}\left(\frac{\mathcal{I}^{\lambda+1,q}(b;c;m)f(z)}{\mathcal{I}^{\lambda,q}(b;c;m)f(z)}\right) \prec (1+Bz)^{\frac{\zeta(A-B)}{B}}.$$

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Department of Mathematics,
Faculty of Science, Damietta University,
New Damietta 34517, Egypt.

current Address:

Department of Mathematics,
College of Science and Arts, Al- Badaya,
Qassim University, Buraidah 51452, Saudi Arabia.
e-mail: shezaeldeeb@yahoo.com

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