Surveys in Mathematics and its Applications

ISSN 1842-6298 (electronic), 1843-7265 (print)
Volume 16 (2021), 371 – 383

ON GENERALIZED G-RECURRENT MANIFOLDS

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Abstract. In this paper, we define a type of Riemannian manifold called generalized G-recurrent manifold, and study the various properties of such a manifold. Among others, it is shown that if a generalized G-recurrent manifold is Einstein, then its associated 1-forms are closed and that if a generalized G-recurrent manifold with constant scalar curvature is conformally flat, then the manifold is semisymmetric. Furthermore a sufficient condition for a generalized G-recurrent manifold to be quasi Einstein is obtained.

1 Introduction

The Riemannian curvature tensor R is one of the most important objects in Riemannian geometry. The curvature tensor R possesses the several symmetric and skew symmetric properties as well as the cyclic ones. Some of which may be described as follows:

$$(1)R(X,Y,Z,W) = -R(Y,X,Z,W),$$
$$(2)R(X,Y,Z,W) = -R(X,Y,W,Z),$$
$$(3)R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0,$$

where X,Y,Z,W are vector fields. A (0,4)-tensor T is said to be a curvature-like tensor if it satisfies the above identities. It is known that if T is a curvature-like tensor then it follows that $T(X,Y,Z,W) = T(Z,W,X,Y)$. For instance, conformal, quasiconformal, conharmonic, concircular and M-projective curvature tensors are curvature-like tensors [9, 10, 17, 26, 28, 30]. However, projective, pseudo-projective, $W^*, W_1, W_2, ..., W_9$ curvature tensors are not curvature-like tensors [16, 17, 18, 19, 20, 26, 29]. In [16], Pokhariyal and Mishra introduced a type of curvature-like tensor (namely G-curvature tensor) and studied its relativistic significance. According to them, a G-curvature tensor G on a Riemannian manifold (M^n, g) is defined by

$$G(X,Y,Z,W) = R(X,Y,Z,W) - \frac{1}{2(n-1)}(r \bullet g)(X,Y,Z,W), \qquad (1.1)$$

2020 Mathematics Subject Classification: 53C15; 53C25
Keywords: generalized G-recurrent manifold; Einstein; conformally flat; semisymmetry; quasi Einstein

where r is the Ricci tensor of (M^n, g) and the symbol $\bullet$ is the Nomizu-Kulkarni product of symmetric (0,2)-tensors generating a curvature type tensor:

$$(h \bullet k)(X, Y, Z, W) = h(X, Z)k(Y, W) + h(Y, W)k(X, Z) - h(X, W)k(Y, Z) - h(Y, Z)k(X, W).$$

In this connection, they introduced a W^*-curvature tensor defined on the line of Weyl projective curvature tensor. The G-curvature tensor G has been defined by breaking W^* into skew-symmetric parts [11, 16, 17, 18].

In [12], the author studies the several geometric properties of a Riemannian manifold with G-curvature tensor on which some pseudo symmetric conditions are imposed.

A non-flat n-dimensional Riemannian manifold (M^n, g) is said to be a generalized recurrent manifold if its curvature tensor R of type $(0, 4)$ satisfies the following:

$$(\nabla_U R)(X, Y, Z, W) = A(U)R(X, Y, Z, W) + B(U)(g \bullet g)(X, Y, Z, W), \qquad (1.2)$$

where ∇ denotes the Levi-Civita connection and A,B are 1-forms of which B is non-zero.

Similarly one may define another type of Riemannian manifold (M^n, g) which is called a generalized Ricci recurrent manifold having a defining condition

$$(\nabla_U r)(Y, Z) = A(U)r(Y, Z) + B(U)g(Y, Z), \qquad (1.3)$$

where A and B are 1-forms of which B is non-zero.

Such manifolds have received a good deal of attention as special types of manifolds. These manifolds were studied in considerable detail by many authors [1, 5, 6, 7, 8, 13, 14, 15, 23]. The present paper deals with a type of Riemannian manifold which is called a generalized G-recurrent manifold realizing the following relation

$$(\nabla_U G)(X, Y, Z, W) = A(U)G(X, Y, Z, W) + B(U)(g \bullet g)(X, Y, Z, W), \qquad (1.4)$$

where A and B are 1-forms of which B is non-zero.

This paper is organized as follows. Section 2 deals with preliminaries. In section 3, we have established some relations among several generalized recurrent manifolds. Section 4 is concerned with an Einstein and generalized G-recurrent manifold. In section 5, it is shown that a conformally flat and generalized G-recurrent manifold with constant scalar curvature is semisymmetric. Finally in the last section, we have inquired under what condition a generalized G-recurrent manifold will be quasi Einstein.

2 Preliminaries

Let (M^n, g) be a Riemannian manifold. The Riemannian curvature tensor R, Ricci tensor r and scalar curvature s are defined by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z. \tag{2.1}$$

And

$$R(X,Y,Z,W) = g(R(X,Y)Z,W). \tag{2.2}$$

$$r(Y,Z) = \sum_{i=1,\dots,n} R(e_i,Y,Z,e_i), \tag{2.3}$$

$$s = \sum_{i=1,\dots,n} r(e_i,e_i), \tag{2.4}$$

where $\{e_i\}_{i=1,..,n}$ is an orthonormal frame. The Riemannian manifold (M^n, g) is called an Einstein manifold [2] if its Ricci tensor r is proportional to the metric tensor g, i.e.,

$$r = \frac{s}{n}g. \tag{2.5}$$

The Weyl curvature tensor of type $(0,4)$ is defined by

$$C(X,Y,Z,W) = R(X,Y,Z,W) + \frac{1}{(n-2)}(r \bullet g)(X,Y,Z,W) - \frac{s}{2(n-1)(n-2)}(g \bullet g)(X,Y,Z,W). \tag{2.6}$$

$C = 0$ if and only if (M^n, g) is conformally flat. The Weyl curvature tensor depends only on the conformal class of (M^n, g). Moreover, it satisfies the curvature symmetries and so we can treat it as a conformal curvature tensor.
A non-flat Riemannian manifold (M^n, g) is said to be quasi Einstein [3, 4, 11, 22] if its Ricci tensor r is not identically zero and satisfies the condition

$$r(X,Y) = ag(X,Y) + bu(X)u(Y) \tag{2.7}$$

for some smooth functions a and $b \neq 0$, where u is a non-zero 1-form such that

$$g(X,U) = u(X), g(U,U) = u(U) = 1.$$

The 1-form u is called the associated 1-form and the unit vector field U is called the generator of the quasi Einstein manifold (M^n, g). The notion of quasi Einstein manifolds arose during the study of exact solutions of the Einstein field equations as well as during considerations of quasi umbilical hypersurfaces. Investigations in [3] and others have revealed that a conformally flat quasi Einstein manifold has the geometric structure of quasi constant curvature. Also it has been found that a

manifold of quasi constant constant is a natural subclass of quasi Einstein manifold [4].
For a $(0, k)$-tensor field T on M^n, we define a $(0, k+2)$-tensor field $R \cdot T$ [23, 24] by

$$(R \cdot T)(X_1, \ldots, X_k; X, Y) = (R(X, Y) \cdot T)(X_1, \ldots, X_k)$$

$$= -T(R(X, Y)X_1, X_2, \ldots, X_k) - \ldots - T(X_1, \ldots, X_{k-1}, R(X, Y)X_k). \tag{2.8}$$

If a Riemannian manifold (M^n, g) satisfies the condition $R \cdot R$=0 (resp. $R \cdot r$=0), then (M^n, g) is said to be semisymmetry (resp. Ricci-semisymmetry)[24, 25]. It is well known that the class of semisymmetric manifolds includes the set of locally symmetric manifolds ($\nabla R = 0$) as a proper subset and that the class of Ricci-semisymmetric manifolds includes the set of Ricci-symmetric manifolds ($\nabla r = 0$) as a proper subset [23, 24].

3 Some properties of generalized G-recurrent manifold

In this section, we show that the class of generalized G-recurrent manifold includes the set of generalized recurrent manifold, while the set of generalized G-recurrent manifold belongs to the class of generalized Ricci recurrent manifold.

Theorem 1. *Every generalized recurrent manifold is generalized G-recurrent.*

Proof. Since (M^n, g) is a generalized recurrent manifold, we have

$$(\nabla_U R)(X, Y, Z, W) = A(U)R(X, Y, Z, W) + B(U)(g \bullet g)(X, Y, Z, W). \tag{3.1}$$

Contracting 3.1 on X and W, we obtain

$$(\nabla_U r)(Y, Z) = A(U)r(Y, Z) + B(U)2(1 - n)g(Y, Z). \tag{3.2}$$

By virtue of 1.1, 3.1 and 3.2 we have

$$(\nabla_U G)(X, Y, Z, W) = (\nabla_U R)(X, Y, Z, W) - \frac{1}{2(n-1)}(\nabla_U r \bullet g)(X, Y, Z, W)$$

$$= A(U)G(X, Y, Z, W) + 2B(U)(g \bullet g)(X, Y, Z, W),$$

which implies that the manifold is a generalized G-recurrent manifold. This completes the proof. □

Theorem 2. *Every generalized G-recurrent manifold is generalized Ricci recurrent.*

Proof. Let (M^n, g) be a generalized G-recurrent manifold. Taking account of 1.1 and 1.4, we have

$$(\nabla_U R)(X, Y, Z, W) - \frac{1}{2(n-1)}(\nabla_U r \bullet g)(X, Y, Z, W)$$

$$= A(U)[R(X, Y, Z, W) - \frac{1}{2(n-1)}(r \bullet g)(X, Y, Z, W)]$$

$$+ B(U)(g \bullet g)(X, Y, Z, W). \tag{3.3}$$

Contracting 3.3 on X and W, we get

$$(\nabla_U r)(Y, Z) - \frac{1}{2(n-1)}((2-n)(\nabla_U r)(Y, Z) - ds(U)g(Y, Z))$$

$$= A(U)[r(Y, Z) - \frac{1}{2(n-1)}((2-n)r(Y, Z) - sg(Y, Z))] + B(U)2(1-n)g(Y, Z),$$

which yields

$$(\nabla_U r)(Y, Z) = A(U)r(Y, Z) + \bar{B}(U)g(Y, Z), \tag{3.4}$$

where $\bar{B}(U) = \frac{s}{3n-4}A(U) - \frac{4(n-1)^2}{3n-4}B(U) - \frac{1}{3n-4}ds(U)$.
Therefore the manifold is generalized Ricci recurrent. □

Corollary 3. *Let (M^n, g) be a generalized G-recurrent manifold. If the scalar curvature s of (M^n, g) is constant, then $s \neq 0$.*

Proof. Contracting 3.4 on Y and Z, we get from s=constant

$$0 = \frac{4(n-1)s}{3n-4}A(U) - \frac{4n(n-1)^2}{3n-4}B(U).$$

Therefore we have

$$B(U) = \frac{s}{n(n-1)}A(U). \tag{3.5}$$

Since $B = 0$ is inadmissible by the defining condition of generalized G-recurrent manifold, we find $s \neq 0$. □

4 Einstein and generalized G-recurrent manifolds

A Riemannian manifold (M^n, g) is said to be an Einstein and generalized G-recurrent manifold if the manifold is simultaneously an Einstein manifold and a generalized G-recurrent manifold. Concerning an Einstein and generalized G-recurrent manifold, we have the following theorems:

Theorem 4. *Let (M^n, g) $(n \geq 3)$ be an Einstein and generalized G-recurrent manifold. Then the manifold is generalized recurrent.*

Proof. By virtue of the given Einstein condition, we know that the scalar curvature and Ricci tensor of the manifold are constant with respect to cavariant differentiation [2]. Therefore we have

$$(\nabla_U G)(X, Y, Z, W) = (\nabla_U R)(X, Y, Z, W) \tag{4.1}$$

From 1.1, 1.4 and 2.5 , it follows

$$(\nabla_U G)(X, Y, Z, W) = A(U)[R(X, Y, Z, W) - \frac{s}{2(n-1)n}(g \bullet g)(X, Y, Z, W)]$$
$$+ B(U)(g \bullet g)(X, Y, Z, W). \tag{4.2}$$

Taking account of 4.1 and 4.2 together, we get

$$(\nabla_U R)(X, Y, Z, W) = A(U)R(X, Y, Z, W) + [-\frac{s}{2(n-1)n}A(U) + B(U)](g \bullet g)(X, Y, Z, W)$$
$$= A(U)R(X, Y, Z, W) + \widehat{B}(U)(g \bullet g)(X, Y, Z, W), \tag{4.3}$$

where $\widehat{B}(U) = -\frac{s}{2(n-1)n}A(U) + B(U)$.
This completes the proof. □

Theorem 5. *Let (M^n, g) $(n \geq 3)$ be an Einstein and generalized G-recurrent manifold. Then either its associated 1-forms A and B are closed or the manifold is a space of constant curvature.*

Proof. It follows from 3 and the Einstein condition that s=constant $\neq 0$. Therefore contracting 4.3 on X and W, and then on Y and Z, we get

$$0 = sA(U) + 2n(1-n)\widehat{B}(U),$$

which implies from the last relation and the definition of $\widehat{B}$ in 4.3

$$B(U) = \frac{s}{n(n-1)}A(U). \tag{4.4}$$

Therefore we have from 4.3 and 4.4

$$(\nabla_U R)(X,Y,Z,W) = A(U)R(X,Y,Z,W) + \frac{s}{2n(n-1)}A(U)(g \bullet g)(X,Y,Z,W) \tag{4.5}$$

From Walker's Lemma [27], we have

$$\begin{aligned}(\nabla_U\nabla_V R)(X,Y,Z,W) - (\nabla_V\nabla_U R)(X,Y,Z,W)\\ +(\nabla_Z\nabla_W R)(U,V,X,Y) - (\nabla_W\nabla_Z R)(U,V,X,Y)\\ +(\nabla_X\nabla_Y R)(Z,W,U,V) - (\nabla_Y\nabla_X R)(Z,W,U,V) = 0.\end{aligned} \tag{4.6}$$

By virtue of s=constant, 4.5 and 4.6, we get

$$\begin{aligned}dA(U,V)[R(X,Y,Z,W) + \frac{s}{2n(n-1)}(g \bullet g)(X,Y,Z,W)]\\ +dA(Z,W)[R(U,V,X,Y) + \frac{s}{2n(n-1)}(g \bullet g)(U,V,X,Y)]\\ +dA(X,Y)[R(Z,W,U,V) + \frac{s}{2n(n-1)}(g \bullet g)(Z,W,U,V)] = 0.\end{aligned} \tag{4.7}$$

Note that $K(X,Y,Z,W) = R(X,Y,Z,W) + \frac{s}{2n(n-1)}(g \bullet g)(X,Y,Z,W)$ is a symmetric (0,4) tensor with respect to the first pair of two indices and the last pair of two indices. Consequently by virtue of Walker's Lemma [27], we have either $dA = 0$ or $K = 0$. In case of $dA = 0$, considering 4.4 and s = constant, we also have $dB = 0$. The other case, K=0 implies that (M^n, g) is a space of constant curvature. This completes the proof. □

5 Conformally flat and generalized G-recurrent manifolds

A Riemannian manifold (M^n, g) is said to be a conformally flat and generalized G-recurrent manifold if the manifold is simultaneously a conformally flat manifold and a generalized G-recurrent manifold. Concerning a conformally flat and generalized G-recurrent manifold, we have the following theorems:

Theorem 6. *A conformally flat and generalized G-recurrent manifold is generalized recurrent.*

Proof. From 1.1 and 1.4, we have

$$(\nabla_U R)(X,Y,Z,W) - \frac{1}{2(n-1)}(\nabla_U r \bullet g)(X,Y,Z,W)$$

$$= A(U)[R(X,Y,Z,W) - \frac{1}{2(n-1)}(r \bullet g)(X,Y,Z,W)] + B(U)(g \bullet g)(X,Y,Z,W).$$

Contracting the last relation on X and W, we get

$$(\nabla_U r)(Y,Z) - \frac{1}{2(n-1)}((2-n)(\nabla_U r)(Y,Z) - ds(U)g(Y,Z))$$

$$= A(U)[r(Y,Z) - \frac{1}{2(n-1)}((2-n)r(Y,Z) - sg(Y,Z))] + B(U)2(1-n)g(Y,Z),$$

which yields

$$(\nabla_U r)(Y,Z) = A(U)r(Y,Z) + \bar{B}(U)g(Y,Z), \tag{5.1}$$

where $\bar{B}(U) = \frac{s}{3n-4}A(U) - \frac{4(n-1)^2}{3n-4}B(U) - \frac{1}{3n-4}ds(U)$.
From 2.6 and conformal flatness, it follows

$$R(X,Y,Z,W) = -\frac{1}{n-2}(r \bullet g)(X,Y,Z,W)$$

$$+ \frac{s}{2(n-1)(n-2)}(g \bullet g)(X,Y,Z,W). \tag{5.2}$$

Taking the covariant differentiation of 5.2, we get

$$(\nabla_U R)(X,Y,Z,W) = -\frac{1}{n-2}(\nabla_U r \bullet g)(X,Y,Z,W)$$

$$+ \frac{ds(U)}{2(n-1)(n-2)}(g \bullet g)(X,Y,Z,W), \tag{5.3}$$

which yields from 5.1, 5.2 and 5.3

$$(\nabla_U R)(X,Y,Z,W) = A(U)R(X,Y,Z,W) + \widetilde{B}(U)(g \bullet g)(X,Y,Z,W), \tag{5.4}$$

where $\widetilde{B}(U) = -\frac{1}{n-2}\bar{B}(U) - \frac{s}{2(n-1)(n-2)}A(U) + \frac{1}{2(n-1)(n-2)}ds(U)$.
This completes the proof. □

Theorem 7. *Let (M^n, g) be a conformally flat and generalized G-recurrent manifold with constant scalar curvature. Then the manifold is semisymmetric.*

Proof. Contracting 5.4 on X, W, and then on Y, Z we have from $s=$ constant

$$0 = sA(U) + 2n(1-n)\widetilde{B}(U). \tag{5.5}$$

Taking account of s=constant, 5.4 and 5.5, we get

$$(\nabla_U \nabla_V R)(X,Y,Z,W) - (\nabla_V \nabla_U R)(X,Y,Z,W)$$

$$= dA(U,V)[R(X,Y,Z,W) + \frac{s}{2n(n-1)}(g \bullet g)(X,Y,Z,W)]. \tag{5.6}$$

By virtue of Walker's Lemma [27] and 5.6, we get

$$\begin{aligned}
&(\nabla_U\nabla_V R)(X,Y,Z,W) - (\nabla_V\nabla_U R)(X,Y,Z,W)\\
&+(\nabla_Z\nabla_W R)(U,V,X,Y) - (\nabla_W\nabla_Z R)(U,V,X,Y)\\
&+(\nabla_X\nabla_Y R)(Z,W,U,V) - (\nabla_Y\nabla_X R)(Z,W,U,V)\\
&= dA(U,V)[R(X,Y,Z,W) + \frac{s}{2n(n-1)}g \bullet g(X,Y,Z,W)]\\
&+dA(Z,W)[R(U,V,X,Y) + \frac{s}{2n(n-1)}g \bullet g(U,V,X,Y)]\\
&+dA(X,Y)[R(Z,W,U,V) + \frac{s}{2n(n-1)}g \bullet g(Z,W,U,V)] = 0,
\end{aligned}$$

which yields either $dA = 0$ or $R + \frac{s}{2n(n-1)}g \bullet g = 0$ by the same reason in the course of the proof of the 5. Contracting 5.4 on X and W, we get

$$(\nabla_U r)(Y,Z) = A(U)r(Y,Z) + 2(1-n)\widetilde{B}(U)g(Y,Z).$$

From the last relation, it follows

$$\begin{aligned}
(R(U,V))\cdot r)(Y,Z) &= (\nabla_U\nabla_V r)(Y,Z) - (\nabla_V\nabla_U r)(Y,Z) - (\nabla_{[U,V]}r)(Y,Z)\\
&= dA(U,V)r(Y,Z) + 2(1-n)d\widetilde{B}(U,V)g(Y,Z)\\
&\quad + 2(1-n)[A(V)\widetilde{B}(U) - A(U)\widetilde{B}(V)]g(Y,Z).
\end{aligned} \tag{5.7}$$

In case of $dA = 0$, we have $d\widetilde{B} = 0$ from s=constant and 5.5. Therefore from $dA = d\widetilde{B} = 0$, 5.5 and 5.7 it follows that

$$R(U,V)\cdot r = 0.$$

The other case, $R + \frac{s}{2n(n-1)}g \bullet g = 0$ implies that the manifold is a space of constant curvature. And hence we get

$$R(U,V)\cdot r = 0$$

because the Ricci tensor of the manifold is constant with respect to covariant differentiation.

Therefore in any case, the manifold is Ricci-semisymmetric, which implies the manifold is semisymmetric , i.e.,

$$R(U,V)\cdot R = 0$$

because of s=constant and conformal flatness. This completes the proof. □

6 Condition for generalized G-recurrent manifold to be quasi Einstein

A vector field V on a Riemannian manifold (M^n, g) is said to be a concircular vector field [21] if it satisfies the condition:

$$\nabla_X V = \alpha X + \Omega(X)V$$

for every vector field X on M^n, where α is a non-zero scalar and Ω is a closed 1-form. Note that we denote the associated vector field of 1-form A as $A^\sharp$, that is, $A(X) = g(A^\sharp, X)$. In this section we establish a sufficient condition for a generalized G-recurrent manifold to be quasi Einstein. More precisely, we can state the following theorem:

Theorem 8. *Let (M^n, g) be a generalized G-recurrent manifold with s=constant. If the associated vector field $A^\sharp$ of 1-form A in (1.4) is a unit concircular vector field with α=constant, then the manifold is quasi Einstein. In particular, if $s = n(1-n)\alpha^2$, then the manifold is Einstein.*

Proof. By virtue of 3.4 we get

$$(\nabla_U r)(Y, Z) = A(U)r(Y, Z) + \bar{B}(U)g(Y, Z), \tag{6.1}$$

where $\bar{B}(U) = \frac{s}{3n-4}A(U) - \frac{4(n-1)^2}{3n-4}B(U) - \frac{1}{3n-4}ds(U)$.
Since s=constant, we have from contracting 6.1 on Y and Z

$$0 = A(U)s + \bar{B}(U)n. \tag{6.2}$$

By the given conditions, we know that $g(A^\sharp, A^\sharp) = 1$ and

$$g(\nabla_X A^\sharp, Y) = \alpha g(X, Y) + \Omega(X)A(Y). \tag{6.3}$$

Hence we get

$$0 = 2g(\nabla_X A^\sharp, A^\sharp) = 2\alpha A(X) + 2\Omega(X),$$

which leads to

$$-\alpha A(X) = \Omega(X). \tag{6.4}$$

Therefore we know from 6.3 and 6.4

$$g(\nabla_X A^\sharp, Y) = \alpha[g(X, Y) - A(X)A(Y)],$$

which yields

$$\nabla_X A^\sharp = \alpha[X - A(X)A^\sharp]. \tag{6.5}$$

Considering 6.5, we obtain

$$R(X, Y)A^\sharp = \nabla_X \nabla_Y A^\sharp - \nabla_Y \nabla_X A^\sharp - \nabla_{[X,Y]} A^\sharp = -\alpha^2 A(Y)X + \alpha^2 A(X)Y$$

and hence we get

$$g(R(X,Y)A^{\sharp},Z) = -g(R(X,Y)Z,A^{\sharp}) = -\alpha^2 A(Y)g(X,Z) + \alpha^2 A(X)g(Y,Z).$$

Contracting the last relation on X and Z, we have

$$r(Y,A^{\sharp}) = -\alpha^2 nA(Y) + \alpha^2 A(Y) = \alpha^2(1-n)A(Y). \tag{6.6}$$

Since

$$(\nabla_X r)(Y,A^{\sharp}) = X(r(Y,A^{\sharp})) - r(\nabla_X Y,A^{\sharp}) - r(Y,\nabla_X A^{\sharp}), \tag{6.7}$$

we have from 6.1, 6.2, 6.4, 6.5 and 6.7

$$r(X,Y) = \alpha^2(1-n)g(X,Y) + \frac{1}{\alpha}(\frac{s}{n} - \alpha^2(1-n))A(X)A(Y),$$

which implies that the manifold is quasi Einstein.
In particular, if $s = \alpha^2 n(1-n)$ then the manifold is Einstein. This completes the proof. □

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