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REPRESENTATIONS OF (QUASI-)COMPLEMENTED ALGEBRAS

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Abstract. We deal with (pre-Hilbert and or Hilbert space) representations of algebraic structures equipped with compatible topologies. Among them we explore the representation theory in the context of certain left (quasi-)complemented (non-normed) topological algebras. To face this problem, we apply either the Kakutani-Mackey-like theorem (a kind of a partial (non-normed) generalization of a classical Kakutani-Mackey theorem) or the Arens-Michael decomposition.

1 Introduction and preliminaries

In [3, p. 463, Theorem 3.6], the authors proved that a B^*-algebra is (left, right) complemented if and only if it is a dual algebra. In the non-normed case, the previous result was generalized in [12, p. 226, Theorem 3.1]. More precisely, a Hausdorff locally C^*-algebra is an annihilator algebra if and only if it is a dual algebra if and only if it is a complemented algebra with left (resp. right) complementor $\perp_\ell = * \circ \mathcal{A}_r$ (resp. $\perp_r = * \circ \mathcal{A}_\ell$); where $\mathcal{A}_r, \mathcal{A}_\ell$ denote the right and the left annihilator maps. Besides, a B^*-algebra which is complemented is isometrically $*$-isomorphic to a certain sum of its minimal closed two-sided ideals, say $I_\lambda, \lambda \in \Lambda$. Moreover, each I_λ is actually isometrically $*$-isomorphic to the algebra $LC(H_\lambda)$ of all compact operators on a Hilbert space H_λ [3, p. 464, Theorem 3.9]. Then, by dropping duality, the authors dealt with (left, right) complemented B^*-algebras which can be viewed as topological algebras of the form $LC(H)$ for some Hilbert space H. When H is infinite-dimensional, then all complementors on $A = LC(H)$ are continuous, and the continuity of a complementor is then equivalent with the fact that this complementor is expressed in the previous form (via the annihilator and some involution) [3, p. 471, Theorem 6.8 and p. 473, Theorem 6.11]. In the case A is a complemented B^*-algebra that has no minimal left ideals of dimension less than three, then every continuous complementor p has the form $p = \sharp \circ \mathcal{A}_\ell$ for some other involution $\sharp$ in A) [ibid. p. 477, Theorem 7.4]. Also, in [2], Alexander considers

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the same restriction on the dimension of the minimal left ideals in complemented Banach algebras A. For the primitive case, such an algebra A admits a faithful continuous strictly dense Hilbert space representation when A is endowed with a continuous complementor [ibid. p. 391, Theorem 3.4]. Besides, in the semisimple case, the algebra gains a faithful continuous Hilbert space representation [ibid. p. 394, Theorem 4.4]. Moreover, in both cases, the primitive and the semisimple one, the complementor of a closed right ideal is related with the inner product of the Hilbert space.

The present paper deals with a representation theory for various classes of non-normed topological complemented algebras (Theorems 6, 8, 18, 25 and Corollary 26). In each case, key assumptions are imposed to succeed these pre-Hilbert (or yet Hilbert) space representations, which reveal the power of the behavior of the ideals of these topological algebras or yet of the normed factors in their Arens-Michael decomposition.

The notion of semilinearity is used to define automorphically perfect pairs, through which the existence of inner products is succeeded, and with the help of them, certain quasi-complemented linear spaces become pre-Hilbert ones (viz. spaces with an inner product). If E is a topologically simple, left complemented locally m-convex algebra, then, via an automorphically perfect pair, for a minimal closed right ideal $R = eE$ with e a minimal element which is not a left topological divisor of zero, E is continuously representable on a pre-Hilbert space (Theorem 6). Actually, the Kakutani-Mackey-like theorem (Theorem 5) is key for the aforementioned representation theorem. Based on Theorem 6, the image of the representation inherits from the initial topological algebra the complementarity (Scholium 7, (iii)). A variation of the same theorem assures the faithfulness of the representation, without the hypothesis that e is not a left topological divisor of zero (Theorem 8). Here we stress the fact that, in Theorem 8, *as ring multiplication* of $\mathcal{L}(R)$ (the algebra of all linear and continuous mappings on R), we define the *anti-composition*. In that case, we have a further information about the relationship between complements and orthogonal complements (that induces the inner product of the pre-Hilbert space).

The problem of the existence of (Hilbert space) representations of an algebraic-topological structure E, is sometimes taken through induced representations. This fact takes on a special importance for the cases in which there are representable structures related to E. In analogy, and having as a tool the Arens-Michael decomposition, a certain left complemented (or a certain quasi-complemented) locally m-convex algebra is continuously representable on a Hilbert space. This is done via a representation of a certain factor of the aforementioned decomposition. The advantage here is that we get some important information either relative to the relationships between the complementor and that of the orthogonal one (induced by the inner product of the Hilbert space (Theorem 18) or about the compactness of the images of the representation concerned (Theorem 25, and Corollary 26).

All vector spaces and algebras, considered throughout, are taken over the field

$\mathbb{C}$ of complexes. By $\mathcal{V}_X$ we denote the set of all closed subspaces of a topological vector space X.

The following definition is referred to some types of complemented topological vector spaces, which we deal with in the sequel.

Definition 1. Let X be a topological vector space.

(i) X is called a *quasi-complemented space* if there is a mapping $\sigma : \mathcal{V}_X \to \mathcal{V}_X$ with $\sigma(M) = M^\sigma$, satisfying the following conditions:

1. If $M_1, M_2 \in \mathcal{V}_X$ with $M_1 \subseteq M_2$, then ${M_2}^\sigma \subseteq {M_1}^\sigma$;
2. If $M \in \mathcal{V}_X$, then $M^{\sigma\sigma} = M$;
3. If $M \in \mathcal{V}_X$, then $M \cap M^\sigma = \{0\}$.

σ is called a *quasi-complementor* and M^σ a *quasi-complement* of M.

(ii) X is called a *complemented space* if it is a quasi-complemented space and satisfies the condition:

$$If \ \ M \in \mathcal{V}_X, \ \ then \ \ X = M \oplus M^\sigma.$$

Of course, the latter condition contains (3). σ is called a (*vector*) *complementor on* X and M^σ a (*vector*) *complement* of M.

In what follows, (X, σ) stands for a quasi-complemented space or a complemented (vector) space X with a quasi-complementor or a complementor σ.

For a terminology comment on the terms "*quasi-complement*" and "*complement*", see after Definition 1.3 in [15].

A *topological algebra* E is an algebra which is a topological vector space and the ring multiplication is separately continuous (see e.g., [24, p. 4, Definition 1.1]). An algebra E equipped with a submultiplicative seminorm p is named an *m-seminormed algebra* and it is denoted by (E, p). In particular, E is named a *locally m-convex algebra* if its topology is defined by a family $(p_\alpha)_{\alpha\in\Lambda}$ of submultiplicative seminorms and it is denoted by $(E, (p_\alpha)_{\alpha\in\Lambda})$. In that case, the ring multiplication is jointly continuous. $\overline{S}$ stands for the topological closure of a subset S in a topological algebra.

Analogous notions, as in Definition 1, can be given by considering topological algebras.

Taking a topological algebra E, $\mathcal{L}_\ell(E) \equiv \mathcal{L}_\ell$ (resp. $\mathcal{L}_r(E) \equiv \mathcal{L}_r$) stands for the set of all closed left (right) ideals of E.

A topological algebra E is called *left complemented* if there exists a mapping $\perp : \mathcal{L}_\ell \longrightarrow \mathcal{L}_\ell : I \mapsto I^\perp$, such that

$$\text{if } I \in \mathcal{L}_\ell, \text{ then } E = I \oplus I^\perp. \tag{1.1}$$

$I^{\perp}$ is called a *complement* of I.

$$\text{If } I, J \in \mathcal{L}_{\ell}, I \subseteq J, \text{ then } J^{\perp} \subseteq I^{\perp}. \tag{1.2}$$

$$\text{If } I \in \mathcal{L}_{\ell}, \text{ then } (I^{\perp})^{\perp} = I. \tag{1.3}$$

$\perp$ as before is called a *left complementor* on E. In what follows, we denote by $(E, \perp)$ a left complemented algebra with a left complementor $\perp$.

A *right complemented algebra* is defined analogously and we talk about a *right complementor*. A left and right complemented algebra is simply called a complemented algebra.

A topological algebra E is called a *left quasi-complemented algebra* if there exists a mapping $\perp : \mathcal{L}_{\ell} \longrightarrow \mathcal{L}_{\ell} : I \mapsto I^{\perp}$, satisfying (1.2), (1.3), and $I \cap I^{\perp} = (0), I \in \mathcal{L}_{\ell}$ (see [11, p. 3727, Scholium 2.16]). We refer to a *left quasi-complemented locally m-convex algebra* E, if the topology of E is defined by a family of submultiplicative seminorms.

A topological algebra E is named *left precomplemented*, if for every $I \in \mathcal{L}_{\ell}$, there exists $I' \in \mathcal{L}_{\ell}$ such that $E = I \oplus I'$. Similarly a *right precomplemented algebra* is defined. A left and right precomplemented algebra is called a *precomplemented algebra*. All the results appeared in this paper hold true by interchanging "left" by "right". The next is used in the proof of Lemma 20.

Remark 2. *Let E be a left precomplemented algebra and $I, I' \in \mathcal{L}_{\ell}$ such that $E = I \oplus I'$. Then there exists a linear map $T : E \to E$ with $T^2 = T$ (projection) such that Im $T = I$ and* $\ker T = I'$. Indeed, if x is an element in E, there are unique $y \in I$ and $z \in I'$ with $x = y + z$. We define $T(x) = y$. T is then a well defined linear map having the aforementioned properties. Moreover, for $y \in$ Im T, there is $x \in E$ with $T(x) = y$ and thus $T(y) = T(T(x)) = T^2(x) = T(x) = y$. Therefore $T(y) = y$ for all $y \in$ Im T. Also, *T is unique with respect to the preceding properties.* To indicate that T is unique, for the ideals $I, I' \in \mathcal{L}_{\ell}$, we employ the symbol $T(I, I')$. *The previous also hold, more general, for vector spaces X with $X = V \oplus V'$ where V, V' are vector subspaces of X.*

For the existence of a projection P of a linear space X whose range is a given linear manifold M (alias a subspace) of X (see e.g., [26, p. 10, and p. 241 Definition and Theorem 4.8-A]).

The next definition refers to an extended notion of linearity, and it constitutes a prerequisite in defining the automorphically perfect pairs (Definition 4), through which the existence of inner products is succeeded. For the existence of this type of mappings see [15, p. 514, Lemma 3.5]. In the context of [ibid., p. 514, Theorem 3.6], we get semilinear transformations from a vector space to its dual. As we shall see, under further properties, the vector space is equipped with an inner product that leads to a representation of a certain topological algebra (Theorem 6).

Definition 3. Let X, Y be vector spaces. A mapping $T : X \to Y$ is named *semilinear* or a *semilinear transformation* (with respect to ϕ), if there exists an automorphism ϕ on $\mathbb{C}$ such that

$$T(\lambda x + \mu y) = \phi(\lambda)T(x) + \phi(\mu)T(y)$$

for any scalars λ, μ and every $x, y \in X$. If, in particular, $\phi(\kappa) = \overline{\kappa}$, where $\overline{\kappa}$ is the complex conjugate of κ, then T is named *conjugate linear*.

In the sequel, we need some more notation. If S is a subset of a vector space X, then $[S]$ stands for the subspace of X, generated by S. We also use the symbol $[S]_X$, in the case, we want to indicate the vector space X. If, in particular, $S = \{x\}, x \in X$, we use the symbol $[x]$ or, yet, $[x]_X$.

Definition 4. Let X be an infinite-dimensional topological vector space and X' its dual. Suppose there exists an $1-1$ correspondence τ between the one-dimensional subspaces of X and X' that respects the linearly independence. If for any semilinear mapping $T : X \to X'$ with $\tau([x]_X) = [T(x)]_{X'}$ the automorphism ϕ, that corresponds to T, is continuous, then the pair (X, τ) is called *automorphically perfect*.

The previous definition is realized in the context of Banach vector spaces (see [20, p. 728, Lemma 2 and p. 729, the proof of Theorem 1]).

The following result was stated in [15, p. 517, Theorem 3.10] and it is a partial generalization of a classical Kakutani-Mackey theorem for Banach complemented (in some sense) vector spaces (see [20, p. 729, Theorem 1]). Here, we note that actually Kakutani and Mackey employed topological vector spaces of the type (i) of Definition 1. Besides, as we mentioned, the Kakutani-Mackey-like theorem contributes to state an important theorem concerning continuous pre-Hilbert representations of appropriate complemented algebras (see Theorem 6).

Let (X, σ) be a Hausdorff quasi-complemented vector space. Consider the correspondence τ from the one-dimensional subspaces of X into one-dimensional subspaces of X' with $\tau(M) = M^\tau$, where M is an one-dimensional subspace of X and $M^\tau = \{f \in X' : M^\sigma \subseteq \ker f\}$ of X'.

We note that τ, as before, is well defined, $1-1$, onto and respects linearly independence (see [15, p. 513, Proposition 3.3] see also [16, p. 93, Note 1 and Theorem 1]). Under this notation, we have the next.

Theorem 5 (A Kakutani-Mackey-like theorem)**.** *Let (X, σ) be a Hausdorff quasi-complemented vector space. Suppose that the pair (X, τ) is automorphically perfect. Then there is defined an inner product $\langle\cdot,\cdot\rangle$, on X, that is, the pair $(X, \langle\cdot,\cdot\rangle)$ is a pre-Hilbert space. Moreover, σ is, finally, a correspondence between orthogonal complements. Namely, for every closed subspace M of X, the following holds.*

$$M^\sigma = \{x \in X : \langle x, y\rangle = 0 \ \textit{for every} \ y \in M\}.$$

An element x of an algebra E, is called *idempotent* if $x = x^2$. Throughout we employ non-zero idempotents. A *minimal element* of E is a non-zero idempotent x such that xEx is a division algebra. An algebra E is called *semisimple* if its Jacobson radical is (0). A topological algebra is named *topologically simple* if it has no proper, nonzero, closed two-sided ideals.

We denote by $\mathcal{A}_\ell(S) = \{x \in E : xS = \{0\}\}$ (resp. $\mathcal{A}_r(S)$) the left (right) annihilator of a (non empty) subset S of an algebra E, being a left (resp. right) ideal of E. If S is a left (resp. right) ideal of E, then the ideals $\mathcal{A}_\ell(S)$ and $\mathcal{A}_r(S)$ are two-sided. We say that E is a *left* (resp. *right*) *preannihilator algebra*, if $\mathcal{A}_\ell(E) = (0)$ (resp. $\mathcal{A}_r(E) = (0)$). E is named *preannihilator* if it is both left and right preannihilator. In the sequel, (0) will stand either for the set containing only the zero element or for the zero ideal of an algebra.

2 An application of the Kakutani-Mackey-like theorem in the representation of complemented topological algebras

We investigate a representation theory in the context of certain left complemented (non-normed) topological algebras. As a tool, we employ the Kakutani-Mackey-like theorem, as stated in Theorem 5.

We remind that a *left topological divisor of zero* of a Hausdorff topological algebra E is an element $(0 \neq)x \in E$, such that there is a net $(x_\delta)_{\delta\in\Delta}$ in E with $x_\delta \nrightarrow 0$ but $xx_\delta \to 0$. This concept is due to R. Arens and we note that in [25, p. 46, Definition 11.1], E.A. Michael uses, for the same notion, the term *strong (left) topological divisor of zero*. Besides, in the context of locally m-convex algebras whose topology is defined by a saturated family of submultiplicative seminorms, Michael uses the term *(left) topological divisor of zero*, to define a weaker notion (of topological divisor of zero) with respect to a Banach factor in the Arens-Michael decomposition [ibid. p. 47, Definition 11.2], see also [24, p. 90, Definition 3.1] and [22, pp. 296-297]. It is proved that a strong (left) topological divisor of zero is a (left) topological divisor of zero, in the sense of Michael (see [25, p. 47, Proposition 11.3]).

In Theorem 6, below, we employ a certain locally m-convex algebra R, having an element, say e, which is not a strong left topological divisor of zero. But, we keep the Arens term *left topological divisor of zero*, as above.

Concerning the next result, we note that, the element e, it easily seen that, it is not a left topological divisor of zero, as an element of R. But, what we really need, is that element not to be a left topological divisor of zero in E.

Also, our assumption that the pair (R, τ) is automorphically perfect " means that we do suppose that R is infinite-dimensional and, in turn, E is infinite-dimensional. For primitive Banach algebras, the conditions "R be infinite-dimensional" and "E itself be infinite-dimensional" are equivalent. See [1, p. 40, before the statement of Theorem 1]. Concerning the next theorem, in (ii) of Example 17, we give a topological algebra in which there exists a minimal ideal $R = eE$.

Theorem 6. (Representation) *Let $(E, \perp)$ be a Hausdorff topologically simple, left complemented locally m-convex algebra and e a minimal element in E, which is not a left topological divisor of zero. Consider the (minimal closed right) ideal $R = eE$ of E, and the correspondence τ from the one-dimensional subspaces of R into the one-dimensional subspaces of R' with $\tau(M) = M^\tau$, where M is an one-dimensional subspace of R and $M^\tau = \{f \in R' : M^p \subseteq \ker f\}$ of R'. Suppose that the pair (R, τ) is automorphically perfect. Then E has a faithful, continuous representation on a space with inner product (pre-Hilbert space).*

Proof. We first note that R, as a vector subspace of the locally m-convex algebra E, is locally convex. Thus R is complemented with complementor p (see [14, the comments after Definition 15, and Theorem 14]). By hypothesis, the pair (R, τ) is automorphically perfect. Moreover, by [15, p. 513, Proposition 3.3], see also [16, p. 93, Note 1 and Theorem 1], τ is well defined, $1-1$, onto and preserves the linearly independence. By the Kakutani-Mackey-like theorem (Theorem 5), there is defined an inner product, say $\langle\cdot,\cdot\rangle$, on R, such that $(R, \langle\cdot,\cdot\rangle)$ is a pre-Hilbert space. Thus there is defined also the norm $\|\cdot\| = \langle\cdot,\cdot\rangle^{1/2}$. Moreover, p is, finally, a correspondence between orthogonal complements. Namely, for any closed vector subspace M of R, it holds

$$M^p = \{x \in R : \langle x, y\rangle = 0 \text{ for every } y \in M\}.$$

Consider the algebra

$$\mathcal{L}(R) = \{f : R \to R, f \text{ linear and continuous}\}.$$

Then, on $\mathcal{L}(R)$, there is defined the operator norm. Namely,

$$|f| = \sup\{\|f(x)\| : x \in R, \|x\| \le 1\}.$$

Thus, $(\mathcal{L}(R), |\cdot|)$ is a normed (topological) algebra. Define now, the mapping

$$T : E \to \mathcal{L}(R), a \mapsto T(a) \equiv T_a \text{ with } T_a(ex) = eax, \tag{2.1}$$

for any $x \in E$, (actually, for every $ex \in R$), which is well defined, and continuous, provided that e is not a left topological divisor of zero. By the separate continuity of the (ring) multiplication of E, T_a is a continuous endomapping on R. Moreover, T_a is linear since, for any $x, y \in E$ and $\lambda \in \mathbb{C}$, we have $T_a(ex + ey) = ea(x + y) = T_a(ex) + T_a(ey)$ and $T_a(\lambda ex) = \lambda T_a(ex)$. Claim that T is a representation of E on R. Indeed, if $a, b \in E$ and $\lambda \in \mathbb{C}$, then, for any $x \in E$, we have.

$$T_{a+b}(ex) = e(a + b)x = T_a(ex) + T_b(ex) = (T_a + T_b)(ex).$$

Similarly, $T_{\lambda a}(ex) = \lambda T_a(ex)$ and

$$T_{ab}(ex) = e(ab)x = ea(bx) = T_a(ebx) = T_a(T_b(ex)) = (T_a \circ T_b)(ex).$$

Therefore

$$T(a+b) = T(a) + T(b), T(\lambda a) = \lambda T(a) \quad \text{and} \quad T(ab) = T(a) \circ T(b)$$

for any $a, b \in E$ and $\lambda \in \mathbb{C}$. For the continuity of T, consider a net $(a_\delta)_{\delta \in \Delta}$ in E with $a_\delta \to 0$, and take the respective net $(T(a_\delta))_{\delta \in \Delta}$ in $\mathcal{L}(R)$. If $x \in E$, then for any $\delta \in \Delta$, we get $T_{a_\delta}(ex) = ea_\delta x$, and, due to the separate continuity of the multiplication in E, we get

$$\lim_\delta T_{a_\delta}(ex) = \lim_\delta (ea_\delta x) = \lim_\delta (ea_\delta)x = 0 = T_0(ex).$$

Thus $\lim_\delta T(a_\delta) = T(0) = T(\lim_\delta a_\delta)$, that assures the continuity of T.

We finally prove that T is faithful (namely, $1-1$). So, if $T_a = 0$, then for any $x \in E$, we get $T_a(ex) = 0$ or $eax = 0$, that yields $ea \in \mathcal{A}_\ell(E)$. Since E is topologically simple, then it is preannihilator [14, Lemma 6]. Hence, $ea = 0$, so since a left divisor of zero is a left topological divisor of zero, we get that e is not a left divisor of zero as well, and thus $a = 0$. □

Scholium 7. *Consider the context of Theorem 6. Suppose moreover, that the following hold.*

(α) *Every closed left ideal of the algebra* $T(E)$ (see (2.1)) *is closed (as set) in the algebra* $\mathcal{L}(R)$ *(with respect to the operator norm).* Here, we note that $T(E)$ becomes a topological algebra, endowed with the relative topology from $\mathcal{L}(R)$.

(β) *T maps closed left ideals to closed subsets of* $T(E)$.

Then the following hold.

(i) If $J \in \mathcal{L}_\ell(T(E))$, then $I = T^{-1}(J) \in \mathcal{L}_\ell(E)$.

Indeed, by (α), J is a closed subset in $\mathcal{L}(R)$. By the continuity of T, I is a closed subset of E. Since T is an $1-1$ linear mapping, T^{-1} is defined (on $T(E)$), and it is linear. Therefore $I = T^{-1}(J)$ is a vector subspace of E. Moreover, if $x \in E$, $a \in I$, then $T(x) \in T(E)$ and $T(a) \in J$. So, since J is a left ideal in the algebra $T(E)$, we get $T(xa) = T(x) \circ T(a) \in J$, and $xa \in T^{-1}(J) = I$, which completes the assertion.

(ii) If $K \in \mathcal{L}_\ell(E)$, then $T(K) \in \mathcal{L}_\ell(T(E))$.

It is easy to be proved.

(iii) *The complementor* $\perp$ *on* E *induces a mapping, say* q, *on* $T(E)$ *so that the pair* $(T(E), q)$ *is a left complemented algebra.*

Indeed, we define

$$q : \mathcal{L}_\ell(T(E)) \to \mathcal{L}_\ell(T(E)) : J \mapsto J^q = T((T^{-1}(J))^\perp).$$

By the above, q is well defined. We show that q satisfies (1.1)-(1.3). Consider $J \in \mathcal{L}_\ell(T(E))$ and put $I = T^{-1}(J)$. Then $E = I \oplus I^\perp$, and by the definition of q, we get $J^q = T(I^\perp)$. Hence

$$T(E) = T(I \oplus I^\perp) \subseteq T(I) + T(I^\perp) = J + J^q \subseteq T(E)$$

and thus $T(E) = J + J^q$. The latter sum is direct. Indeed, take $f \in J \cap J^q$. Then there is $x \in I = T^{-1}(J)$ with $f = T(x)$, and since $J^q = T(I^\perp)$, there is $y \in I^\perp$ with $f = T(y)$. So, since T is $1-1$, we get $x = y$, and thus $x \in I \cap I^\perp = (0)$. Hence $f = T(0) = 0$, and $T(E) = J \oplus J^q$.

Now, take $J, K \in \mathcal{L}_\ell(T(E))$ with $J \subseteq K$. Then we have, in turn, $T^{-1}(J) \subseteq T^{-1}(K)$, $(T^{-1}(K))^\perp \subseteq (T^{-1}(J))^\perp$, and therefore

$$T((T^{-1}(K))^\perp) \subseteq T((T^{-1}(J))^\perp).$$

Namely, $K^q \subseteq J^q$.

If $J \in \mathcal{L}_\ell(T(E))$, then $J^q = T((T^{-1}(J))^\perp)$ that yields in turn

$$T^{-1}(J^q) = (T^{-1}(J))^\perp, \ \ (T^{-1}(J^q))^\perp = T^{-1}(J) \ \text{ and } \ T((T^{-1}(J^q))^\perp) = J.$$

Namely, $(J^q)^q = J$. The previous argumentation assures that $T(E)$ is a left complemented algebra with left complementor q.

Concerning the proof of Theorem 6, instead of (2.1), we could employ the following mapping

$$T : E \to \mathcal{L}(R), a \mapsto T(a) \equiv T_a \ \text{ with } \ T_a(ex) = exa \ \text{ for any } \ x \in E,$$

actually for any $ex \in R$. T_a is an endomapping on R linear and continuous. Moreover, T is a continuous representation of E, on R, which is faithful, *without the hypothesis that e is not a left topological divisor of zero*. In particular, in the present case, *as ring multiplication of $\mathcal{L}(R)$ we define the anti-composition* (viz. $fg = g \circ f$, where $\circ$ is the composition). In that case, we have a further information about the relationship between complements and orthogonal complements (that induces the inner product of the pre-Hilbert space). Thus, we have the following theorem.

Theorem 8. (Representation) *Let $(E, \perp)$ be a Hausdorff topologically simple, left complemented locally m-convex algebra. Consider the (minimal closed right) ideal $R = eE$ with e a minimal element in E, and the respective complemented vector space (R, p)* (see Theorem 6)*. Suppose that the pair (R, τ) is automorphically perfect, where τ is the correspondence between the one-dimensional subspaces of R and R', as it defined in* Theorem 6*. Then E has a faithful, continuous representation T on a pre-Hilbert space $(H, \langle\cdot,\cdot\rangle)$ such that the following holds.*

$$I^\perp = \{a \in E : T_a(H) \perp_{\langle\cdot,\cdot\rangle} T_I(H)\}, \tag{2.2}$$

for every $I \in \mathcal{L}_\ell(E)$*. If, in particular,* E *has an involution* $*$ *and the above inner product satisfies*

$$\langle xa, y\rangle = \langle x, ya^*\rangle \quad \text{for } x, y \in H \ \text{ and } a \in E, \tag{2.3}$$

then the representation T *preserves the involution.*

Proof. As in the proof of Theorem 6, there is defined an inner product, say $\langle \cdot, \cdot \rangle$, on R, such that $(R, \langle \cdot, \cdot \rangle)$ is a pre-Hilbert space, denoted by $(H, \langle \cdot, \cdot \rangle)$. Thus there is also defined the norm $\| \cdot \| = \langle \cdot, \cdot \rangle^{1/2}$. Moreover, p is finally a correspondence between orthogonal complements [ibid.]. Namely, for any closed vector subspace M of $H(= R)$, it holds

$$M^p = \{x \in H : \langle x, y\rangle = 0 \ \text{ for every } \ y \in M\}.$$

Namely, $M^p = M^{\perp_{\langle \cdot, \cdot \rangle}}$. Now, the algebra

$$\mathcal{L}(H) = \{f : H \to H, f \ \text{ linear and continuous}\}$$

becomes a normed (topological) algebra, when it is endowed with the operator norm

$$|f| = \sup\{\|f(x)\| : x \in H, \|x\| \leq 1\}.$$

We note that on $\mathcal{L}(H)$ as the ring multiplication we consider here, the anti-composition. Define the mapping

$$T : E \to \mathcal{L}(H), a \mapsto T(a) \equiv T_a \ \text{ with } \ T_a(ex) = exa, x \in E, \tag{2.4}$$

actually, for any $ex \in H$. By applying a proof analogous to that of Theorem 6, we get that T_a is a linear and continuous endomapping on H, and T is a continuous representation of E on H.

We now prove that T is faithful. So, if $T_a = 0$, then, for any $x \in E$, we get $T_a(ex) = 0$ or $exa = 0$. Thus $a \in \mathcal{A}_r(eE)$. By the topologically simpleness of E, the closed two-sided ideal $\mathcal{A}_r(eE)$ is either (0) or E. In the later case, $e \in \mathcal{A}_r(eE)$ that gives $(eE)e = 0$, and in particular, $e = 0$ which is a contradiction. Therefore $\mathcal{A}_r(eE) = (0)$ and $a = 0$. We now prove the relation (2.2). To this end, take a closed subspace S of $H = eE$. Consider the closed ideal $\overline{ES}$ of E generated by S. From Proposition 13 in [14], we get the relations:

$$H \cap I = HI \ \text{ for every } \ I \in \mathcal{L}_\ell(E) \tag{2.5}$$

and

$$\overline{ES} = \{x \in E : Hx \subseteq S\} \ \text{ for every } \ S \in \mathcal{V}_H. \tag{2.6}$$

From the comments preceding Theorem 6, the mappings

$$s : \mathcal{L}_\ell(E) \to \mathcal{V}_H \ \text{ with } \ s(I) := I \cap H$$

and

$$j : \mathcal{V}_H \to \mathcal{L}_\ell(E) \text{ with } j(S) := \overline{ES}$$

are inverses of each other. Therefore, for every $I \in \mathcal{L}_\ell(E)$, we get $(j \circ s)(I) = I$ and thus $\overline{E(I \cap H)} = I$. In view of (2.5), we get

$$\overline{E(HI)} = I \text{ for every } I \in \mathcal{L}_\ell(E). \tag{2.7}$$

By [14, p. 101, Theorem 16 (see also its proof)], the (vector) complementor on H is

$$p : \mathcal{V}_H \to \mathcal{V}_H \text{ with } S \mapsto S^p := H \cap (\overline{ES})^\perp.$$

Finally, for every closed left ideal I of E, and since $p = \perp_{\langle\cdot,\cdot\rangle}$ (see [15, p. 517, Theorem 3.10]), by applying (2.5) and (2.7), we have

$$(HI)^{\perp_{\langle\cdot,\cdot\rangle}} = (HI)^p = H \cap (\overline{E(HI)})^\perp = H(\overline{E(HI)})^\perp = HI^\perp. \tag{2.8}$$

For $a \in I^\perp$, and in view of (2.8), we get $Ha \subseteq HI^\perp = (HI)^{\perp_{\langle\cdot,\cdot\rangle}}$, whence $T_a(H) \subseteq (T_I(H))^{\perp_{\langle\cdot,\cdot\rangle}}$. Therefore $T_a(H) \perp_{\langle\cdot,\cdot\rangle} T_I(H)$ and hence

$$I^\perp \subseteq \{a \in E : T_a(H) \perp_{\langle\cdot,\cdot\rangle} T_I(H)\}.$$

For the inverse relation of the latter, take $a \in E$ with $T_a(H) \perp_{\langle\cdot,\cdot\rangle} T_I(H)$. Then $Ha \perp_{\langle\cdot,\cdot\rangle} HI$. Thus, applying also (2.8), we get

$$Ha \subseteq (HI)^{\perp_{\langle\cdot,\cdot\rangle}} = HI^\perp.$$

Consider the closed subspace $[Ha]$ of H, generated by Ha. Then $[Ha] \subseteq HI^\perp$, and thus $\overline{E[Ha]} \subseteq \overline{E(HI^\perp)} = I^\perp$ (for the last equality see (2.7)). But, by (2.6), we get $\overline{E[Ha]} = \{x \in E : Hx \subseteq [Ha]\}$. Therefore $a \in \overline{E[Ha]}$. The previous argumentation yields $a \in I^\perp$ and this completes the assertion.

Now, take $a \in E$ and T_a^* the conjugate linear mapping of T_a. Then, for any $x, y \in H$, we have $\langle T_a(x), y\rangle = \langle x, T_a^*(y)\rangle$, and from (2.3),

$$\langle x, ya^*\rangle = \langle xa, y\rangle = \langle x, T_a^*(y)\rangle.$$

Therefore $T_a^*(y) = ya^*$ for every $y \in H$. Namely, $T_a^*(y) = T_{a^*}(y)$ for every $y \in H$, and $T_a^* = T_{a^*}$. So T preserves the involution. □

Scholium 9. *The condition* (2.3) *is, for instance, satisfied for all elements in a Hausdorff locally convex* H^*-*algebra* (see [9, Theorems 1.2 and 1.3]). In case the pair $(R = eE, \tau)$ is automorphically perfect, then in view of the Kakutani-Mackey-like theorem, the "axial ideal" $H = R = eE$, as a vector subspace, is equipped with an inner product and its respective norm, so that $\mathcal{L}(H)$, under the norm operator turns to be, as already we know, a normed algebra. This contributes to get an (algebra-topological) complementarity of an appropriate subalgebra of the algebra of continuous linear operators on an axial ideal. In that way, we are able to face the problem in what extent, through axial substructures of a complemented topological algebra E, one can get complementarity for subalgebras of topological algebras of continuous operators on such a substructure (see [16]).

3 Representation theorems via the Arens-Michael decomposition

In this section, we get a continuous representation on a Hilbert space of an appropriate left complemented (resp. quasi-complemented) locally m-convex algebra E by employing certain factors in the Arens-Michael decomposition of E. In that case, we also get (Theorem 18) the relationship between the complementor and that of the orthogonal one (induced by the inner product of the Hilbert space) on one hand, and on the other, the images of the representations turn to be compact operators (Theorem 25 and Corollary 26).

We refer briefly to the Arens-Michael decomposition which will be used in the sequel. Let $(E, (p_\alpha)_{\alpha\in\Lambda})$ be a complete locally m-convex algebra and

$$\rho_\alpha : E \to E/\ker p_\alpha \equiv E_\alpha : x \mapsto \rho_\alpha(x) := x + \ker p_\alpha \equiv x_\alpha,\ \ \alpha \in \Lambda$$

the respective quotient maps. Then, $\|x_\alpha\|_\alpha := p_\alpha(x)$, $x \in E$, $\alpha \in \Lambda$, defines on E_α an algebra norm, so that E_α is a normed algebra and the morphisms ρ_α, $\alpha \in \Lambda$ are continuous. $\tilde{E}_\alpha$, $\alpha \in \Lambda$, stands for the completion of E_α (with respect to $\|\cdot\|_\alpha$). Λ is endowed with a partial order by putting $\alpha \leq \beta$ if and only if $p_\alpha(x) \leq p_\beta(x)$ for every $x \in E$. Thus, $\ker p_\beta \subseteq \ker p_\alpha$ and hence the continuous (onto) morphism

$$f_{\alpha\beta} : E_\beta \to E_\alpha : x_\beta \mapsto f_{\alpha\beta}(x_\beta) := x_\alpha, \quad \alpha \leq \beta$$

is defined. Moreover, $f_{\alpha\beta}$ is extended to a continuous morphism

$$\bar{f}_{\alpha\beta} : \tilde{E}_\beta \to \tilde{E}_\alpha, \quad \alpha \leq \beta.$$

Thus, $(E_\alpha, f_{\alpha\beta})$, $(\tilde{E}_\alpha, \bar{f}_{\alpha\beta})$, $\alpha, \beta \in \Lambda$, with $\alpha \leq \beta$ are projective systems of normed (resp. Banach) algebras, so that

$$E \cong \varprojlim E_\alpha \cong \varprojlim \tilde{E}_\alpha \ \ (\textit{Arens-Michael decomposition}), \tag{3.1}$$

within topological algebra isomorphisms (cf., for instance, [24, p. 88, Theorem 3.1 and p. 90, Definition 3.1] and/or [25, p. 20, Theorem 5.1]).

Note.- The fact that E_α is a normed algebra holds for any locally m-convex algebra, not necessarily complete (see [24, p. 85, (3.1) and the comments before it]).

The next lemmas, analogous of which hold true for right ideals, are useful in the sequel.

Concerning the proof of the next lemma, we note that *any left ideal of* $E/\ker p$ *is actually of the form*

$$\{x + \ker p : x \in M \subseteq E \ \text{ and } \ \ker p \subseteq M\}.$$

Indeed, let $J = \{x + \ker p : x \in K \subseteq E\}$ be a left ideal of $E/\ker p$ with $K \nsubseteq \ker p$. We consider the set

$J_1 = \{y + \ker p : y \in K \cup \ker p \subseteq E\}$. Obviously, $J \subseteq J_1$. Take $y + \ker p \in J_1$ with $y \in K \cup \ker p$. If $y \in K$, then $y + \ker p \in J$. If $y \notin K$, then $y \in \ker p$ and thus $y + \ker p = \ker p$ (since J is an ideal with $\ker p$ as its zero element). Therefore, $J_1 \subseteq J$ and hence $J_1 = J$.

Lemma 10. *Let (E, p) be an m-seminormed algebra. Then the only left ideals of the normed algebra $E/\ker p$ are of the form $I/\ker p$ with I a left ideal of E that contains* $\ker p$.

Proof. It is easily seen that if I is a left ideal of E containing $\ker p$, then $I/\ker p$ is a left ideal of $E/\ker p$.

Let now $J = \{x + \ker p : x \in K \subseteq E\}$ be a left ideal of the algebra $E/\ker p$. According to the discussion, preceding the statement of the lemma, we suppose, without loss of generality, that $\ker p \subseteq K$. Take the left ideal I of E, generated by K. Then $\ker p \subseteq K \subseteq I$ and thus

$$J = \{x + \ker p : x \in K \subseteq E\} \subseteq I/\ker p.$$

Take $(\sum_{i\in A} \lambda_i x_i + \sum_{j\in B} a_j x_j) + \ker p \in I/\ker p$, where A, B are finite sets of indices, $\{x_i\}_{i\in A}, \{x_j\}_{j\in B}$ families of elements in K, $\{a_j\}_{j\in B}$ a family of elements in E and $\{\lambda_i\}_{i\in A}$, a family of elements in $\mathbb{C}$.

Since J is a left ideal, we obviously have

$$(\sum_{i\in A} \lambda_i x_i + \sum_{j\in B} a_j x_j) + \ker p \in J,$$

that yields $I/\ker p \subseteq J$ and thus, $J = I/\ker p$. Therefore every left ideal of the algebra $E/\ker p$ has the form $I/\ker p$ with I a left ideal of E that contains $\ker p$. For the uniqueness of this form see also the scholium that follows. □

Scholium 11. By Lemma 10 and [7, p. 316, Proposition B.5.4], *there is an* $1-1$ *and onto correspondence between the ideals I of E, as in Lemma 10, which contain* $\ker p$*, and the ideals in $E/\ker p$, with respect to the mutually inverse correspondences $I \mapsto \rho(I)$ and $J \mapsto \rho^{-1}(J)$, where ρ is the canonical epimorphism $\rho : E \to E/\ker p$ with $\rho(x) = x + \ker p$. The last correspondences preserve the relations of "containing" (proper or not).*

Concerning the next lemma, we note that (3.2) is obvious, if p is a norm.

Lemma 12. *Let (E, p) be an m-seminormed algebra. Then the following hold.*

$$\text{If } I \text{ is a left ideal of } E \text{ that contains } \ker p \text{ and } x + \ker p \in I/\ker p, \text{ then } x \in I. \tag{3.2}$$

$$\textit{If } I_1, I_2 \textit{ are left ideals of } E \textit{ that contain } \ker p \\ \textit{and } I_1/\ker p \subseteq I_2/\ker p, \textit{ then } I_1 \subseteq I_2. \tag{3.3}$$

Moreover (3.2) *and* (3.3) *are equivalent.*

Proof. Let I be a left ideal of E that contains $\ker p$. If $x + \ker p \in I/\ker p$, then there exists $y \in I$ with $x + \ker p = y + \ker p$. Thus $x - y \in \ker p$, and since $\ker p \subseteq I$, we finally get $x = (x - y) + y \in I$. Thus (3.2) holds true.

To prove (3.3), take $x \in I_1$, then $x + \ker p \in I_1/\ker p$. So, $x + \ker p \in I_2/\ker p$, and in view of (3.2), $x \in I_2$. Therefore $I_1 \subseteq I_2$.

The latter argumentation also assures that (3.2) implies (3.3). For the inverse implication, apply (3.3) for $I_1 = [x] + \ker p$, where $[x]$ stands for the left ideal of E, generated by x, and $I_2 = I$, where I is a left ideal of E with $\ker p \subseteq I$ and $x + \ker p \in I/\ker p$. □

Scholium 13. *If I, I_1, I_2 are left ideals as in Lemma 12 that contain $\ker p$, then the following are obvious.*

$$x + \ker p \in I/\ker p \Leftrightarrow x \in I, \;\; I_1/\ker p \subseteq I_2/\ker p \Leftrightarrow I_1 \subseteq I_2$$

$$\textit{and } I_1/\ker p = I_2/\ker p \Leftrightarrow I_1 = I_2.$$

Now, if K is a left ideal of an m-seminormed algebra (E, p), then $K + \ker p$ is a left ideal of E, as well. Notice that $(K + \ker p)/\ker p = \{x + \ker p, x \in K\}$. A similar relation holds when K is a right ideal of E.

The next result is useful by its own right.

Lemma 14. *Let (E, p) be an m-seminormed algebra and I a left ideal of E. Then the following hold.*

(i) If I contains $\ker p$ and the left ideal $I/\ker p$ of $E/\ker p$ is minimal, then I is minimal with respect to the property $\ker p \subsetneq I$.

(ii) If $I \cap \ker p = (0)$ and the left ideal $(I + \ker p)/\ker p$ of $E/\ker p$ is minimal, then I is minimal, as well.

Proof. (i) We first note that, the minimality of the ideal $I/\ker p$ implies that $\ker p \subsetneq I$. Let now J be a left ideal of E with $\ker p \subsetneq J \subseteq I$. Then

$$\ker p \neq J/\ker p \subseteq I/\ker p,$$

that yields $J/\ker p = I/\ker p$ and thus, in view of Scholium 13, $J = I$.

(ii) Consider a left ideal J of E with $(0) \neq J \subseteq I$. Then

$$\ker p \neq J + \ker p \subseteq I + \ker p,$$

whence

$$(J + \ker p)/\ker p \subseteq (I + \ker p)/\ker p.$$

Thus, by the minimality of $(I + \ker p)/\ker p$, we get

$$(J + \ker p)/\ker p = (I + \ker p)/\ker p.$$

Therefore, from (3.3) and Scholium 13, we get

$$J + \ker p = I + \ker p.$$

Take now $x \in I$ and $y \in \ker p$. Then there exist $x_1 \in J$ and $y_1 \in \ker p$ with $x + y = x_1 + y_1$, that yields $x - x_1 = y_1 - y \in I \cap \ker p$, and by hypothesis, $x = x_1 \in J$ and hence $J = I$. □

The next result concerns quasi-complementarity of certain quotient algebras. Relative to the normed algebra $E/\ker p_{\alpha_0}$, see also the Note before Lemma 10.

According to (3.4) below, certain closed ideals of the algebra E are projected to closed ideals of the quotient algebra, in question, and the sum of two certain closed subspaces is closed as well, in the initial topological algebra. For a realization of the first part of (3.4), see Proposition 21, below.

Theorem 15. *Let $(E, (p_\alpha)_{\alpha\in\Lambda}, \perp)$ be a Hausdorff left quasi-complemented locally m-convex algebra, $I \in \mathcal{L}_\ell(E)$ and $\alpha_0 \in \Lambda$. Suppose that the following conditions hold:*

$$\text{If } \ker p_{\alpha_0} \subseteq I, \text{ then the ideal } I/\ker p_{\alpha_0} \text{ of } E/\ker p_{\alpha_0} \text{ is closed, and the ideal } I^\perp + \ker p_{\alpha_0} \text{ of } E \text{ is closed.} \tag{3.4}$$

$$If \quad \ker p_{\alpha_0} \subseteq I, \quad then \quad (I^\perp + \ker p_{\alpha_0})^\perp + \ker p_{\alpha_0} = I. \tag{3.5}$$

Then the normed algebra $E/\ker p_{\alpha_0}$ is left quasi-complemented. In particular, if $(E, (p_\alpha)_{\alpha\in\Lambda}, \perp)$ is left complemented, then $E/\ker p_{a_0}$ is left complemented, as well.

Proof. Take a left ideal J of E with $\ker p_{\alpha_0} \subseteq J$. We claim that if the ideal $J/\ker p_{\alpha_0}$ is closed, then J is closed, as well. Indeed, take $x \in \overline{J}$. Then there exists a net $(x_\delta)_{\delta\in\Delta}$ in J with $x_\delta \to x$. Thus $p_\alpha(x_\delta - x) \to 0$ for every $\alpha \in \Lambda$. In particular, $p_{\alpha_0}(x_\delta - x) \to 0$ and thus

$$\|(x_\delta + \ker p_{\alpha_0}) - (x + \ker p_{\alpha_0})\|_{\alpha_0} \to 0,$$

which implies $x_\delta + \ker p_{\alpha_0} \underset{\|\cdot\|_{\alpha_0}}{\longrightarrow} x + \ker p_{\alpha_0}$. Thus, by the closeness of the ideal $J/\ker p_{\alpha_0}$, we get $x + \ker p_{\alpha_0} \in J/\ker p_{\alpha_0}$, and by Lemma 12 (3.2) (applied for (E, p_{α_0})), $x \in J$. Namely, J is closed. So, in view of (3.4), we have the next.

$$I \in \mathcal{L}_\ell(E) \Leftrightarrow I/\ker p_{\alpha_0} \in \mathcal{L}_\ell(E/\ker p_{\alpha_0}) \quad \text{with} \quad \ker p_{\alpha_0} \subseteq I.$$

The normed algebra $E/\ker p_{\alpha_0}$ becomes left quasi-complemented. Indeed, we define

$$\perp_{p_{\alpha_0}}: \mathcal{L}_\ell(E/\ker p_{\alpha_0}) \to \mathcal{L}_\ell(E/\ker p_{\alpha_0})$$

with

$$(I/\ker p_{\alpha_0})^{\perp_{p_{\alpha_0}}} := (I^\perp + \ker p_{\alpha_0})/\ker p_{a_0} \text{ and } \ker p_{\alpha_0} \subseteq I. \tag{3.6}$$

By Lemma 10 (see also its proof) and (3.4), the mapping $\perp_{p_{\alpha_0}}$ is well defined. Claim that the last mapping reverses the inclusion. So, take

$$I/\ker p_{\alpha_0}, J/\ker p_{\alpha_0} \in \mathcal{L}_\ell(E/\ker p_{\alpha_0})$$

with I, J closed left ideals in E that contain $\ker p_{\alpha_0}$ and such that $I/\ker p_{\alpha_0} \subseteq J/\ker p_{\alpha_0}$. Then (Lemma 12) $I \subseteq J$, so $J^\perp \subseteq I^\perp$ and hence $J^\perp + \ker p_{\alpha_0} \subseteq I^\perp + \ker p_{\alpha_0}$. Thus, by Scholium 13,

$$(J^\perp + \ker p_{a_0})/\ker p_{a_0} \subseteq (I^\perp + \ker p_{a_0})/\ker p_{a_0}$$

or

$$(J + \ker p_{\alpha_0})^{\perp_{p_{\alpha_0}}} \subseteq (I + \ker p_{\alpha_0})^{\perp_{p_{\alpha_0}}}.$$

By applying (3.5), we easily get that $\perp_{p_{\alpha_0}}$ satisfies (1.3).

Now, take $x + \ker p_{\alpha_0} \in (I/\ker p_{\alpha_0}) \cap (I/\ker p_{\alpha_0})^{\perp_{p_{\alpha_0}}}$. Then, by the definition of $\perp_{p_{\alpha_0}}$ and Lemma 12 (3.2), and since $\ker p_{\alpha_0} \subseteq I^\perp + \ker p_{\alpha_0}$ and $\ker p_{\alpha_0} \subseteq I$, we easily get

$$x \in I \cap (I^\perp + \ker p_{\alpha_0}) = \ker p_{\alpha_0}.$$

Thus $x + \ker p_{\alpha_0} = \ker p_{\alpha_0}$ and hence

$$(I/\ker p_{\alpha_0}) \cap (I/\ker p_{\alpha_0})^{\perp_{p_{\alpha_0}}} = \ker p_{\alpha_0}.$$

The previous argumentation assures the left quasi-complementarity of the algebra $(E/\ker p_{\alpha_0}, \perp_{p_{\alpha_0}})$.

Suppose now that $\perp$ is a left complementor on E. Take $I/\ker p_{\alpha_0} \in \mathcal{L}_\ell(E/\ker p_{\alpha_0})$ with $\ker p_{\alpha_0} \subseteq I$ and $x + \ker p_{\alpha_0} \in E/\ker p_{\alpha_0}$. Since $E = I \oplus I^\perp$, there are (unique) $y \in I, z \in I^\perp$ with $x = y + z$, so that

$$x + \ker p_{\alpha_0} = (y + z) + \ker p_{\alpha_0} \in (I/\ker p_{\alpha_0}) + (I^\perp + \ker p_{\alpha_0})/\ker p_{\alpha_0}.$$

Therefore $E/\ker p_{\alpha_0} = (I/\ker p_{\alpha_0}) + (I/\ker p_{\alpha_0})^{\perp_{p_{\alpha_0}}}$. Actually,

$$E/\ker p_{\alpha_0} = (I/\ker p_{\alpha_0}) \oplus (I/\ker p_{\alpha_0})^{\perp_{p_{\alpha_0}}},$$

and finally $\perp_{p_{\alpha_0}}$ is a complementor. □

Scholium 16. Concerning Theorem 15, we remark that, if $(E,(p_\alpha)_{\alpha\in\Lambda},\perp)$ is left quasi-complemented, then condition (3.5) is used only to prove that $\perp_{p_{\alpha_0}}$ satisfies (1.3) . In case $(E,(p_\alpha)_{\alpha\in\Lambda},\perp)$ is left complemented, then what we really need is a mild version of (3.5), as follows.

$$If \ \ \ker p_{\alpha_0} \subseteq I, \ \text{ then } \ (I^\perp + \ker p_{\alpha_0})^\perp + \ker p_{\alpha_0} \subseteq I. \tag{3.7}$$

In that case, we modify the proof of (1.3). Indeed, by (3.6), (3.7) and Scholium 13, we get

$$\begin{aligned} ((I/\ker p_{\alpha_0})^{\perp_{p_{\alpha_0}}})^{\perp_{p_{\alpha_0}}} &= ((I^\perp + \ker p_{\alpha_0})/\ker p_{\alpha_0})^{\perp_{p_{\alpha_0}}} \\ ((I^\perp + \ker p_{\alpha_0})^\perp &+ \ker p_{\alpha_0})/\ker p_{\alpha_0} \subseteq I/\ker p_{\alpha_0}. \end{aligned} \tag{3.8}$$

Now, by the proof of Theorem 15, we have

$$E/\ker p_{\alpha_0} = (I/\ker p_{\alpha_0}) \oplus (I/\ker p_{\alpha_0})^{\perp_{p_{\alpha_0}}},$$

for every $I/\ker p_{\alpha_0} \in \mathcal{L}_\ell(E/\ker p_{\alpha_0})$. In particular, we get

$$E/\ker p_{\alpha_0} = (I/\ker p_{\alpha_0})^{\perp_{p_{\alpha_0}}} \oplus ((I/\ker p_{\alpha_0})^{\perp_{p_{\alpha_0}}})^{\perp_{p_{\alpha_0}}}. \tag{3.9}$$

Moreover, by the proof of Theorem 15,

$$(I/\ker p_{\alpha_0}) \cap (I/\ker p_{\alpha_0})^{\perp_{p_{\alpha_0}}} = \ker p_{\alpha_0}.$$

By the last relation and (3.8), $I/\ker p_{\alpha_0}$ contains $((I/\ker p_{\alpha_0})^{\perp_{p_{\alpha_0}}})^{\perp_{p_{\alpha_0}}}$ and intersects $(I/\ker p_{\alpha_0})^{\perp_{p_{\alpha_0}}}$ to the zero element $\ker p_{\alpha_0}$ of $E/\ker p_{\alpha_0}$. Therefore, by applying [13, p. 154, Lemma 3.17], see also (3.9), we get

$$((I/\ker p_{\alpha_0})^{\perp_{p_{\alpha_0}}})^{\perp_{p_{\alpha_0}}} = I/\ker p_{\alpha_0}.$$

Namely, (1.3) holds true.

Example 17. *(i) Every Hausdorff annihilator locally C^*-algebra $(E,(p_\alpha)_{\alpha\in\Lambda})$ satisfies (3.7) for any $\alpha\in\Lambda$.*

According to [12, p. 226, Theorem 3.1], a Hausdorff locally C^*-algebra E is an annihilator algebra if and only if it is a dual algebra, if and only if it is complemented with a left (resp. right) complementor $\perp = * \circ \mathcal{A}_r$ (resp. $\perp = * \circ \mathcal{A}_l$), where $\mathcal{A}_r$ (resp. $\mathcal{A}_\ell$) is the right (left) annihilator of E.

By an *annihilator algebra* we mean a topological algebra E, which is preannihilator and satisfies the conditions: If $\mathcal{A}_\ell(I) = (0)$ with $I \in \mathcal{L}_r$, then $I = E$ and if $\mathcal{A}_r(J) = (0)$ with $J \in \mathcal{L}_\ell$, then $J = E$. A topological algebra E is named a *dual algebra*, if $\mathcal{A}_\ell(\mathcal{A}_r(I)) = I$ for every $I \in \mathcal{L}_\ell$, and $\mathcal{A}_r(\mathcal{A}_\ell(J)) = J$ for every $J \in \mathcal{L}_r$. (see [12, p. 220]).

To prove (3.7), for $I \in \mathcal{L}_\ell$ with $\ker p \subseteq I$, we use the obvious equality $\mathcal{A}_r(I)^* = \mathcal{A}_\ell(I^*)$, and by the definition of a dual algebra, we employ the equality $\mathcal{A}_\ell(\mathcal{A}_r(I)) = I$. Also, to simplify the notation, we use p in place of p_α. So, using the definition of $\perp$, as above, we have

$$(I^\perp + \ker p)^\perp + \ker p = (\mathcal{A}_\ell(I^*) + \ker p)^\perp + \ker p = \mathcal{A}_\ell((\mathcal{A}_\ell(I^*) + \ker p)^*) + \ker p$$

$$= \mathcal{A}_\ell(\mathcal{A}_\ell(I^*)^* + (\ker p)^*) + \ker p = \mathcal{A}_\ell(\mathcal{A}_r(I) + (\ker p)^*) + \ker p.$$

It is easily seen that, for any linear subspaces S, F in E, it holds $\mathcal{A}_\ell(S + F) = \mathcal{A}_\ell(S) \cap \mathcal{A}_\ell(F)$. So, we get

$$(I^\perp + \ker p)^\perp + \ker p = (\mathcal{A}_\ell(\mathcal{A}_r(I)) \cap \mathcal{A}_\ell((\ker p)^*)) + \ker p$$

$$= (I \cap \mathcal{A}_\ell((\ker p)^*)) + \ker p \subseteq I + I \subseteq I.$$

Thus,

$$(I^\perp + \ker p)^\perp + \ker p \subseteq I.$$

(ii) *If in particular, E is commutative and its spectral radius satisfies $r_E(x) \leq p_{\alpha_0}(x)$ for every $x \in E$ and some $\alpha_0 \in \Lambda$, then E has a minimal closed ideal $R = eE(= Ee)$.* Indeed, by [8, p. 87, Corollary 6.22] E is a Q-algebra, and by [ibid. p. 117, Theorem 9.3 (2)], E is semisimple. Thus the assertion follows from the fact that E is a semisimple annihilator Q-algebra (see [10, p. 154, Theorem 3.8]).

A terminology comment and remarks.- Let $(E, (p_\alpha)_{\alpha\in\Lambda}, \perp)$ be a left quasi-complemented locally m-convex algebra. Then condition (3.5), which is related with a certain seminorm on E, leads, under certain conditions, to the validity of (1.3), namely to the reflexivity of a complementor, which is well defined by (3.4) (see e.g. the proof of Theorem 15). Thus, in what follows, by a *reflexive seminorm*, in a topological algebra as before, we mean a seminorm p_{α_0} that satisfies (3.4), and (3.5).

(i) If p_{α_0} is a norm, then (3.4), and (3.5) are trivially true. Thus p_{α_0} is reflexive. In particular, *in every quasi-complemented normed algebra $(E, \|\cdot\|, \perp)$, its norm is reflexive.* Thus, *a left quasi-complemented locally m-convex algebra, such that in the family $(p_\alpha)_{\alpha\in\Lambda}$, which defines its locally m-convex topology, there is an index $\alpha_0 \in \Lambda$, such that p_{α_0} is a norm, satisfies the two aforementioned conditions.* This means that the results that follow take a more neat form, when we suppose that, in the family $(p_\alpha)_{\alpha\in\Lambda}$ there is an $\alpha_0 \in \Lambda$ such that p_{α_0} is a norm.

(ii) In [21], the author uses the notion of a K-Fréchet-Stein algebra in which the Fréchet-topology is defined by a family of norms (see e.g., [ibid., Theorem (Schneider-Teitelbaum)]. This constitutes an important application of (locally m-convex algebras) whose topology is defined by a family of norms, in the study of distributions. On the other hand, a locally convex topology defined by a family of norms is close related with spectral measure triples in the sense of [27, p. 299, Definition 1.4 and p. 304]. We also refer to an example of a family $(p_n)_n$ of norms that defines the metric locally convex topology of the quotient field $c(t)$ of the polynomial algebra $c[t]$ of polynomials in t with complex coefficients (see [5, p. 214, Example 4.9-1 and p. 277, 4.6]). Also, for an example of a complete metrizable locally m-convex algebra, whose topology is defined by a family of norms, see e.g., [8, p. 18, 2.4 (1)].

(iii) The second part of (3.4), concerns the problem of closedness of the sum of two closed subspaces of a topological space. This issue is of a great interest and several authors have been dealt with it. For instance, Luxemburg gave conditions under which this is true. So, in the framework of a Banach space E, he proved that the algebraic sum of two closed subspaces X, Y is (norm) closed if and only if the mapping $(x, y) \to x + y : X \times Y \to E$ is open (see [23, p. 239, Theorem 2.5]). For a necessary and sufficient condition for the sum of two closed subspaces to be closed see also [6, p. 397, Theorem 4] and [28, p. 62, Theorem 1.1].

Concerning the reference [2, p. 394, Theorem 4.4], in the proof of the next result, we note that, Theorem 4.4 *holds true by interchanging "right" by "left"*. See also [ibid. p. 386].

Theorem 18. (Representation) *Let $(E, (p_\alpha)_{\alpha\in\Lambda}, \perp)$ be a Hausdorff, left complemented, complete locally m-convex algebra. Suppose there exists $\alpha_0 \in \Lambda$, such that the seminorm p_{α_0} is reflexive and the normed algebra $E/\ker p_{\alpha_0}$ is semisimple, complete with no minimal right ideals of dimension less than 3, and the complementor $\perp_{p_{\alpha_0}}$ as in (3.6) is continuous. Then E has a continuous representation T on a Hilbert space $(H, \langle\cdot,\cdot\rangle)$. Moreover, for every closed left ideal I of E that contains $\ker p_{\alpha_0}$ the following hold.*

$$(T_I H)^{\perp_{\langle\cdot,\cdot\rangle}} = \overline{T_{I^\perp} H} \tag{3.10}$$

and

$$(I^\perp + \ker p_{\alpha_0})/\ker p_{\alpha_0} = \left\{x \in E : T_x H \perp_{\langle\cdot,\cdot\rangle} T_I H\right\} / \ker p_{\alpha_0}. \tag{3.11}$$

Proof. By hypothesis, the complemented algebra $E/\ker p_{\alpha_0}$ (see Theorem 15) is Banach semisimple and has no minimal right ideals of dimension less than 3. Thus, by [2, p. 394, Theorem 4.4, see also its proof], $E/\ker p_{\alpha_0}$ has a faithful continuous representation S on a Hilbert space $(H, \langle\cdot,\cdot\rangle)$, namely $x + \ker p_{\alpha_0} \to S_{x+\ker p_{\alpha_0}}$. Take the Arens-Michael decomposition of E (see (3.1)). By [24, p. 88, Theorem 3.1], the algebra-topological isomorphism, say Φ, between E and $\varprojlim E/\ker p_\alpha$ is defined by the relation $\Phi(x) = (x + \ker p_\alpha)_{\alpha\in\Lambda}$. Consider the mapping

$$f_{\alpha_0} : \varprojlim E/\ker p_\alpha \to E/\ker p_{\alpha_0}$$

with

$$f_{\alpha_0}(([x]_\alpha)_{\alpha\in\Lambda}) = f_{\alpha_0}((x + \ker p_\alpha)_{\alpha\in\Lambda}) = x + \ker p_{\alpha_0},$$

which obviously is a well defined continuous morphism. Moreover, $f_{\alpha_0} \circ \Phi = \rho_{\alpha_0}$, where $\rho_{\alpha_0} : E \to E/\ker p_{\alpha_0}$ with $\rho_{\alpha_0}(x) = x + \ker p_{\alpha_0}$ (the canonical epimorphism). Since ρ_{α_0} is onto, f_{α_0} is onto, as well (on $\Phi(E) = \varprojlim E/\ker p_\alpha$). Therefore we get that $T = S \circ f_{\alpha_0} \circ \Phi$ defines a continuous representation of E on the aforementioned Hilbert space H.

Let now I be a closed left ideal of E that contains $\ker p_{\alpha_0}$. Then, since p_{α_0} is reflexive, by (3.4), $I/\ker p_{\alpha_0}$ is a closed left ideal of $E/\ker p_{\alpha_0}$, and thus by [2, p. 394, Theorem 4.4], the following relations hold true.

$$(S_{I/\ker p_{\alpha_0}} H)^{\perp_{\langle\cdot,\cdot\rangle}} = \overline{S_{(I/\ker p_{\alpha_0})^{\perp_{p_{\alpha_0}}}} H} \text{ and } (I/\ker p_{\alpha_0})^{\perp_{p_{\alpha_0}}} \tag{3.12}$$
$$= \left\{ x + \ker p_{\alpha_0} \in E/\ker p_{\alpha_0} : S_{x+\ker p_{\alpha_0}} H \perp_{\langle\cdot,\cdot\rangle} S_{I/\ker p_{\alpha_0}} H \right\}.$$

Since $(f_{\alpha_0} \circ \Phi)(I) = I/\ker p_{\alpha_0}$ and $(f_{\alpha_0} \circ \Phi)^{-1}(I/\ker p_{\alpha_0}) = I$, we get (using also (3.12)) that

$$\begin{aligned}(T_I H)^{\perp_{\langle\cdot,\cdot\rangle}} = ((S \circ f_{\alpha_0} \circ \Phi)_I H)^{\perp_{\langle\cdot,\cdot\rangle}} = (S_{(f_{\alpha_0}\circ\Phi)(I)} H)^{\perp_{\langle\cdot,\cdot\rangle}} = (S_{I/\ker p_{\alpha_0}} H)^{\perp_{\langle\cdot,\cdot\rangle}} \\ = \overline{S_{(I/\ker p_{\alpha_0})^{\perp_{p_{\alpha_0}}}} H} = \overline{S_{(I^\perp + \ker p_{\alpha_0})/\ker p_{\alpha_0}} H} \\ = \overline{(S \circ f_{\alpha_0})_{(I^\perp + \ker p_{\alpha_0})} H} = \overline{(S \circ f_{\alpha_0} \circ \Phi)_{I^\perp} H} = \overline{T_{I^\perp} H}.\end{aligned}$$

Namely, (3.10) holds. Moreover, based on (3.12) and the definition of $\perp_{p_{\alpha_0}}$, we get

$$\begin{aligned}(I/\ker p_{\alpha_0})^{\perp_{p_{\alpha_0}}} = (I^\perp + \ker p_{\alpha_0})/\ker p_{a_0} \\ = \left\{ x + \ker p_{\alpha_0} \in E/\ker p_{\alpha_0} : S_{x+\ker p_{\alpha_0}} H \perp_{\langle\cdot,\cdot\rangle} S_{I/\ker p_{\alpha_0}} H \right\} \\ = \left\{ x + \ker p_{\alpha_0} \in E/\ker p_{\alpha_0} : (S \circ f_{\alpha_0} \circ \Phi)_x H \perp_{\langle\cdot,\cdot\rangle} (S \circ f_{\alpha_0} \circ \Phi)_I H \right\} \\ = \left\{ x + \ker p_{\alpha_0} \in E/\ker p_{\alpha_0} : T_x H \perp_{\langle\cdot,\cdot\rangle} T_I H \right\} \\ = \left\{ x \in E : T_x H \perp_{\langle\cdot,\cdot\rangle} T_I H \right\} / \ker p_{a_0}.\end{aligned}$$

Namely, (3.11) holds, and this completes the proof. □

Remarks 19. 1) *In the previous result, the hypothesis "the complete algebra $E/\ker p_{\alpha_0}$ has no minimal right ideals of dimension less than 3" can not be removed.* An example is given in [2, p. 396, (2)] in which there is defined a continuous complementor on a two-dimensional Hilbert space H, which is not the orthogonal mapping (with respect to none inner product on H).

2) *When the family $(p_\alpha)_{\alpha\in\Lambda}$ consists only of one seminorm p_α, such that the topological algebra (E, p_α) is Hausdorff, then the condition (3.4) holds automatically,* since p_α is actually a norm (see [24, p. 13]). *Moreover, in that case, the results of the respective theorems lead to the classical context of Banach algebras.* See also the remarks before Theorem 18.

The next result refers to an expecting relation between appropriate linear projections on the complemented algebras $(E,(p_\alpha)_{\alpha\in\Lambda})$ and $E/\ker p_{\alpha_0}$.

Lemma 20. *Consider the framework of* Theorem 15 *in the case where the algebras are left complemented. Let I be a closed left ideal of E with* $\ker p_{\alpha_0}\subseteq I$. *Consider the linear projections $P=P(I,I^\perp)$, $Q=Q\left(I/\ker p_{\alpha_0},(I/\ker p_{\alpha_0})^{\perp_{\alpha_0}}\right)$ on the left complemented algebras $(E,(p_\alpha)_{\alpha\in\Lambda},\perp)$ and $\left(E/\ker p_{\alpha_0},\perp_{p_{\alpha_0}}\right)$, respectively with*

$\mathrm{Im}P=I$, $\ker P=I^\perp$ *and* $\mathrm{Im}\,Q=I/\ker p_{\alpha_0}$, $\ker Q=(I/\ker p_{\alpha_0})^{\perp_{\alpha_0}}$.

Then

$$Q(x+\ker p_{\alpha_0})=P(x)+\ker p_{\alpha_0}\ \text{ for every } x\in E.$$

Proof. If $x\in E$, there exist unique $y\in I, z\in I^\perp$ with $x=y+z$. Obviously, $P(x)=y$. Furthermore, we have

$$x+\ker p_{\alpha_0}=(y+\ker p_{\alpha_0})+(z+\ker p_{\alpha_0})$$

with $y+\ker p_{\alpha_0}\in I+\ker p_{\alpha_0}$. Therefore, by the uniqueness of the projections P and Q (see Remark 2), we get

$$Q(x+\ker p_{\alpha_0})=y+\ker p_{\alpha_0}=P(x)+\ker p_{\alpha_0}.$$

□

The next result concerns a realization of the first part of (3.4).

Proposition 21. *Let $(E,(p_\alpha)_{\alpha\in\Lambda})$ be a Hausdorff complete locally m-convex algebra. Suppose that for $\alpha_0\in\Lambda$, the following condition holds.*

$$\begin{aligned}&\text{If } (x_\delta+\ker p_{\alpha_0})_{\delta\in\Delta} \text{ is a Cauchy net in } E/\ker p_{\alpha_0},\\ &\qquad\text{then the net } (x_\delta)_{\delta\in\Delta} \text{ is a Cauchy net in } E.\end{aligned}\tag{3.13}$$

Then, the first part of (3.4) *holds, as well.*

Proof. Take $I\in\mathcal{L}_\ell(E)$ that contains $\ker p_{\alpha_0}$. We show that $\overline{I/\ker p_{\alpha_0}}\subseteq I/\ker p_{\alpha_0}$. So, take $z\in\overline{I/\ker p_{\alpha_0}}$. Then there exists a net $(x_\delta+\ker p_{\alpha_0})_{\delta\in\Delta}$ in $I/\ker p_{\alpha_0}$ with $z=\lim\limits_\delta(x_\delta+\ker p_{\alpha_0})$. The net $(x_\delta+\ker p_{\alpha_0})_{\delta\in\Delta}$, as a convergent one is Cauchy in $E/\ker p_{\alpha_0}$, and by (3.13), the net $(x_\delta)_{\delta\in\Delta}$ is Cauchy in E. Thus, by the completeness of E, there exists $x_0\in E$ with $x_0=\lim\limits_\delta x_\delta$. Since $x_\delta\in I$ for all $\delta\in\Delta$ (see (3.2)), and since I is closed, we get $x_0\in I$. Therefore, for every $\alpha\in\Lambda$, $p_\alpha(x_\delta-x_0)\underset{\delta}{\to}0$ and hence $p_{\alpha_0}(x_\delta-x_0)\underset{\delta}{\to}0$. Namely, $\|(x_\delta+\ker p_{\alpha_0})-(x_0+\ker p_{\alpha_0})\|_{\alpha_0}\to 0$. Therefore $z=\lim\limits_\delta(x_\delta+\ker p_{\alpha_0})=x_0+\ker p_{\alpha_0}\in I/\ker p_{\alpha_0}$, which completes the assertion. □

Remarks 22. *(i)* (3.13) *is trivially true for any Hausdorff complete m-seminormed algebra (E, p).*

(ii) A *strong spectrally bounded algebra* (the term is due to A. Mallios) is a locally m-convex algebra $(E, (p_\alpha)_{\alpha\in\Lambda})$ such that $\sup_{\alpha\in\Lambda} p_\alpha(x) < \infty$ for all $x \in E$ (see [8, p. 252]). In this context, the following modification of (3.13) holds true.

$$\begin{aligned} &\textit{If } (x_\delta + \ker \|\cdot\|_b)_{\delta\in\Delta} \textit{ is a Cauchy net in } E/\ker\|\cdot\|_b, \\ &\textit{then the net } (x_\delta)_{\delta\in\Delta} \textit{ is a Cauchy net in } (E, (p_\alpha)_{\alpha\in\Lambda}). \end{aligned} \tag{3.14}$$

Here $\|\cdot\|_b$ denotes the submultiplicative norm on E, defined by $\|x\|_b = \sup_\alpha p_\alpha(x)$ for all $x \in E$. We note that, if a net in E is Cauchy with respect to the topology on E, defined by $\|\cdot\|_b$, then it is Cauchy with respect to the topology, defined by $(p_\alpha)_{\alpha\in\Lambda}$. This assures the consistency of (3.14) in connection with the two topologies defined on E. If moreover, E is complete, then the first part of (3.4) holds, too (see Proposition 21).

According to [12, p. 226, Theorem 3.1], a Hausdorff annihilator locally C^*-algebra E is left complemented with a complementor $\perp = * \circ \mathcal{A}_r$ (where $\mathcal{A}_r$ is the right annihilator of E). We also know that every locally C^*-algebra is embedded, as a closed *-subalgebra, in some $L(H)$, where H is a locally Hilbert space (see [18, p. 231, Definition 5.2 and p. 232, Theorem 5.1] and [8, p. 114, Theorem 8.5]). Furthermore, if T is a representation of a locally C^*-algebra $(E, (p_\alpha)_{\alpha\in\Lambda})$ on a Hilbert space H, then there is $\alpha \in \Lambda$ and a representation T_α of $E_\alpha = E/\ker p_\alpha$ (the latter is complete, see [4, p. 32, Theorem 2.4]), with $T = T_\alpha \circ f_\alpha \circ \Phi$ where Φ denotes the algebra-topological isomorphism in the Arens-Michael decomposition of E and $f_\alpha : \varprojlim E_\alpha \to E_\alpha$ with $f_\alpha(x) = x + \ker p_\alpha$, $x \in \varprojlim E_\alpha$. Actually, T_α is a representation of E_α induced by T (see [19, p. 1924, before Definition 2.1]). The next result concerns a kind of inverse, in view of which, there is a representation of a certain locally C^*-algebra from a respective one of an appropriate factor in its Arens-Michael decomposition.

Corollary 23. *Let $(E, (p_\alpha)_{\alpha\in\Lambda})$ be a Hausdorff annihilator locally C^*-algebra. Suppose there exists $\alpha_0 \in \Lambda$, such that the seminorm p_{α_0} is reflexive and the normed algebra $E/\ker p_{\alpha_0}$ is semisimple with no minimal right ideals of dimension less than 3, and the complementor $\perp_{p_{\alpha_0}}$, as in* (3.6)*, is continuous. Then E has a continuous representation $x \to T_x$ on a Hilbert space $(H, \langle\cdot,\cdot\rangle)$ such that, for every closed left ideal I of E that contains $\ker p_{\alpha_0}$,* (3.10) *and* (3.11) *hold true.*

Proof. It is immediate by the discussion preceding the statement, and Theorem 18. □

The following result is useful in the sequel and it generalizes a respective one in [1, p. 39, Lemma 3] stated for Banach algebras. See also [12, Theorem 2.4 and the comment that follows]. Also, we note that, if $x \in E$ is idempotent, then Ex is closed and $x \in Ex$.

Lemma 24. *Let E be a right preannihilator left precomplemented algebra. Then $x \in \overline{Ex}$ for all $x \in E$.*

Proof. For $x \in E$, $\overline{Ex}$ is a closed left ideal, so there is $I \in \mathcal{L}_\ell(E)$ with $E = \overline{Ex} \oplus I$. Therefore there are $y \in \overline{Ex}$ and $z \in I$ with $x = y + z$. So, for every $w \in E$ $wz = wx - wy$. Obviously, $wx - wy \in \overline{Ex}$, and $wz \in \overline{Ex} \cap I = (0)$, whence $wz = 0$. Since E is right preannihilator, $z = 0$, and $x = y \in \overline{Ex}$. □

The next three results generalize a respective one in [17, p. 146, Theorem 4.3] from right quasi-complemented Banach algebras to Hausdorff, left quasi-complemented, complete locally m-convex algebras.

Theorem 25. (Representation) *Let $(E, (p_\alpha)_{\alpha\in\Lambda}, \perp)$ be a Hausdorff, left quasi-complemented, complete locally m-convex algebra. Suppose there exists $\alpha_0 \in \Lambda$, such that the seminorm p_{α_0} is reflexive and the normed algebra $E/\ker p_{\alpha_0}$ is primitive, complete and satisfies the condition*

$$z \in \overline{(E/\ker p_{\alpha_0})z} \quad \textit{for every} \quad z \in E/\ker p_{\alpha_0}. \tag{3.15}$$

Then there is a continuous representation of E on a Hilbert space $(H, \langle\cdot,\cdot\rangle)$ such that every image of it is a compact operator.

Proof. By hypothesis and Theorem 15, the normed algebra $E/\ker p_{\alpha_0}$ is left quasi-complemented with quasi-complementor $\perp_{p_{\alpha_0}}$ as in (3.6). Let R be a minimal closed right ideal of the normed algebra $E/\ker p_{\alpha_0}$. By hypothesis, the normed algebra is primitive Banach and satisfies (3.15). Then, by [17, p. 145, Lemma 4.2 (i), and the discussion preceding its statement], in R there is defined an inner product, such that R turns to be a Hilbert space. Take the canonical right representation $w \to S_w$ of $E/\ker p_{\alpha_0}$ on the Hilbert space R. Namely, if $w \in E/\ker p_{\alpha_0}$, then

$$S_w(x + \ker p_{\alpha_0}) = (x + \ker p_{\alpha_0})w \quad \text{for every} \quad x + \ker p_{\alpha_0} \in R.$$

S is a continuous isomorphism of $E/\ker p_{\alpha_0}$ on the algebra

$$F = \{S_w : w \in E/\ker p_{\alpha_0}\},$$

which consists of compact operators on the Hilbert space R [ibid., p. 146, Theorem 4.3, notice that the hypothesis "*every maximal closed right ideal is modular*" is not used here]. Applying now a proof analogous to that of Theorem 18 and since the set of the images of the representation of the algebra coincides with the set of the images of the representation S (with respect to f_{α_0}; see the proof of Theorem 18), we get the assertion. □

Condition (3.15) is, for instance, fulfilled in the context of Lemma 24. Thus, we get the next refinement version of Theorem 25.

Corollary 26. (Representation) *Let $(E, (p_\alpha)_{\alpha\in\Lambda}, \perp)$ be a Hausdorff, left quasi-complemented, complete locally m-convex algebra. Suppose there exists $\alpha_0 \in \Lambda$, such that the seminorm p_{α_0} is reflexive and the normed algebra $E/\ker p_{\alpha_0}$ is primitive, complete and left precomplemented. Then there is a continuous representation of E on a Hilbert space $(H, \langle\cdot,\cdot\rangle)$ such that every image of it is a compact operator.*

Proof. By hypothesis, the (topological) algebra $E/\ker p_{\alpha_0}$ is primitive. So, (0) is a primitive ideal. As it is known, the Jacobson radical is the intersection of all primitive ideals. This yields that the aforementioned algebra is semisimple, so it is topologically semiprime and thus preannihilator (see [10, p. 150, Lemma 2.3]). So, being also left precomplemented, it satisfies (3.15) (see Lemma 24). Theorem 25 completes the assertion. □

As a direct consequence, of the last result, we get.

Corollary 27. (Representation) *Let $(E, (p_\alpha)_{\alpha\in\Lambda}, \perp)$ be a Hausdorff, left quasi-complemented, complete locally m-convex algebra. Suppose there exists $\alpha_0 \in \Lambda$, such that the seminorm p_{α_0} is reflexive and the normed algebra $E/\ker p_{\alpha_0}$ is primitive, complete and the quasi-complementor $\perp_{p_{\alpha_0}}$, as in (3.6), is a complementor. Then there is a continuous representation of E, as in* Corollary 26.

According to Theorem 15, the normed algebra $E/\ker p_{\alpha_0}$ is left complemented. Thus, we also get the next.

Corollary 28. (Representation) *Let $(E, (p_\alpha)_{\alpha\in\Lambda}, \perp)$ be a Hausdorff, left complemented, complete locally m-convex algebra. Suppose there exists $\alpha_0 \in \Lambda$, such that the seminorm p_{α_0} is reflexive and the normed algebra $E/\ker p_{\alpha_0}$ is primitive and complete. Then there is a continuous representation of E, as in* Corollary 26.

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