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APPROXIMATE θ-MULTIPLIERS ON BANACH ALGEBRAS

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Abstract. We show that each approximate θ-multiplier $T : A \longrightarrow A$ on without order Banach algebra A is an exact θ-multiplier, and for every approximate θ-multiplier T there corresponds a unique θ-multiplier near to T. Moreover, each mapping $S : A \longrightarrow A$ which is near to θ-multiplier T is approximate θ-multiplier. Some useful results about θ-multiplier of Banach algebras are given as well.

1 Introduction

Let A be a Banach algebra. A mapping $T : A \longrightarrow A$ is called *left multiplier*(*centralizer*), [*right multiplier*] if for all $a, b \in A$,

$$T(ab) = T(a)b, \qquad [T(ab) = aT(b)],$$

and T is called *multiplier*(*centralizer*), if it is both left and right multiplier.

J. G. Wendel in [17] considered centralizer mapping for group algebra, and S. Helgason [5] used multiplier instead of centralizer. The general theory of (centralizers) multipliers on Banach algebras has been developed by Johnson [7]. He proved that each multiplier on without order Banach algebra A is linear and continuous. One may refer to the monograph [10], for the theory of multipliers.

Let A be a algebra and let θ be an algebra endomorphism of A. A mapping $T : A \longrightarrow A$ is called *left θ-multiplier*[*right θ-multiplier*], if for all $a, b \in A$,

$$T(ab) = T(a)\theta(b), \qquad [T(ab) = \theta(a)T(b)].$$

If T is left and right θ-multiplier, then it is natural to call T a θ-multiplier. This concept introduced by E. Albas [1], and some interesting generalization of multiplier to the case of θ-multiplier on semiprime rings obtained. Clearly, every multiplier is a special case of a θ-multiplier with $\theta = id$, the identity map on A.

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There is a weaker definition of θ-multiplier. Indeed, a mapping $T : A \longrightarrow A$ is said to be θ-multiplier if for every $a, b \in A$,

$$\theta(a)T(b) = T(a)\theta(b). \tag{1.1}$$

Obviously, if T is a left and right θ-multiplier, then T is a θ-multiplier as in (1.1), but in general, the converse is fails. See [18, Example 1.1] for the case $\theta = id$. Next we give another example for which θ is not identity map.

Example 1. Let

$$A = \left\{ \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} : \quad a, b, c \in \mathbb{R} \right\},$$

and define $\theta, T : A \longrightarrow A$ by

$$\theta\left(\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad T\left(\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & c & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{bmatrix}.$$

Then, $\theta(uv) = \theta(u)\theta(v)$ for all $u, v \in A$, and hence θ is a endomorphism. Moreover,

$$\theta(u)T(v) = T(u)\theta(v) = 0,$$

for all $u, v \in A$. Thus, T is a θ-multiplier as in (1.1), but it is not left (right) θ-multiplier, because $T(uv) \neq 0$, in general.

We recall that the Banach algebra A is called *without order*, if for all $x \in A$, $xA = \{0\}$ $[Ax = \{0\}]$ implies $x = 0$. If θ is epimorphism, then for without order Banach algebra A, every θ-multiplier is both left and right θ-multiplier. Indeed, let $\theta(a)T(b) = T(a)\theta(b)$, for every $a, b \in A$. Then for all $x \in A$,

$$\theta(x)\big(T(ab)\big) = T(x)\theta(ab) = T(x)\theta(a)\theta(b) = \theta(x)T(a)\theta(b).$$

Consequently, $\theta(x)\big(T(ab) - T(a)\theta(b)\big) = 0$. Since θ is surjective and A is without order, we deduce $T(ab) = T(a)\theta(b)$, for all $a, b \in A$. So T is a left θ-multiplier. Similarly, T is a right θ-multiplier.

A classical question in the theory of functional equations is that "when is it true that a mapping which approximately satisfies a functional equation Δ must be somehow close to an exact solution of Δ?" Such a problem was formulated by Ulam [16] in 1940 and solved in the next year for the Cauchy functional equation by Hyers [4]. It gave rise to the stability theory for functional equations.

Rassias [13] considered a generalized version of the Hyers's result as follows.

Theorem 2. *Let X and Y be two real Banach spaces, $\varepsilon \geqslant 0$ and $0 \leqslant p < 1$. If a mapping $f : X \longrightarrow Y$ satisfies*

$$\|f(x+y) - f(x) - f(y)\| \leqslant \varepsilon(\|x\|^p + \|y\|^p),$$

for all $x, y \in X$, then there is a unique additive mapping $F : X \longrightarrow Y$ such that

$$\|F(x) - f(x)\| \leqslant \frac{2\varepsilon}{|2 - 2^p|}\|x\|^p, \qquad (x \in X).$$

Indeed, Hyers [4] obtained the result for $p = 0$, and then Rassias [13] generalized the above result of Hyers to the case where $0 \leqslant p < 1$. In [3], Gajda proved that Theorem 2 is valid for $p > 1$. He also gave an example showing that a similar result to the above does not hold for $p = 1$. Rassias and Semrl [14] independently obtained a different example. If $p < 0$, then $\|x\|^p$ is meaningless for $x = 0$, in this case if we assume that $\|x\|^p = \infty$, then the proof given in [13] also works for $x \neq 0$. Moreover, with minor changes in the proof, the result is also valid for $p < 0$. Thus, the Hyers-Ulam-Rassias stability holds for all $p \in \mathbb{R}$, with $p \neq 1$.

A linear map ϕ between Banach algebras A and B is called *almost homomorphism* if there exists $\varepsilon \geqslant 0$ such that for all $a, b \in A$,

$$\|\phi(ab) - \phi(a)\phi(b)\| \leqslant \varepsilon\|a\|\|b\|.$$

Jarosz in [6], introduced the concept of an almost homomorphism between Banach algebras. Some results on such maps obtained in [9] and [15].

The Hyers-Ulam-Rassias stability of multipliers on Banach algebras was proved by Miura et al. in [11]. In this paper, by applying a theorem of Rassias [13] and Gajda [3], we prove the Hyers-Ulam-Rassias stability of θ-multipliers, which generalizes the results of [11].

We also show that each mapping $S : A \longrightarrow A$ which is near to approximate θ-multiplier T is an exact approximate θ-multiplier. Some results about connection of approximate θ-multiplier and almost homomorphism are given as well.

2 Approximate θ-multipliers

We commence with the concept of (*approximate*) *θ-multiplier* of Banach algebras.

Definition 3. *Let A be a Banach algebra, $\theta : A \longrightarrow A$ be an algebra endomorphism. A map $T : A \longrightarrow A$ is called left θ-multiplier [right θ-multiplier], if for all $a, b \in A$,*

$$T(ab) = T(a)\theta(b), \quad [T(ab) = \theta(a)T(b)],$$

and T is called θ-multiplier, if $\theta(a)T(b) = T(a)\theta(b)$, for every $a, b \in A$.

Moreover, $T : A \longrightarrow A$ is called approximate left θ-multiplier, if there exists $\varepsilon \geqslant 0$ such that for every $a, b \in A$,

$$\|T(ab) - T(a)\theta(b)\| \leqslant \varepsilon\|a\|\|b\|.$$

Approximate (right) θ-multiplier can be defined analogously. Note that each θ-multiplier is approximate θ-multiplier for all $\varepsilon \geqslant 0$, and every approximate θ-multiplier turns out to be θ-multiplier, whenever $\varepsilon = 0$.

From this point up to the last section θ is a epimorphism of A.

Theorem 4. *Suppose that T is a approximate θ-multiplier of A. If θ is continuous and $S : A \longrightarrow A$ is a mapping such that*

$$\|S(a) - T(a)\| \leqslant \varepsilon\|a\|, \tag{2.1}$$

for every $a \in A$, then S is a approximate θ-multiplier.

Proof. By assumption

$$\|\theta(a)T(b) - T(a)\theta(b)\| \leqslant \varepsilon_1\|a\|\|b\|,$$

for some $\varepsilon_1 \geqslant 0$. Hence for every $a, b \in A$, we have

$$\begin{aligned}\|\theta(a)S(b) - S(a)\theta(b)\| \leqslant& \|\theta(a)S(b) - \theta(a)T(b)\| + \|\theta(a)T(b) - T(a)\theta(b)\| \\ &+ \|T(a)\theta(b) - S(a)\theta(b)\| \\ \leqslant& \varepsilon\|\theta(a)\|\|b\| + \varepsilon_1\|a\|\|b\| + \varepsilon\|\theta(b)\|\|a\| \\ \leqslant& (2\varepsilon\|\theta\| + \varepsilon_1)\|a\|\|b\|.\end{aligned}$$

Thus,

$$\|\theta(a)S(b) - S(a)\theta(b)\| \leqslant \delta\|a\|\|b\|,$$

where $\delta = 2\varepsilon\|\theta\| + \varepsilon_1$. This completes the proof. □

Corollary 5. *Let T be a θ-multiplier of A and $S : A \longrightarrow A$ be a mapping such that the inequality (2.1) satisfies. Then S is a approximate θ-multiplier.*

Next we show the superstability of θ-multiplier on without order Banach algebras.

Theorem 6. *Let A be a without order Banach algebra, and $T : A \longrightarrow A$ be a mapping such that*

$$\|\theta(a)T(b) - T(a)\theta(b)\| \leqslant \varepsilon\|a\|^r\|b\|^r, \quad (a, b \in A), \tag{2.2}$$

for some $\varepsilon \geqslant 0$, and $r \geqslant 0$ with $r \neq 1$. Then T is a θ-multiplier.

Proof. We first prove that $T(\mu a) = \mu T(a)$ for all $\mu \in \mathbb{C}$ and $a \in A$. Let $x \in A$ be arbitrary. Take $s = -sgn(r-1)$. Then for all $n \in \mathbb{N}$, it follows from (2.2) that

$$\begin{aligned}\|\theta(n^s x)[T(\mu a) - \mu T(a)]\| &\leqslant \|\theta(n^s x)[T(\mu a)] - [T(n^s x)]\theta(\mu a)\| \\ &\quad + \|[T(n^s x)]\theta(\mu a) - \theta(n^s x)(\mu T(a))\| \\ &\leqslant \varepsilon \|n^s x\|^r \|\mu a\|^r + \|[T(n^s x)]\mu\theta(a) - \theta(n^s x)(\mu T(a))\| \\ &\leqslant \varepsilon \|n^s x\|^r \|\mu a\|^r + \varepsilon|\mu| \|n^s x\|^r \|a\|^r \\ &\leqslant n^{sr}\varepsilon\big(|\mu|^r + |\mu|\big)\|x\|^r \|a\|^r,\end{aligned}$$

and hence

$$\|\theta(x)[T(\mu a) - \mu T(a)]\| \leqslant n^{s(r-1)}\varepsilon\big(|\mu|^r + |\mu|\big)\|x\|^r\|a\|^r. \tag{2.3}$$

Since $s(r-1) < 0$, by letting $n \to \infty$ in (2.3), we get

$$\theta(x)[T(\mu a) - \mu T(a)] = 0.$$

Since θ is surjective and A is without order, we deduce $T(\mu a) = \mu T(a)$, for all $\mu \in \mathbb{C}$ and $a \in A$. Thus, $T(a) = n^{-s}T(n^s a)$ for all $n \in \mathbb{N}$. Hence for all $a, b \in A$,

$$\begin{aligned}\|\theta(a)T(b) - T(a)\theta(b)\| &= n^{-s}\|n^s\theta(a)T(b) - T(n^s a)\theta(b)\| \\ &= n^{-s}\|\theta(n^s a)T(b) - T(n^s a)\theta(b)\| \\ &\leqslant n^{-s}\varepsilon\|n^s a\|^r\|b\|^r \\ &= n^{s(r-1)}\varepsilon\|a\|^r\|b\|^r,\end{aligned}$$

By letting $n \to \infty$, we obtain $T(a)\theta(b) = \theta(a)T(b)$. Consequently, T is a θ-multiplier. □

The following example provided that the linearity and surjectivity of θ in Theorem 6 is essential.

Example 7. Let $r \geqslant 0$. Define $T, \theta : \mathbb{R} \longrightarrow \mathbb{R}$ by $T(a) = |sin(a)|^r$ and $\theta(a) = |a|^r$ for all $a \in \mathbb{R}$. Then $\theta(ab) = \theta(a)\theta(b)$ for every $a, b \in \mathbb{R}$, but neither θ is linear nor it is surjective. Since $|sin(a)| \leqslant |a|$ for every $a \in \mathbb{R}$, we have

$$|T(a)\theta(b) - \theta(a)T(b)| \leqslant 2|a|^r|b|^r, \quad (a, b \in \mathbb{R}).$$

Therefore the inequality (2.2) is holds for $\varepsilon = 2$. However, T is not θ-multiplier.

Next we show that Theorem 6 is also true for the case $r < 0$, with the extra condition that $T(0) = 0$.

Theorem 8. *Let A be a without order Banach algebra, and $T : A \longrightarrow A$ be a mapping such that $T(0) = 0$ and*

$$\|\theta(a)T(b) - T(a)\theta(b)\| \leqslant \varepsilon\|a\|^r\|b\|^r, \quad (a, b \in A),$$

for some $\varepsilon \geqslant 0$, and $r < 0$. Then T is a θ-multiplier.

Proof. We show that $\theta(a)T(b) = T(a)\theta(b)$, for all $a, b \in A$. The result is clear, if $a = 0$ or $b = 0$. Let $a, b \in A \setminus \{0\}$. Then the inequality (2.3) is satisfied and $s(r-1) < 0$ because $r < 0$, and hence

$$\theta(x)[T(\mu a) - \mu T(a)] = 0, \tag{2.4}$$

for all $\mu \in \mathbb{C}$ and $a \in A$ with $a \neq 0$. The equality (2.4) is also valid for $x = 0$. Since θ is surjective and A is without order, we get $T(\mu a) = \mu T(a)$, for all $\mu \in \mathbb{C}$ and $a \in A$. The rest of proof is similar to the proof of Theorem 6. □

Theorem 9. *Suppose that A is a Banach algebra, which need not be without order, and $f : A \longrightarrow A$ is a mapping such that*

$$\|f(a+b) - f(a) - f(b)\| \leqslant \varepsilon(\|a\|^r + \|b\|^r), \tag{2.5}$$

$$\|\theta(a)f(b) - f(a)\theta(b)\| \leqslant \varepsilon\|a\|^r\|b\|^r, \tag{2.6}$$

for some $\varepsilon \geqslant 0$, and $r \in \mathbb{R}$. If $r \geqslant 0$ and $r \neq 1$, or $r < 0$ and $f(0) = 0$, then there exist a unique θ-multiplier $T : A \longrightarrow A$ such that

$$\|f(a) - T(a)\| \leqslant \frac{2\varepsilon}{|2 - 2^r|}\|a\|^r, \quad (a \in A). \tag{2.7}$$

Proof. Assume that $r \neq 1$. It follows from (2.5) and a theorem of Rassias [13] and Gajda [3] that there exists a unique additive mapping $T : A \longrightarrow A$ such that (2.7) holds. We show that $\theta(a)T(b) = T(a)\theta(b)$ for all $a, b \in A$. Since T is additive, $T(0) = 0$ and hence it is enough to consider $a, b \in A \setminus \{0\}$. Put $s = -sgn(r-1)$. Since T is additive, we obtain $T(a) = n^{-s}T(n^s a)$ for all $n \in \mathbb{N}$. Using (2.7) we get

$$\|n^{-s}f(n^s b) - T(b)\| \leqslant n^{-s}\frac{2\varepsilon}{|2-2^r|}\|n^s b\|^r = n^{s(r-1)}\frac{2\varepsilon}{|2-2^r|}\|b\|^r \longrightarrow 0,$$

and hence

$$\|n^{-s}f(n^s b) - T(b)\| \longrightarrow 0. \tag{2.8}$$

On the other hand,

$$\begin{aligned}\|n^{-s}\theta(a)f(n^s b) - f(a)\theta(b)\| &= n^{-s}\|\theta(a)f(n^s b) - f(a)\theta(n^s b)\| \\ &\leqslant n^{-s}\varepsilon\|a\|^r\|n^s b\|^r \\ &= n^{s(r-1)}\varepsilon\|a\|^r\|b\|^r \longrightarrow 0.\end{aligned}$$

Therefore

$$\|n^{-s}\theta(a)f(n^s b) - f(a)\theta(b)\| \longrightarrow 0. \tag{2.9}$$

Now

$$\begin{aligned}\|\theta(a)T(b) - T(a)\theta(b)\| \leqslant{}& \|\theta(a)T(b) - n^{-s}\theta(a)f(n^s b)\| + \|n^{-s}\theta(a)f(n^s b) - f(a)\theta(b)\| \\ &+ \|f(a)\theta(b) - T(a)\theta(b)\|.\end{aligned}$$

Thus, by (2.7), (2.8) and (2.9),

$$\|\theta(a)T(b) - T(a)\theta(b)\| \leqslant \frac{2\varepsilon}{|2-2^r|}\|a\|^r\|\theta(b)\|. \tag{2.10}$$

Applying (2.10), for all $a, b \in A$ we get

$$\begin{aligned}\|\theta(a)T(b) - T(a)\theta(b)\| &= n^{-s}\|n^s\theta(a)T(b) - T(n^s a)\theta(b)\| \\ &= n^{-s}\|\theta(n^s a)T(b) - T(n^s a)\theta(b)\| \\ &\leqslant n^{-s}\frac{2\varepsilon}{|2-2^r|}\|n^s a\|^r\|\theta(b)\| \\ &= n^{s(r-1)}\varepsilon\|a\|^r\|b\|.\end{aligned}$$

By letting $n \to \infty$, we conclude that $T(a)\theta(b) = \theta(a)T(b)$, for all $a, b \in A$. Hence T is a θ-multiplier. □

Recall that an algebra A is called *semisimple* if $radA = \{0\}$, where $radA$ is the intersection of all maximal left (right) ideals in A.

Proposition 10. *Let A be a semisimple Banach algebra with left approximate identity. If $T : A \longrightarrow A$ is a left θ-multiplier, then T is almost homomorphism.*

Proof. It follows from Theorem 2.1 of [12] that T is linear and continuous. Hence there exist $k_1 > 0$ such that $\|T\| \leqslant k_1$. On the other hand, θ is continuous by a classical result of Johnson, which states that every surjective homomorphism from Banach algebra A onto semisimple Banach algebra B is automatically continuous [8]. Thus, there exist $k_2 > 0$ such that $\|\theta\| \leqslant k_2$. Now for every $a, b \in A$, we have

$$\begin{aligned}\|T(ab) - T(a)T(b)\| =&\|T(a)\theta(b) - T(a)T(b)\| \\ \leqslant&\|T(a)\|\|\theta(b) - T(b)\| \\ \leqslant&\|T(a)\|\big(\|\theta(b)\| + \|T(b)\|\big) \\ \leqslant&k_1\|a\|\big(k_2\|b\| + k_1\|b\|\big) \\ \leqslant&k_1(k_1+k_2)\|a\|\|b\|.\end{aligned}$$

Therefore

$$\|T(ab) - T(a)T(b)\| \leqslant \varepsilon\|a\|\|b\|,$$

where $\varepsilon = k_1(k_1 + k_2)$. This completes the proof. □

Recall that every C^*-algebra is semisimple and has a bounded approximate identity [2]. Hence we get the next result.

Corollary 11. *Every left θ-multiplier T on C^*-algebra A is almost homomorphism.*

Proposition 12. *Let A be a Banach algebra and $T : A \longrightarrow A$ be a continuous linear map. Suppose that*

$$\|T(x) - \theta(x)\| \leqslant \delta\|x\|,$$

for some $\delta > 0$ and for all $x \in A$. Then T is approximate left θ-multiplier if and only if it is almost homomorphism.

Proof. Let T be an approximate left θ-multiplier on A. Then

$$\|T(ab) - T(a)\theta(b)\| \leqslant \xi\|a\|\|b\|,$$

for some $\xi > 0$. Since T is continuous, hence there exist $k > 0$ such that $\|T\| \leqslant k$. Now for every $a, b \in A$, we have

$$\begin{aligned}\|T(ab) - T(a)T(b)\| \leqslant & T(ab) - T(a)\theta(b)\| + \|T(a)\theta(b) - T(a)T(b)\| \\ \leqslant & \xi\|a\|\|b\| + \delta k\|a\|\|b\| \\ \leqslant & (\xi + \delta k)\|a\|\|b\|.\end{aligned}$$

If we take $\varepsilon = \xi + \delta k$, then

$$\|T(ab) - T(a)T(b)\| \leqslant \varepsilon\|a\|\|b\|,$$

and hence T is almost homomorphism. The converse is similar. $\square$

If T is a left and right approximate θ-multiplier, then T is an approximate θ-multiplier, but in general, the converse is fails. Next we prove the converse statement with additional hypotheses on Banach algebra A.

Proposition 13. *Suppose that A is a semisimple Banach algebra with a bounded left approximate identity. If $T : A \longrightarrow A$ is a approximate θ-multiplier, then T is approximate left (right) θ-multiplier.*

Proof. By assumption

$$\|\theta(a)T(b) - T(a)\theta(b)\| \leqslant \varepsilon\|a\|\|b\|,$$

for some $\varepsilon \geqslant 0$ and every $a, b \in A$. Let $\{e_\lambda\}_{\lambda\in\Lambda}$ be a bounded left approximate identity for A.

Since θ is a surjective homomorphism, it follows from Johnson's theorem that θ is continuous and hence we can choose a bounded net $\{x_\lambda\}_{\lambda\in\Lambda}$ in A such that $\theta(x_\lambda) = e_\lambda$ for each $\lambda \in \Lambda$. Put $m = \sup_{\lambda\in\Lambda} \|x_\lambda\|$ and $k = \|\theta\|$. Now for every $a, b, c \in A$, we have

$$\begin{aligned}\|\theta(c)T(ab) - \theta(c)T(a)\theta(b)\| \leqslant & \|\theta(c)T(ab) - T(c)\theta(ab)\| \\ & + \|T(c)\theta(ab) - \theta(c)T(a)\theta(b)\| \\ \leqslant & \varepsilon\|ab\|\|c\| + \|T(c)\theta(a) - \theta(c)T(a)\|\|\theta(b)\| \\ \leqslant & \varepsilon\|a\|\|b\|\|c\| + k\varepsilon\|a\|\|b\|\|c\| \\ \leqslant & \varepsilon(1 + k)\|a\|\|b\|\|c\|.\end{aligned}$$

Replacing c by $\{x_\lambda\}_{\lambda\in\Lambda}$ and using $\theta(x_\lambda) = e_\lambda$, we get

$$\|e_\lambda T(ab) - e_\lambda T(a)\theta(b)\| \leqslant \varepsilon m(1+k)\|a\|\|b\|.$$

By letting $\lambda \to \infty$, we obtain

$$\|T(ab) - T(a)\theta(b)\| \leqslant \delta\|a\|\|b\|,$$

where $\delta = \varepsilon m(1+k)$. Thus, T is approximate left θ-multiplier. □

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