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SUBCLASSES OF MEROMORPHIC p-VALENT FUNCTIONS DEFINED BY q-DERIVATIVE OPERATOR ASSOCIATED WITH CASSINIAN OVALS

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Abstract. In the present investigation we introduce two new subclasses $\mathscr{MS}^*_q(p,b;c)$ and $\mathscr{MK}_q(p,b;c)$ of meromorphic multivalent functions by using q-derivative operator defined in the punctured unit disc. We use the Cassinian Oval $\sqrt{1+cz}$ with $c \in (0,1]$ as a subordinant function in this treatise. Also we derive several properties including convolution properties, the necessary and sufficient condition and coefficient estimates for these subclasses.

1 Introduction and definitions

Let Σ_p denote the class of meromorphic functions of the form

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_k z^{k-p} \qquad (p \in \mathbb{N}), \tag{1.1}$$

which are analytic and p-valent in the punctured unit disc $\mathbb{U}^* = \mathbb{U}\backslash\{0\}$, where $\mathbb{U} = \{z : z \in \mathbb{C}, |z| < 1\}$. Let g and f be two analytic functions in $\mathbb{U}$, then function g is said to be subordinate to f if there exists an analytic function w in the unit disk $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $g(z) = f(w(z))$ $(z \in \mathbb{U})$. We denote this subordination by $g \prec f$. In particular, if the function f is univalent in $\mathbb{U}$ the above subordination is equivalent to $g(0) = f(0)$ *and* $f(\mathbb{U}) \subset g(\mathbb{U})$.

Quantum calculus, or q-calculus is the subject of a extend investigation constructed on the different applications perceived for it over striking mathematical fields and in expansion to centrality to hypothetical physics. Not long ago, innumerable authors have presented modern classes of analytic functions utilizing q-calculus. The q-calculus is an ordinary calculus without notion of limit point. The application of q-calculus was initiated by Jackson [6, 7, 8] to begin with investigated q-calculus

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applications, efficiently creating q-derivative and q-integral. By making use of q-calculus various functions classes in Geometric Functions Theory are introduced and investigated from different view points and perspectives (see [1], [12], [17], [19], [20], [23] and references therein). Purpose of this treatise is to introduce and study two new subclasses of p-valent meromorphic functions by applying q-derivative operators in conjunction with the principle of subordinations.

For $0 < q < 1$, the q-derivative of a function f is defined by (see [5, 6, 7, 8])

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z} \qquad (z \in \mathbb{U}), \tag{1.2}$$

provided that $f'(0)$ exists.

From (1.2), it can be easily obtain that

$$D_q f(z) = \frac{-[p]_q}{q^p z^{p+1}} + \sum_{k=1}^{\infty} [k-p]_q a_k z^{k-p-1},$$

where

$$[k]_q = \frac{1-q^k}{1-q}.$$

As q$\to 1^-$, $[k]_q \to k$ and $\lim_{q\to 1^-} D_q f(z) = f'(z)$. Also, we have

$$[k+p]_q = [k]_q + q^k [p]_q = q^p [k]_q + [p]_q,$$

$$[k-p]_q = q^{-p}[k]_q - q^{-p}[p]_q,$$

$$[0]_q = 0, [1]_q = 1.$$

For $f \in \Sigma_p$ given by (1.1) and $g \in \Sigma_p$ given by

$$g(z) = z^{-p} + \sum_{k=1}^{\infty} b_k z^{k-p} \qquad (p \in \mathbb{N}),$$

the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z^{-p} + \sum_{k=1}^{\infty} a_k b_k z^{k-p} = (g * f)(z).$$

Motivated essentially due to the work of Aouf et *al*. [3], Kant and Vyas [9], Seoudy [15], Seoudy and Aouf [16] and Srivastava et *al.* [21], we define the following two subclasses of Σ_p by using the q-derivative operator D_q and the principle of subordination between analytic functions:

Definition 1. *Let* $0 < q < 1$, $c \in (0, 1]$ *and* $b \in \mathbb{C}\backslash\{0\}$. *A function* f *belonging to* Σ_p *is said to be in the class* $\mathscr{MS}^*_q(p, b; c)$ *if it satisfies*

$$1 - \frac{1}{b}\left[\frac{zD_q f(z)}{f(z)} + \frac{[p]_q}{q^p}\right] \prec q_c(z). \tag{1.3}$$

Definition 2. *Let* $0 < q < 1$, $c \in (0, 1]$ *and* $b \in \mathbb{C}\backslash\{0\}$. *A function* f *belonging to* Σ_p *is said to be in the class* $\mathscr{MK}_q(p, b; c)$ *if it satisfies*

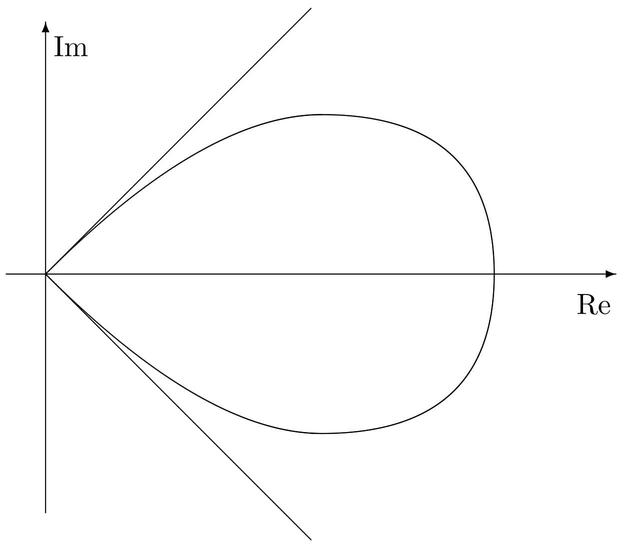
$$1 - \frac{1}{b}\left[\frac{D_q\big(zD_q f(z)\big)}{D_q f(z)} + \frac{[p]_q}{q^p}\right] \prec q_c(z), \tag{1.4}$$

where $q_c(z) = \sqrt{1 + cz} \quad (z \in \mathbb{U})$.

We also verify from both above definitions that

$$f \in \mathscr{MK}_q(p, b; c) \Leftrightarrow -\frac{q^p}{[p]_q} zD_q f \in \mathscr{MS}^*_q(p, b; c). \tag{1.5}$$

Notice that for $c \in (0, 1)$ the set $q_c(\mathbb{U}) = \left\{\omega \in \mathbb{C} : Re\, \omega > 0 \cap |\omega^2 - 1| < c\right\}$ is the interior of the right half of the Cassini's curve $\left|\omega^2 - 1\right| = c$ and in the special case $c = 1$ this curve is the Bernoulli's lemniscate; see [11], pp. 231-235.



The image of the unit circle under the function $q_1(z)$

For detailed study about Cassinian ovals and related topics one can refer [2, 4, 10, 13, 14, 18, 22]. In the present investigations, we derive several properties including convolution properties, the necessary and sufficient condition and coefficient estimates for functions belonging to the subclasses $\mathscr{MS}^*_q(p, b; c)$ and $\mathscr{MK}_q(p, b; c)$.

2 Main Results

Unless otherwise mentioned, we assume throughout this section that $0 < q < 1$, $c \in (0,1]$, $b \in \mathbb{C}\backslash\{0\}$ and $\theta \in [0, 2\pi)$.

Theorem 3. *If $f \in \Sigma_p$, then $f \in \mathscr{MS}^*_q(p,b;c)$ if and only if*

$$z^p \left[f(z) * \frac{1 + \big[M(\theta) - q\big] z}{z^p(1-z)(1-qz)} \right] \neq 0 \qquad (z \in \mathbb{U}^*), \tag{2.1}$$

where

$$M(\theta) = \frac{e^{-i\theta}(1 + \sqrt{1 + ce^{i\theta}})}{cbq^p}. \tag{2.2}$$

Proof. It is easy to verify that for any function $f \in \Sigma_p$, we have

$$f(z) = f(z) * \frac{1}{z^p(1-z)} \tag{2.3}$$

and

$$-\frac{q^p}{[p]_q} z D_q f(z) = f(z) * \frac{1 - \left(q + \frac{1}{[p]_q}\right) z}{z^p(1-z)(1-qz)}. \tag{2.4}$$

First, if $f \in \mathscr{MS}^*_q(p,b;c)$, in order to prove that (2.1) holds we will write (1.3) by using the definition of the subordination, that is

$$-\frac{q^p}{[p]_q} \frac{zD_q f(z)}{f(z)} = 1 - b\frac{q^p}{[p]_q}\left(1 - \sqrt{1 + c\omega(z)}\right) \qquad (z \in \mathbb{U}^*),$$

where w is a Schwarz function, hence

$$z^p \left[-q^p z D_q f(z) - \left\{ [p]_q + bq^p(\sqrt{1 + ce^{i\theta}} - 1) \right\} f(z) \right] \neq 0 \qquad (z \in \mathbb{U}^*). \tag{2.5}$$

Now from (2.3) and (2.4), we may write (2.5) as

$$z^p \Bigg[\left(f(z) * \frac{\{1 - (q + \frac{1}{[p]_q})z\}[p]_q}{z^p(1-z)(1-qz)} \right)$$
$$- \left\{ [p]_q + bq^p(\sqrt{1 + ce^{i\theta}} - 1) \right\} \left(f(z) * \frac{1}{z^p(1-z)} \right) \Bigg] \neq 0 \qquad (z \in \mathbb{U}^*),$$

which is equivalent to

$$z^p \left[f(z) * \frac{1 + \left(-q + \frac{1}{bq^p(\sqrt{1+ce^{i\theta}}-1)} \right) z}{z^p(1-z)(1-qz)} \Big[-bq^p(\sqrt{1 + ce^{i\theta}} - 1) \Big] \right] \neq 0$$

or

$$z^p\left[f(z)*\frac{1+\left(\frac{e^{-i\theta}\left(1+\sqrt{1+ce^{i\theta}}\right)}{cbq^p}-q\right)z}{z^p(1-z)(1-qz)}\right]\neq 0 \qquad (z\in\mathbb{U}^*),$$

which leads to (2.1), that proves the necessary part of Theorem 3.
Reversely, suppose that $f\in\Sigma_p$ satisfy the condition (2.1). Since it was shown in the first part of the proof that assumption (2.1) is equivalent to (2.5), we obtain that

$$-\frac{q^p}{[p]_q}\frac{zD_qf(z)}{f(z)}\neq 1-\frac{bq^p}{[p]_q}(1-\sqrt{1+ce^{i\theta}}) \qquad (z\in\mathbb{U}^*), \tag{2.6}$$

and let us assume that

$$\varphi(z)=-\frac{q^p}{[p]_q}\frac{zD_qf(z)}{f(z)} \quad and \quad \psi(z)=1-\frac{bq^p}{[p]_q}(1-\sqrt{1+cz}).$$

The relation (2.6) means that

$$\varphi(\mathbb{U}^*)\cap\psi(\partial\mathbb{U}^*)=\emptyset.$$

Thus, the simply connected domain is included in a connected component of $\mathbb{C}\backslash\psi(\partial\mathbb{U}^*)$. Therefore, using the fact that $\varphi(0)=\psi(0)$ and the univalence of the function ψ, it follows that $\varphi(z)\prec\psi(z)$, which implies that $f\in\mathscr{MS}_q^*(p,b;c)$. Thus, the proof of Theorem 3 is now completed. □

Theorem 4. *A necessary and sufficient condition for the function f defined by (1.1) to be in the class $\mathscr{MS}_q^*(p,b;c)$ is that*

$$1+\sum_{k=1}^{\infty}\left[1+\frac{e^{-i\theta}(1+\sqrt{1+ce^{i\theta}})[k]_q}{cbq^p}\right]a_kz^k\neq 0 \quad (z\in\mathbb{U}^*). \tag{2.7}$$

Proof. From Theorem 3, we find that $f\in\mathscr{MS}_q^*(p,b;c)$ if and only if (2.1) holds. Since

$$\frac{1}{z^p(1-z)(1-qz)}=\frac{1}{z^p}+(1+q)z^{1-p}+(1+q+q^2)z^{2-p}+(1+q+q^2+q^3)z^{3-p}+\cdots, \quad (z\in\mathbb{U}^*),$$

hence

$$\frac{1+\left[M(\theta)-q\right]z}{z^p(1-z)(1-qz)}=\frac{1}{z^p}+\sum_{k=1}^{\infty}\left(1+M(\theta)[k]_q\right)z^{k-p},$$

where $M(\theta)$ is given by (2.2).
Now a simple computation shows that (2.1) is identical to (2.7). Thus, the proof of Theorem 4 is completed. □

Theorem 5. *If $f \in \Sigma_p$ satisfies the inequality*

$$\sum_{k=1}^{\infty}\left[[k]_q\left(|1+\sqrt{1+ce^{i\theta}}|\right)+c|b|q^p\right]|a_k|<c|b|q^p \tag{2.8}$$

then $f \in \mathscr{MS}_q^*(p,b;c)$.

Proof. Since

$$\begin{aligned}&\left|1+\sum_{k=1}^{\infty}\left(\frac{cbq^p+e^{-i\theta}(1+\sqrt{1+ce^{i\theta}})[k]_q}{cbq^p}\right)a_kz^k\right|\\ &\geq 1-\left|\sum_{k=1}^{\infty}\left(\frac{cbq^p+e^{-i\theta}(1+\sqrt{1+ce^{i\theta}})[k]_q}{cbq^p}\right)a_kz^k\right|\\ &\geq 1-\sum_{k=1}^{\infty}\left(\frac{c|b|q^p+(|1+\sqrt{1+ce^{i\theta}}|)[k]_q}{c|b|q^p}\right)|a_k|>0.\end{aligned}$$

Thus, the inequality (2.8) holds and our result follows from Theorem 4. □

Theorem 6. *If $f \in \Sigma_p$, then $f \in \mathscr{MK}_q(p,b;c)$ if and only if*

$$z^p\left[f(z)*\frac{1-\left[\frac{1+M(\theta)}{[p]_q}-\left(M(\theta)-q-q^2\right)\right]z-\left(M(\theta)-q\right)\left(q+\frac{1}{[p]_q}\right)qz^2}{z^p(1-z)(1-qz)(1-q^2z)}\right]\neq 0 \qquad (z\in\mathbb{U}^*), \tag{2.9}$$

where $M(\theta)$ is given by (2.2).

Proof. From (1.5) it follows that $f \in \mathscr{MK}_q(p,b;c)$ if and only if $-\frac{q^p}{[p]_q}zD_qf \in \mathscr{MS}_q^*(p,b;c)$. Then from Theorem 3, the function $-\frac{q^p}{[p]_q}zD_qf \in \mathscr{MS}_q^*(p,b;c)$ if and only if

$$z^p\left[-\frac{q^p}{[p]_q}zD_qf*g(z)\right]\neq 0, \qquad (z\in\mathbb{U}^*), \tag{2.10}$$

where

$$g(z)=\frac{1+[M(\theta)-q]z}{z^p(1-z)(1-qz)}.$$

On a basic computation we note that

$$\begin{aligned}D_qg(z)&=\frac{g(qz)-g(z)}{(q-1)z}\\ &=\frac{-[p]_q+\left[1+M(\theta)-[p]_q\left(M(\theta)-q-q^2\right)\right]z+\left(M(\theta)-q\right)\left(q[p]_q+1\right)qz^2}{q^pz^{p+1}(1-z)(1-qz)(1-q^2z)}\end{aligned}$$

and therefore

$$-\frac{q^p}{[p]_q} zD_q g(z) = \frac{1 - \left[\frac{1+M(\theta)}{[p]_q} - \left(M(\theta) - q - q^2\right)\right] z - \left(M(\theta) - q\right)\left(q + \frac{1}{[p]_q}\right) q z^2}{z^p (1-z)(1-qz)(1-q^2 z)}.$$

Using the above relation and the identity

$$\left(-\frac{q^p}{[p]_q} zD_q f(z)\right) * g(z) = f(z) * \left(-\frac{q^p}{[p]_q} zD_q g(z)\right),$$

it is simple to check that (2.10) is identical to (2.9). Thus, the proof of Theorem 6 is completed. □

Theorem 7. *A necessary and sufficient condition for the function f defined by (1.1) to be in the class $\mathscr{MK}_q(p,b;c)$ is that*

$$1 + \sum_{k=1}^{\infty} \left(\frac{cbq^p + e^{-i\theta}(1+\sqrt{1+ce^{i\theta}})[k]_q}{cbq^p}\right)\left(1 - \frac{[k]_q}{[p]_q}\right) a_k z^k \neq 0 \qquad (z \in \mathbb{U}^*). \quad (2.11)$$

Proof. From Theorem 6, we find that $f \in \mathscr{MK}_q(p,b;c)$ if and only if (2.9) holds.

Since

$$\frac{1}{z^p(1-z)(1-qz)(1-q^2z)} = \frac{1}{z^p} + (1+q+q^2)z^{1-p} + (1+q+2q^2+q^3+q^4)z^{2-p} + (1+q+2q^2+2q^3+2q^4+q^5+q^6)z^{3-p} + \cdots, \qquad (z \in \mathbb{U}^*),$$

hence

$$\frac{1 - \left[\frac{1+M(\theta)}{[p]_q} - (M(\theta) - q - q^2)\right] z - \left(M(\theta) - q\right)\left(q + \frac{1}{[p]_q}\right) q z^2}{z^p (1-z)(1-qz)(1-q^2 z)} = \frac{1}{z^p} + \sum_{k=1}^{\infty}\left(1 + M(\theta)[k]_q\right)\left(1 - \frac{[k]_q}{[p]_q}\right) z^{k-p} \qquad (z \in \mathbb{U}^*),$$

where $M(\theta)$ is given by (2.2).
Now a simple computation shows that (2.9) is identical to (2.11). Thus, the proof of Theorem 7 is completed. □

Using similar arguments to those in the proof of Theorem 5, we may also prove the next result.

Theorem 8. *If $f \in \Sigma_p$ satisfies the inequality*

$$\sum_{k=1}^{\infty} \left[[k]_q\left(|1+\sqrt{1+ce^{i\theta}}|\right) + c|b|q^p\right]\left(1 - \frac{[k]_q}{[p]_q}\right)|a_k| < c|b|q^p \qquad (2.12)$$

then $f \in \mathscr{MK}_q(p,b;c)$.

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