Surveys in Mathematics and its Applications

ISSN 1842-6298 (electronic), 1843-7265 (print)
Volume 17 (2022), 181 – 203

DYNAMICS OF THE STOCHASTIC WAVE EQUATIONS WITH DEGENERATE MEMORY EFFECTS ON BOUNDED DOMAIN

Abdelmajid Ali Dafallah, Fadlallah Mustafa Mosa, Mohamed Y. A. Bakhet and Eshag Mohamed Ahmed

Abstract. In this work, we consider the behavior of long-time dynamics of a random dynamical system generated wave equations with degenerate memory and additive noise, on bounded domain U, with Dirichlet boundary condition, and nonlinear term f(u) with critical growth conditions, then we establish the existence of a random attractor.

1 Introduction

Let us denote by $U \subset \mathbb{R}^n$ an open bounded domain, then we investigate the existence of random attractors for the following viscoelastic wave equations with past history:

$$\begin{cases} u_{tt} - \triangle u + \displaystyle\int_0^\infty \mu(s) div[a(x)\nabla u(t-s)]ds + b(x)u_t + f(u) = g(x) + c\sum_{j=1}^m h_j \dot{W}_j(t), \\ u(\tau, x) = u_0(x)\ ,\ u_t(\tau, x) = u_1(x),\ x \in U, \tau \in \mathbb{R}, t \geq \tau, \\ u(x,t) = 0,\ on\ \Gamma \times \mathbb{R}^+. \end{cases} \tag{1.1}$$

Where U is a bounded open set of $\mathbb{R}^n$ with Dirichlet boundary condition, and let c is proper a positive constant, $\mu(s) \leq 0$ for every $s \in \ \mathbb{R}^+$, $\triangle$ is the Laplacian with respect to the variable $x \in \mathbb{R}^n$, $u = u(t,x)$ is a real valued function on $U \times [\tau, +\infty)$, the function $g(x) \in Ł^2(U)$ is external force, $h_j \in H_0^1 \bigcap H^2$, $\{W(t)_j\}_{j=1}^m$ are an independent two sided real-valued Wiener processes of probability space, and $a(x), b(x)$ are positive function, where $a(x) \in C^1(\bar{U}), b(x) \in L^\infty(U)$, then following assumption hold

$$a(x) + b(x) \geq \delta > 0,\ x \in\ U. \tag{1.2}$$

2020 Mathematics Subject Classification: 37L55; 35R60; 35B40; 35B41; 35B45.
Keywords: Stochastic Wave Equations; Degenerate Memory; Random Dynamical System; Random Attractor.

We observe that the assumption (1.2) gives us a wide assortment of possibilities from which to choose the functions a(x) and b(x), and the most interesting case occurs when one has simultaneous and complementary damping mechanisms. Taking this point of view into account, a distinctive feature of our paper is exactly to consider different and localized damping mechanisms acting in the domain, but not necessarily strategically localized dissipations as considered in the prior literature.

Evolution models with memory terms are ubiquitous. Natural and social phenomena are often affected not only by its current state, but also by its history. Some classical and modern examples leading to partial differential equations with memory. Refer and more details, see therein references [5, 10, 11, 13, 14, 19]. The problem is degenerate in the sense that the function $a(x) \geq 0$ in the memory term is allowed to vanish is a subset ω_0 of $\bar{U}$ and we shall view the wave model involving the memory term as the above problem(1.1) with combined viscoelastic-frictional dissipation. However, in random case, there is no result obtained the existence of random attractors for problem (1.1)-(1.2).

For obviously,stochastic evolution equation without memory terms, once $(a(X) = 0)$ its existence and uniqueness of solutions for random dynamical systems have been studied extensively in the literature (see [1, 2, 4, 6, 7, 8, 12] and the references therein). But when $(a(X) = a_0)$ with memory terms, there are fewer results and most previous authors have concentrated on this case (see[5]). For example, we may refer to [13, 14, 15, 16, 17, 18] for the stochastic wave equations with memory. All the above mentioned results were considered with linear memory. In this article, by using a similar technique as in [9, 12], we will continue to devote the existence of random attractors for the system (1.1) in space E, under the perturbation of additive noise.

The plan of the paper is as follows. In Section 2, we give the preliminaries for random dynamical systems (RDS). In Section 3, we present the functional setting for this model and show the definition of continuous RDS solutions, along with existence and uniqueness results. Section 4 is devoted to the existence of bounded absorbing sets for the solutions. In last Section, by some asymptotic estimates in advance associated to our problem we obtain the existence of random attractor in spaces E. The type of degenerate memory that we consider is given by an operator of the from

$$Au = div(a(x)\nabla u),$$

where $a \in C^1(\bar{U})$ may vanish is a subset ω_0 of $\bar{U}$. In order to treat the memory term, we introduce a past history variable using the relative displacement history for all $x \in U \subset \mathbb{R}^n$ and $s, t \in \mathbb{R}^+$, so clearly by [5], such that satisfies the following results:

$$\begin{cases} (\eta,\eta)_{\mu,1} = \int_0^\infty \mu(s)(\nabla\eta(s),\nabla\eta(s))ds, \\ \eta(x,t,s) = u(x,t) - u(x,t-s), \\ \eta_t = \dfrac{\partial}{\partial t}\eta \ , \ \eta_s = \dfrac{\partial}{\partial s}\eta. \end{cases} \tag{1.3}$$

Using the relation (1.3), we can transform the problem (1.1) into the following equivalent systems

$$\begin{cases} u_{tt} - div[(1-a(x)k_0)\nabla u - \int_0^\infty \mu(s) div[a(x)\nabla\eta^t(s)]ds + b(x)u_t + f(u) \\ \qquad\qquad = g(x) + c\sum_{j=1}^m h_j \dot{W}_j(t), \\ \eta_t = -\eta_s + u_t. \end{cases} \tag{1.4}$$

With boundary and initial conditions

$$\begin{cases} u(x,t) = 0, & on\ \Gamma\times\mathbb{R}^+, \\ \eta^t(x,t,s) = 0, & on\ \Gamma\times\mathbb{R}^+, \\ u(\tau,x) = u_0(x), & x\in U, \tau\mathbb{R}, \\ u_t(\tau,x) = u_1(x), & x\in U, \tau\mathbb{R}, \\ \eta^0(x,\tau,s) = u_0(x) - u_0(x,-s), & (x,\tau,s)\in U\times\mathbb{R}\times\mathbb{R}^+. \end{cases} \tag{1.5}$$

2 Preliminaries and Abstract Results

In this Section, we recall some basic concepts related to RDS and a random attractor for RDS in [8, 6], which are important for getting our main results. Let $(\Omega,\mathcal{F},P)$ be a probability space and (X,d) be a Polish space with the Borel σ-algebra $B(X)$. Then, the distance between $x\in\ X$ and $B\subseteq X$ is denoted by $d(x,B)$. If $B\subseteq X$ and $C\subseteq X$, the Hausdorff semi-distance from B to C is denoted by $d(B,C) = \sup_{x\in B} d(x,C)$.

Definition 1. *$(\Omega,\mathcal{F},P,(\theta_t)_{t\in\mathbb{R}})$ is called a metric dynamical system if $\theta:\mathbb{R}\times\Omega\longrightarrow\Omega$ is $(\mathcal{B}(\mathbb{R})\times\mathcal{F},\mathcal{F})$- measurable, θ_0 is the identity on Ω, $\theta_{s+t} = \theta_t\circ\theta_s, \forall\ (s,t)\in\mathbb{R}$ and $\theta_0 P = P,\ \forall\ t\in\mathbb{R}$.*

Definition 2. *A mapping $\Phi(t,\tau\ ,\omega,x):\mathbb{R}^+\times\mathbb{R}\times\Omega\times X\to X$ is called continuous cocycle on X over $\mathbb{R}$ and $(\Omega,\mathcal{F},P,(\theta_t)_{t\in\mathbb{R}})$, if for all $\tau\in\mathbb{R}\ ,\omega\in\Omega$ and $t,s\in\mathbb{R}^+$, the following conditions are satisfied :*
i) $\Phi(t,\tau,\omega,x):\mathbb{R}^+\times\mathbb{R}\times\Omega\times X\to X$ *is a* $(\mathcal{B}(\mathbb{R}^+)\times\mathcal{F},\mathcal{B}(\mathbb{R}))$ *measurable mapping,*
ii) $\Phi(0,\tau,\omega,x)$ *is identity on* X*,*
iii) $\Phi(t+s,\tau,\omega,x) = \Phi(t,\tau+s,\theta_s\omega,x)\circ\Phi(s,\tau,\omega,x)$*,*
iv) $\Phi(t,\tau,\omega,x):X\to X$ *is continuous.*

Definition 3. *Let 2^X be the collection of all subsets of X, a set valued mapping $(\tau,\omega) \mapsto \mathcal{D}(t\ \omega) : \mathbb{R} \times \Omega \mapsto 2^X$ is called measurable with respect to $\mathcal{F} \in \Omega$, if $\mathcal{D}(t\ \omega)$ is a ("usually closed") nonempty subset of X and the mapping $\omega \in \Omega \mapsto d(X, B(\tau,\omega))$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$- measurable for every fixed $x \in X$ and $\tau \in \mathbb{R}$. Let $B = B(t,\omega) \in \mathcal{D}(t,\omega) : \tau \in \mathbb{R}, \omega \in \Omega$ is called a random set.*

Definition 4. *A random bounded set $B = \{B(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ of X is called tempered with respect to $\{\theta(t)\}_{t\in\Omega}$, if for p-a.e $\omega \in \Omega$,*

$$\lim_{t\ \mapsto\infty} e^{-\beta t}\ d(B(\theta_{-t}\omega)) = 0\ ,\ \forall\ \beta\ > 0\ ,$$

where

$$d(B) = \sup_{x\in B} \|x\|_X.$$

Definition 5. *Let $\mathcal{D}$ be a collection of random subset of X and $K = \{K(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, then K is called an absorbing set of $\Phi \in \mathcal{D}$, if for all $\tau \in \mathbb{R}, \omega \in \Omega$ and $B \in \mathcal{D}$, there exists, $T = T(\tau,\omega,B) > 0$ such that*

$$\Phi(t\ ,\tau,\theta_{-t}\omega, B(\tau,\theta_{-t}\omega)) \subseteq K(\tau,\omega)\ ,\ \forall\ t\ \geq T.$$

Definition 6. *Let $\mathcal{D}$ be a collection of random subset of X, the Φ is said to be $\mathcal{D}$-pullback asymptotically compact in X if for p-a.e $\omega \in \Omega$, $\{\Phi(t_n\ ,\theta_{-t_n}\omega\ , x_n)\}_{n=1}^{\infty}$ has a convergent subsequence in X when $t_n \mapsto \infty$ and $x_n \in B(\theta_{-t_n}\omega)$ with $\{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$.*

Definition 7. *Let $\mathcal{D}$ be a collection of random subset of X and $\mathcal{A} = \{\mathcal{A}(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$,then $\mathcal{A}$ is called a $\mathcal{D}$-random attractor (or $\mathcal{D}$-pullback attractor) for Φ, if the following conditions are satisfied: for all $t \in \mathbb{R}^+, \tau \in \mathbb{R}$ and $\omega \in \Omega$*

i): $\mathcal{A}(\tau,\omega)$ *is compact, and* $\omega \mapsto d(x, \mathcal{A}(\omega))$ *is measurable for every* $x \in X$,
ii): $\mathcal{A}(\tau,\omega)$ *is invariant, that is*

$$\Phi(t,\tau,\omega,\mathcal{A}(\tau,\omega)) =\ \mathcal{A}(\tau + t, \theta_t\omega), \forall\ t\ \geq \tau,$$

iii): $\mathcal{A}(\tau,\omega)$ *attracts every set in* $\mathcal{D}$, *that is for every* $B = \{B(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$,

$$\lim_{t\ \mapsto\infty} d_X(\Phi(t,\tau,\theta_{-t}\omega, B(\tau,\theta_{-t}\omega)), \mathcal{A}(\tau,\omega)) = 0,$$

where d_X is the Hausdorff semi-distance given by

$$d_X(Y,Z) = \sup_{y\in Y} \inf_{z\in Z} \|y - z\|_X,\quad \forall\ (Y,Z) \in X.$$

Lemma 8. *Let $\mathcal{D}$ be a neighborhood-closed collection of (τ, ω)- parameterized families of nonempty subsets of X and Φ be a continuous cocycle on X over $\mathbb{R}$ and $(\Omega, \digamma, P, (\theta_t)_{t\in\mathbb{R}})$. Then Φ has a pullback $\mathcal{D}$-attractor $\mathcal{A}$ in $\mathcal{D}$ if and only if Φ is pullback $\mathcal{D}$-asymptotically compact in X and Φ has a closed, $\digamma$-measurable pullback $\mathcal{D}$-absorbing set $K \in \mathcal{D}$, the unique pullback $\mathcal{D}$-attractor $\mathcal{A} = \mathcal{A}(\tau\ , \omega)$ is given $\mathcal{A}(\tau, \omega) = \bigcap_{r\geq 0}\overline{\bigcup_{t\geq r}\Phi(t, \tau - t, \theta_{-t}\omega, K(\tau - t, \theta_{-t}\omega))}$ $\tau \in \mathbb{R}\ , \omega \in \Omega$.*

3 Existence and Uniqueness of Solutions

In this section, we introduce some notations used throughout this paper. We also show that the existence and uniqueness of solutions of the systems (1.4)-(1.5). Let us introduce the following operator $A = -\triangle, D(A) \hookrightarrow \mathrm{v}_a^2(U) \hookrightarrow H_0^1(U) \to \mathbb{L}^2(U)$ it satisfies self-adjoint, positive and linear, the eigenvalue $\{\lambda_i\}_{i\in N}$ of A satisfies, $0 < \lambda_1 \leq \lambda_2 \leq .... \leq \lambda_i \leq ...., \lambda_i \to +\infty\ (i \to +\infty)$.

Notation 9. ***The following hypotheses are necessary to obtain our main results:*** *Refer and more details, see to[13, 14, 16, 17]*

$(\mathbf{A}_1)$ *Since $a(x) \in C^1(\bar{U})$ such that*

$$meas\{x \in \Gamma : a(x) > 0\} > 0, \tag{3.1}$$

and

$$\mathrm{v}_a^1 : u \in L^2(U) = \int_U a(x)|\nabla u(x)|^2 dx < \infty, u|\Gamma = 0,$$

is a Hilbert space endowed with the product

$$(\eta, \zeta)_{\mathrm{v}_a^1} := \int_U a(x)\nabla\eta(x, s)|.\nabla\zeta(x, s)|dx.$$

(Two example are given[13]). Above $\psi|\Gamma = 0$ is meant in the sense of trace which is well-defined when $\mathrm{v}_a^1(U) \hookrightarrow W^{1,1}(U)$. In addition, we also assume the following continuous embedding

$$H_0^1(U) \hookrightarrow \mathrm{v}_a^1(U) \hookrightarrow L^2(U)$$

$(\mathbf{A}_2)$ *Suppose that $a(x)$ and $b(x)$ are positive functions, $a(x) \in C^1(\bar{U}), b(x) \in L^\infty(U)$ and δ is a constant hold the following assumption*

$$a(x) + b(x) \geq \delta > 0,\ x \in\ U. \tag{3.2}$$

$(\mathbf{A}_3)$ *Concerning the memory kernel μ is required to verify the following hypotheses:*

$$\begin{cases} \mu \in \mathbb{C}^1(\mathbb{R}^+) \cap \mathbb{L}^1(\mathbb{R}^+), \mu(s) \geq 0, \mu'(s) \leq 0\ , \forall s \in \mathbb{R}^+, \\ \mu'(s) + \delta\mu(s) \leq 0\ , \forall s \in \mathbb{R}^+ and\ \delta > 0, \end{cases} \tag{3.3}$$

and denote

$$k_0 := \int_0^\infty \mu(s)ds < \|a\|_\infty^{-1}, \tag{3.4}$$

let condition (3.1) is standard and apply μ decays exponentially to zero. Due to(3.2)implies that $1 - a(x)k_0 \geq \beta > 0,\ \forall x \in \bar{U}$, such that we obtain

$$\beta = 1 - k_0\|a\|_\infty^{-1}. \tag{3.5}$$

($\mathbf{A}_4$) *For the nonlinear term $f \in \mathbb{C}^2(\mathbb{R})$ with $f(0) = 0$, we assume that f satisfy the following hypotheses:*
(a) For any $u, v \in \mathbb{R}$, there exists a constant $C_1 > 0$ such that

$$|f(u) - f(v)| \leq C_1(1 + |u|^{p-1} + |v|^{p-1})|u - v|. \tag{3.6}$$

Dissipation conditions
(b) There exists constants $\beta > 0$ and $\nu > 0$, and $C_f \in \mathbb{R}$ satisfying

$$\begin{cases} \beta F(u) - \nu u^2 - c_f \leq uf(u),\ \forall\ u \in \mathbb{R}, \\ \beta F(u) \geq -\nu u^2 - c_f,\ \forall\ u \in \mathbb{R}, \end{cases} \tag{3.7}$$

and

$$\lim_{|u|\to\infty} \inf \frac{f(u)}{u} \geq -\lambda_1\ , \forall\ u \in\ \mathbb{R}, \tag{3.8}$$

where $F(s) = \int_0^s f(r)dr$. Concerning the new regularity results described in section4, we additionally let us recall some important results satisfies along with the $\mathbf{A}_1 - \mathbf{A}_4$.
($\mathbf{A}_5$) *Assume $a(x) \in C^1(\bar{U})$ such that*

$$\mathrm{v}_a^2 : u \in L^2(U) = \int_U a(x)\left(|\triangle u(x)|^2 + |u(x)|^2\right) dx < \infty, u|\Gamma = 0,$$

is a Hilbert space endowed with the product

$$(\eta, \zeta)_{\mathrm{v}_a^2} := \int_U a(x)\left(|\triangle\eta(x, s)|.|\triangle\zeta(x, s)| + \eta(x, s).\zeta(x, s)\right) dx,$$

then, let continuous embedding holds

$$D(-\triangle) \hookrightarrow \mathrm{v}_a^2(U) \hookrightarrow H_0^1(U).$$

($\mathbf{A}_6$) *Suppose that strict inequality holds*

$$k_0\delta > 1. \tag{3.9}$$

Remark 10. assumption $(\mathbf{A}_1)$ and $(\mathbf{A}_5)$ allows us to set the space for the past history function $\eta^t(x,s)$. Indeed, define

$$\mathfrak{R}_0 := L^2_\mu(\mathbb{R}^+; \mathrm{v}^1_a) = \left\{ \eta : \int_0^\infty \mu(s) \, \|\eta(x,s)\|^2_{\mathrm{v}^1_a} \, ds < \infty \right\} \tag{3.10}$$

which is Hilbert space with the product

$$(\eta,\zeta)_{\mathfrak{R}_0} := \int_0^\infty \mu(s) \left(\int_U a(x) \nabla \eta(x,s)|.\nabla\zeta(x,s)| dx \right) ds,$$

and

$$\mathfrak{R}_1 := L^2_\mu(\mathbb{R}^+; \mathrm{v}^2_a) = \left\{ \eta : \int_0^\infty \mu(s) \, \|\eta(x,s)\|^2_{\mathrm{v}^2_a} \, ds < \infty \right\}, \tag{3.11}$$

which is Hilbert space with the product

$$(\eta,\zeta)_{\mathfrak{R}_1} := \int_0^\infty \mu(s) \left(\int_U a(x) \triangle \eta(x,s)|.\triangle\zeta(x,s)| dx \right) ds.$$

Finally, we introduce the following Hilbert space

$$E(U) = H^1_0(U) \times L^2(U) \times \mathfrak{R}_0(U)$$

endow with the usual inner product and norms like as $\varphi = (u, v, \eta) \in E$, respectively

$$\|\varphi\|^2_{E(U)} = \|u\|^2_{H^1_0(U)} + \|u\|^2_{L^2(U)} + \|\eta\|^2_{\mathfrak{R}_\mu(U)}.$$

And from Remark10, we also define the following Hilbert space

$$E_1(U) = H^2_0(U) \times H^\star(U) \times \mathfrak{R}_1(U)$$

endow with the norms is given by $\psi = (u, v, \eta) \in E_1$, respectively.

$$\|\varphi\|^2_{E(U)} = \|u\|^2_{H^1_0(U)} + \|u\|^2_{H^\star(U)} + \|\eta\|^2_{\mathfrak{R}_\mu(U)},$$

where $H^\star(U)$ is norm with

$$\left(|\nabla u(x)|^2 + |u(x)|^2\right).$$

Next, it is convenient to study the dynamical behavior of system(1.4)-(1.5), we need to convert the stochastic system into deterministic one with a random parameter, then show that it generates a random dynamical system. Due to Ornstein-Uhlenbeck process driving by the Brownian motion, which satisfying the *Itô* differential equation

$$dz + \delta z dt = dw, \;\; \delta > 0 \tag{3.12}$$

hance the solution is given by

$$z(\theta_t\omega) = z(t,\omega) \;=\; -\delta\int_{\infty}^{0} e^{\delta s}(\theta_t\omega)sds, s\in\mathbb{R},\omega\in\Omega. \tag{3.13}$$

Where the random variable $|z(\omega)|$ is tempered and there is a invariant set $\bar{\Omega}\subseteq\Omega$ of full P measure such that $z(\theta_t\omega) = z(t,\omega)$ is continuous in t for every $\omega\in\bar{\Omega}$, $\delta > 0$. This equation has a random fixed point in the sense of random dynamical systems generating a stationary solution known as the stationary Ornstein-Uhlenbeck process (see [8, 9, 6] for more details). In fact, we dedicate a cocycle of system(1.4)-(1.5), let

$$v = \frac{du}{dt} + \varepsilon u - cz(\theta_t\omega), \tag{3.14}$$

thus, apply (3.14) and (1.4)-(1,5), such that equivalent the following random system

$$\begin{cases} \dfrac{du}{dt} + \varepsilon u - v = cz(\theta_t\omega), \\ \dfrac{dv}{dt} + \varepsilon(\varepsilon - b(x))u - (\varepsilon - b(x))v - div[(1 - a(x)k_0)\nabla u \\ \qquad - \displaystyle\int_0^{\infty} \mu(s)div[a(x)\nabla\eta^t(s)]ds = -f(u) + g(x) + cz(\theta_t\omega)(2\varepsilon - b(x)), \\ \eta_t + \eta_s + \varepsilon u - v = cz(\theta_t\omega), \end{cases} \tag{3.15}$$

with boundary and initial conditions

$$\begin{cases} u(x,t) = 0, & on\ U\times\mathbb{R}^+, \\ \eta^t(x,t,s) = 0, & on\ U\times\mathbb{R}^+, \\ u(\tau,x) = u_0(x), & x\in U, \tau\in\mathbb{R}, \\ u_t(\tau,x) = u_1(x), & (x,\tau)\in U\times\mathbb{R}, \\ \eta^0(x,\tau,s) = u_0(x) - u_0(x,-s), & (x,\tau,s)\in U\times\mathbb{R}\times\mathbb{R}^+. \end{cases} \tag{3.16}$$

Suppose that $y = (u,v,\eta)^\top$, so that the following system is equivalent (3.15)-(3.16)

$$\begin{cases} y' + H(y) = Q(y,\omega) \\ y(0,\omega) = (u_1(x) + \varepsilon u_0(x) - cz(\theta_t\omega), \eta_0(x))^\top = (u_0(x), v_0(x), \eta_0)^\top = y_0, \end{cases} \tag{3.17}$$

where

$$H(y) = \begin{pmatrix} \varepsilon u - v \\ (\varepsilon - b(x))(\varepsilon u - v) - div[(1 - a(x)k_0)\nabla u - \int_0^{\infty}\mu(s)div[a(x)\nabla\eta^t(s)]ds \\ \varepsilon u - v + \eta_s \end{pmatrix},$$

and

$$Q(y,\omega) = \begin{pmatrix} cz(\theta_t\omega) \\ -f(u) + g(x) + cz(\theta_t\omega)(2\varepsilon - b(x)) \\ cz(\theta_t\omega) \end{pmatrix}.$$

In [3, 5], we know that -H is the infinitesimal generator of C° semigroup e^{-Ht} on E for $t > 0$, by the assumptions (3.17), it is easy to check $F(y, \omega; y_0) : E \to E$ is locally lipschitz continuous with respect to y, then by the classical semigroup theory concerning the (local) existence and uniqueness solution of evolution differential equation, we have the following Theorem.

Theorem 11. *Assume that* $(\mathbf{A}_1)$-$(\mathbf{A}_6)$ *and* $g(x) \in L^2(U)$ *hold. For any* $\tau \in \mathbb{R}\omega \in \Omega, t \geq 0$ *and* $y_0 \in E$*, then there exists* $T \geq \tau$ *such that (3.17) has a unique mild function* $y(.) = y(., \omega, y_0) \in C([0, T); E)$*, as so as* y_0 *and* $y(t)$ *satisfies the integral equation*

$$y(t, \omega, y_0) = e^{-H(t)} y_0(\omega) + \int_\tau^t e^{-H(t-r)} Q(y(r), \omega) dr, \tag{3.18}$$

in this case, $y(t, 0, \omega, y_0)$ *is called mild solution of (3.17) if furthermore, satisfies the following properties*
(1)-if $y_0(\omega) \in E$ *then* $y(.) = y(., \omega, y_0) \in C([0, \infty); E)$*,*
(2)-$y(t, \omega, y_0)$ *is jointly continuous into* t *and measurable in* $y_0(\omega)$*,*
(3)-the solution mapping of (3.17) holds the properties of RDS.

Proof. The local existence and uniqueness of solution are proved from [5], the existence of global solution is presenting in Section4 (see Lemma 4.1) and Theorem 3.1 in [5]. Then, the solution $y(., 0, \omega, y_0) \in C([0, +\infty, E)$ can define a continuous random dynamical system over $\mathbb{R}$ and $(\Omega, \digamma, P, (\theta_t)_{t \in \mathbb{R}})$.

$$\Psi : \mathbb{R}^+ \times \Omega \times E \to E, y(\tau, \omega) \to y(t, \omega). \tag{3.19}$$

To show the conjugation of the solutions of the stochastic partial differential equation (1.4)-(1.5) and the random partial differential equation (3.17), we introduce the isomorphism $T^1(\theta_t \omega) z = (z_1, z_2 - \varepsilon z_1 + \delta(\theta_t \omega), \eta_1)^\top$, $z = (z_1, z_2, \eta_1)^\top \in E$ which has inverse isomorphism $T^{-1}(\theta_t \omega) z = (z_1, z_2 + \varepsilon z_1 - \delta(\theta_t \omega), \eta_1)^\top$, then we have the following convert mapping

$$\Phi(t, \omega) = \bar{T} \Psi(t, \omega) \bar{T}^{-1}. \tag{3.20}$$

We will also use determines RDS (3.17) corresponding to equations (1.4)-(1.5), Such that

$$w_1 = u, \ \ w_2 = u_t + \varepsilon u, \tag{3.21}$$

similarly to (3.17), we obtain

$$\begin{cases} w'(t, \tau, w_0) + H(w(t, \tau, w_0)) = Q(w(t, \tau, w_0)) \\ w_0 = (u_0, v_0, \eta_0)^\top, (x, \tau, s) \in (x, \tau) \in U \times \mathbb{R} \times \mathbb{R}^+, \end{cases} \tag{3.22}$$

where $w = (w_1, w_2, \eta)^\top$,

$$H(w) = \begin{pmatrix} \varepsilon w_1 - w_2 \\ (\varepsilon - b(x))(\varepsilon w_1 - w_2) + div[(1 - a(x)k_0)\nabla w_1 - \int_0^\infty \mu(s) div[a(x)\nabla \eta^t(s)]ds \\ \varepsilon w_1 - w_2 + \eta_s \end{pmatrix},$$

and

$$Q(w) = \begin{pmatrix} 0 \\ -f(w_1) + q(x) \\ 0 \end{pmatrix}.$$

Here we also introduce the isomorphism $\bar{T}_\epsilon w = (w_1, w_2 - \varepsilon w_1, \eta_2)^\top$, $w = (w_1, w_2, \eta_2)^\top \in E$, which has inverse isomorphism $\bar{T}_{-\epsilon} w = (w_1, w_2 + \varepsilon w_1, \eta_2)^\top$ it follows that (θ, Ψ) with mapping

$$\bar{\Phi}(t, \omega, y_0) = T_\epsilon \Psi(t, \omega, y_0) T_{-\epsilon}. \tag{3.23}$$

Two RDS are equivalent and corresponding to RDS (3.17) □

.

4 Random Absorbing Set

In this section, we will prove the existence of a random absorbing set for the RDS $\{y(t, \omega, y_0), t \geq 0\}$ in the space E. Let $y = (u, v, \eta)^\top = (u, u_t + \varepsilon u - cz(\theta_t \omega), \eta)^\top$, where ε is chosen as

$$\varepsilon = \frac{\alpha\lambda_1 + \beta_1}{4 + 2(\alpha\lambda_1 + \beta_1)\alpha + \beta_2^2/\lambda_1}, \tag{4.1}$$

hence (3.17) follows, there exists equivalent system such that

$$\begin{cases} y'(t, \omega, y_0) + H\left(y(t, \omega, y_0)\right) = \bar{F}\left(\varphi(t, \omega, y_0), \omega\right), \\ y_0 = (u_0, v_0, \eta_0)^\top, \end{cases} \tag{4.2}$$

where

$$H(y) = \begin{pmatrix} \varepsilon u - v \\ (\varepsilon - b(x))(\varepsilon u - v) - div[(1 - a(x)k_0)\nabla u - \int_0^\infty \mu(s) div[a(x)\nabla \eta^t(s)]ds \\ \varepsilon u - v + \eta_s \end{pmatrix},$$

and

$$\bar{F}(y, \omega, t) = \begin{pmatrix} cz(\theta_t \omega) \\ -f(u) + g(x) + cz(\theta_t \omega)(2\varepsilon - b(x)) \\ cz(\theta_t \omega) \end{pmatrix}.$$

Infer from (4.2), we have the following Lemma

Lemma 12. *For any $y = (u, v, \eta)^\top \in E$ we have*

$$(H(y), y)_E \geq \frac{\varepsilon}{2}(\|u\|_1^2 + \|v\|^2) + \frac{\delta}{2}\|v\|^2 + \frac{\varepsilon}{4}\|\eta\|_{\mu,1}^2, \tag{4.3}$$

Proof. This is easily obtained after simple computation. □

Lemma 13. *Assume that $\mathbf{A}_1 - \mathbf{A}_6$ hold. For any $\tau \in \mathbb{R}, \omega \in \Omega$ and $t \geq 0$, there is exits a closed tempered bounded absorbing ball $B_0\omega = \{y \in E : \|y\| \leq M_0(\omega) = B_E(0, M_0(\omega) \in \mathcal{D}(E)\} \subset E$, centered at 0 with random radius $M_0(\omega) > 0$ such that for each bounded non-random set $B \subset E$, there exists a deterministic time $T_B = T(\omega, B, M_0) \geq \tau$ such as the solution $y(t, \omega; y_0(\omega))$ of system (4.2) with initial value $y_0 = (u_0, v_0, \eta_0)^\top \in B$ satisfies for P-a.s.$\omega \in \Omega, t \geq 0$,*

$$\|y\,(t, \omega; y_0(\omega))\|_E^2 \leq M_0^2(\omega),$$

that is

$$\Phi\,(t, \omega; B(\theta_{-t}\omega)) \subseteq B_0(\omega)\ ,\ \forall\ t \geq \tau. \tag{4.4}$$

Proof. Multiplying (4.2) by y, and integrating over U, then implies to Lemma12, we can get

$$\frac{1}{2}\frac{d}{dt}\|y\|_E^2 + \frac{\varepsilon}{2}(\|u\|_1^2 + \|v\|^2) + \frac{\delta}{2}\|v\|^2 + \frac{\varepsilon}{4}\|\eta\|_{\mu,1}^2 \leq (\bar{F}(y, \omega), y). \tag{4.5}$$

Let us estimate any term of the right-hand side of (4.5) as follow as,

$$\begin{aligned}(\bar{F}(y, \omega), y) = {} & ((cuz(\theta_t\omega), u)) + (-f(u) + g(x) + cz(\theta_t\omega)(2\varepsilon - b(x)), v) \\ & + (cz(\theta_t\omega), \eta)_{\mu,1}.\end{aligned} \tag{4.6}$$

By the Cauchy-Schwartz inequality, we find that

$$((cuz(\theta_t\omega), u)) \leq |c||z(\theta_t\omega)|\|\nabla u\|^2 \leq |c||z(\theta_t\omega)|\|u\|_1^2, \tag{4.7}$$

$$(2\varepsilon cz(\theta_t\omega), v) \leq |\varepsilon|^2|c|^3|z(\theta_t\omega)|^3 + |c||z(\theta_t\omega)|\|v\|^2, \tag{4.8}$$

$$\frac{\delta}{2}\,(cz(\theta_t\omega), v) \leq \frac{\delta}{8}|c|^2|z(\theta_t\omega)|^2 + \frac{\delta}{2}\|v\|^2, \tag{4.9}$$

$$(cz(\theta_t\omega), \eta)_{\mu,1} \leq \frac{1}{4}|c|^3|z(\theta_t\omega)|^3 + |c||z(\theta_t\omega)|\|\eta\|_{\mu,1}^2, \tag{4.10}$$

$$(q(x), v) \leq \|q(x)\|\ \|v\| \leq \frac{1}{(4\delta + \varepsilon)}\|q(x)\|^2 + \frac{(4\delta + \varepsilon)}{4}\|v\|^2, \tag{4.11}$$

next due to (1.6)-(1.10) and the Hölder inequality, it yields

$$\begin{aligned}(f(u), v) &= \left(f(u), \frac{du}{dt} + \varepsilon u - cz(\theta_t\omega)\right) \\ &\geq \tfrac{d}{dt}\tilde{F}(u) + \varepsilon(f(u), u) - (f(u), cz(\theta_t\omega)).\end{aligned} \tag{4.12}$$

Moreover, by assumptions (3.6)-(3.7) then, we have

$$
\begin{aligned}
(f(u), cz(\theta_t\omega)) &\leq C_1 \int_U |c|\left(1+|u|^p\right)|z(\theta_t\omega)|dx \\
&\leq C_1|c|\left(\int_U |u|^{P+1}dx\right)^{\frac{p}{P+1}} |z(\theta_t\omega)|_{P+1} + C_1|c|\int_U |z(\theta_t\omega)|dx \\
&\leq C_1|c|\int_U |z(\theta_t\omega)|dx + \beta C_1^{-\frac{p}{P+1}}\left(\int_U (F(u)+C_1)dx\right)^{\frac{p}{P+1}} |z(\theta_t\omega)|_{P+1} \\
&\leq C_3|z(\theta_t\omega)| + \tfrac{\varepsilon C_2}{2}\int_U F(u)dx + \tfrac{\varepsilon C_1 C_2|U|}{2} + \beta\|z(\theta_t\omega)\|_{P+1}^{P+1}
\end{aligned}
$$

, (4.13)

and from (3.7)-(3.8), there exists positive constant μ_1, so that

$$
(f(u), u) - \beta\tilde{F}(u) + \frac{\lambda}{4}\|u\|_1 + \mu_1 \geq 0, \tag{4.14}
$$

it follows from (4.12)-(4.14) and poincarè inequality, we can get

$$
\begin{aligned}
(f(u), v) \leq \frac{d}{dt}\tilde{F}(u) + \frac{\varepsilon C_2}{2}\tilde{F}(u) - \frac{\varepsilon\lambda}{4}\|u\|_1 \\
+ \varepsilon\mu_1 + C_3|z(\theta_t\omega)| + \tfrac{\varepsilon C_1 C_2|U|}{2} + \beta\,|z(\theta_t\omega)|_{P+1}^{P+1}.
\end{aligned} \tag{4.15}
$$

Where $\tilde{F}(u) = \int_U F(u)dx$. Together form (4.7)-(4.15) and (4.6), we show that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|u\|_1^2 + \|v\|^2 + \|\eta\|_{\mu,1}^2 + 2\tilde{F}(u) + c_0 c_2|U|\right) \\
&\quad\leq -\left(\tfrac{\varepsilon}{4} - |c||z(\theta_t\omega)|\right) \times (\|u\|_1^2 + \|v\|^2 + \|\eta\|_{\mu,1}^2) \\
&\quad - \tfrac{\varepsilon C_2}{2}\tilde{F}(u)\tilde{F}(u) + \tfrac{\varepsilon C_1 C_2|U|}{2} + \varepsilon\mu_1 + C_3|z(\theta_t\omega)| \\
&\quad + \beta|z(\theta_t\omega)|_{P+1}^{P+1} + \tfrac{\delta}{8}|c|^2|z(\theta_t\omega)|^2 \\
&\quad + \tfrac{1}{4}|c|^3|z(\theta_t\omega)|^3 + |\varepsilon|^2|c|^3|z(\theta_t\omega)|^3 + \tfrac{1}{(4\delta+\varepsilon)}\|q(x)\|^2.
\end{aligned} \tag{4.16}
$$

Suppose $\sigma = \min[\frac{\varepsilon}{4}, \frac{\varepsilon\beta}{2}]$ and assume $\|y\|^2 = (\|u\|_1^2 + \|v\|^2 + \|\eta\|_{\mu,1}^2)$, we have the following equivalent system

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|y\|_E^2 + 2\tilde{F}(u) + \frac{C_1 C_2|U|}{2}\right) \\
&\quad\leq (-\sigma + |c||z(\theta_t\omega)|)\left(\|y\|_E^2 + \tilde{F}(u) + \tfrac{\varepsilon C_1 C_2|U|}{2}\right) \\
&\quad + \varepsilon\mu_1 + C_3|z(\theta_t\omega)| + \beta|z(\theta_t\omega)|_{P+1}^{P+1} + \tfrac{\delta}{8}|c|^2|z(\theta_t\omega)|^2 \\
&\quad + \tfrac{1}{4}|c|^3|z(\theta_t\omega)|^3 + |\varepsilon|^2|c|^3|z(\theta_t\omega)|^3 + \tfrac{1}{(4\delta+\varepsilon)}\|q(x)\|^2.
\end{aligned} \tag{4.17}
$$

Since $\Gamma(\omega) = \sigma - |c||z(\theta_t\omega)|$ and $|z(\theta_t\omega)|$ is tempered random variable. Hence by (3.2) and (3.3), we choose the following inequality

$$
\rho(\theta_t\omega) = \alpha(1 + |z(\theta_t\omega)| + |z(\theta_t\omega)|^2 + |c|^3|z(\theta_t\omega)|^3 + |z(\theta_t\omega)|_{P+1}^{P+1} + \|q(x)\|^2),
$$

where α depend only on $\alpha = \max\left[\mu_1, \beta, \frac{\varepsilon C_1 C_2}{2}, \frac{1}{4}, C, |\varepsilon|^2, \frac{\delta}{8}, \frac{1}{(4\delta+\varepsilon)}\right]$.
By applying Gronwall's inequality to (4.17) over $[0,t]$ and replacing ω to $\theta_{-t}\omega$ we have

$$\begin{aligned} \|y(t,\theta_{-t}\omega;y_0)\|_E^2 &\le \left(\|y_0\|_E^2 + 2\tilde{F}(u(0)\right) \\ &\le \left(\|y_0\|_E^2 + 2\tilde{F}(u_0)\right) e^{2\int_\tau^t \Gamma(r-t,\omega)dr} \\ &\quad + \int_\tau^t \rho(\theta_r\omega) e^{2\int_t^r \Gamma(s-t,\omega)ds} dr. \end{aligned} \tag{4.18}$$

Suppose that

$$\begin{aligned} \varphi\left(t,\theta_{-t}\omega;y_0(\theta_{-t}\omega)\right) &= \left(\|y(t,\theta_{-t}\omega;y_0(\theta_{-t}\omega))\|_E^2 + 2\tilde{F}(u(t;y_0(\theta_{-t}\omega)))\right) \\ &\ge \|y(t,\omega;y_0(\theta_{-t}\omega))\|_E^2 \ge 0. \end{aligned}$$

Using (1.8), Young's inequality and the embedding theorem, we have for any bounded set B of E, where $\sup_{y\in B}\|y\|_E \le M(\omega)$, if $y_0 \in B$,

$$\begin{aligned} 2\tilde{F}(u) &\le \beta \int_U (f(u)+1)u dx \\ &\le \beta \int_U (f(u)u dx + C_3 \int_U u dx \\ &\le \beta\|u\|^2 + k\|u\|_{H^1}^{P+1} \le \beta\|u\|^2 + k\|u\|_{H^1}^{P+1} \\ &\le k r_m(\omega), \end{aligned} \tag{4.19}$$

For any set $\{B(\omega):\omega\in\Omega\} \in U$ $y_0 = (u_0(x), u_1(x) + \varepsilon u_0(x) - c u_0 z(\theta_t\omega))^\top \in \{B(\omega):\omega\in\Omega\} \in \mathcal{D}(E)$,
we have

$$\lim_{t\to\infty} \sup(\|y_0(\theta_{-t}\omega)\|_E^2 + 2C_2(\|u_0\|^2 + \|u_0\|_{H^1}^{P+2}))e^{-2\sigma t} = 0, \tag{4.20}$$

which is a tempered random variable, then by (4.19)-(4.20) there $B_0(\omega) = \{y \in E : \|y_0(\theta_{-t}\omega)\|_E \le M^2(\omega)\}$ is closed measurable absorbing ball in $\mathcal{D}(E)$, and there exists $T_B = T(\omega, B, M_0) \ge 0$ such that $y\left(t,\theta_{-t}\omega;y_0(\theta_{-t}\omega)\right) = y_0(\theta_{-t}\omega) \in B_0(\omega),\ \forall\, t \ge 0$, then satisfies the following result

$$\|y\left(t,\theta_{-t}\omega;y_0(\theta_{-t}\omega)\right)\|_E^2 \le M_0^2(\omega).$$

we complete the proof. □ □

5 Decomposition of Equations

In order to obtain regularity estimates later, we decompose the equations (1.4), by decomposing the nonlinear term. At first.
The nonlinearity f we will give the following decomposition

$f = f_0 + f_1$, where $f_0, f_1 \in \mathbb{C}^1$ satisfies the following decomposing properties for some proper constants: there is a constant $C > 0$ such that

$$\begin{cases} |f_0(u) - f_0(v)| \leq C\left(1 + |u|^4 + |v|^4\right)|u - v|\,,\ \forall\ u, v \in\ \mathbb{R}, \\ f_0(s)s \geq F_0(s) \geq 0\ , \\ \exists \beta_0 \geq 1,\ \vartheta_1 \geq 0 such\ that\ \forall\ \vartheta \in (0, \vartheta_1], \\ \forall\ c_\vartheta \in \mathbb{R}\ ,\ \beta_0 F_0(s) + \vartheta s^2 - c_\vartheta \leq s f_0(s),\ \forall\ s \in \mathbb{R}, \\ F_i(s) = \displaystyle\int_0^s f_i(r)dr,\ i = 0, 1. \end{cases} \tag{5.1}$$

and

$$|f_1(s)| \leq C(1 + |s|^\gamma)\ ,\ \forall\ s \in\ \mathbb{R}. \tag{5.2}$$

Where $\beta_0, \vartheta_1, \vartheta, c_\vartheta$ are positive constants. Let $y = (u, v, \eta^t)$ is a solutions of (3.17), then we decompose in two parts

$$y = \tilde{y} + \bar{y},$$

where $\tilde{y} = (\tilde{u}, \tilde{v}, \tilde{\eta})$,$\bar{y} = (\bar{u}, \bar{v}, \bar{\eta})$

$$\begin{cases} \tilde{y}'(t, \omega, \tilde{y}_0) + H\left(\tilde{y}(t, \omega, \tilde{y}_0)\right) = \bar{F}_1\left(\tilde{y}(t, \omega, \tilde{y}_0), \omega\right), \\ \tilde{y}_0 = (u_0, v_0, \eta_0)^\top, \end{cases} \tag{5.3}$$

where

$$H(\tilde{y}) = \begin{pmatrix} \varepsilon\tilde{u} - \tilde{v} \\ (\varepsilon - b(x))\left(\varepsilon\bar{u} - \tilde{v}\right) - div[(1 - a(x)k_0)\nabla\tilde{u} - \int_0^\infty \mu(s)div[a(x)\nabla\tilde{\eta}^t(s)]ds \\ \varepsilon\tilde{u} - \tilde{v} + \tilde{\eta}_s \end{pmatrix}$$

and

$$\bar{F}_1(\tilde{y}, \omega, t) = \begin{pmatrix} 0 \\ -f(\tilde{u}) + g(x) \\ 0 \end{pmatrix}$$

$$\begin{cases} \bar{y}'(t, \omega, y_0) + H\left(\bar{y}(t, \omega, y_0)\right) = \bar{F}_2\left(\bar{y}(t, \omega, y_0), \omega\right), \\ \bar{y}_0 = (u_0, v_0, \eta_0)^\top, \end{cases} \tag{5.4}$$

where

$$H(\bar{y}) = \begin{pmatrix} \varepsilon\bar{u} - \bar{v} \\ (\varepsilon - b(x))\left(\varepsilon\bar{u} - \bar{v}\right) - div[(1 - a(x)k_0)\nabla\bar{u} - \int_0^\infty \mu(s)div[a(x)\nabla\bar{\eta}^t(s)]ds \\ \varepsilon\bar{u} - \bar{v} + \bar{\eta}_s \end{pmatrix},$$

and

$$\bar{F}_2(\bar{y}, \omega, t) = \begin{pmatrix} cz(\theta_t\omega) \\ -(f(u) - f_0(\tilde{u})) + g(x) + cz(\theta_t\omega)(2\varepsilon - b(x)) \\ cz(\theta_t\omega) \end{pmatrix}.$$

To prove the existence of a compact random attractor for the RDS Φ, we get the solutions of systems (5.3) and (5.4) are similar to solution of a system (4.2), which one decays exponentially and another is bounded in higher regular space. In order to get the regularity estimate, we will prove some priori estimate for the solutions u(x,t) of systems (5.3)-(5.4) on $U \times [0, \infty)$ into two parts.
Let $y(t, \theta_{-t}\omega; y_0(\omega)) = \Phi_\varepsilon(t, \omega)y_0(\omega)$ be the solution of (3.17) with $y_0(\omega) \in B_0$ set $y = \tilde{y} + \bar{y}$ are the basis of absorbing set Φ, suppose that $\hat{T} = \hat{T}(B_0, \omega)$

$$\begin{aligned} B_1(\omega) &= \sqcup_{t \geq \hat{T}} y\left(t, \theta_{-t}\omega; y_0(\omega)\right) \\ &= y_0(\omega) \in B_1(\omega) \subseteq B_0(\omega),\ \forall\, t \geq \hat{T}, \end{aligned} \tag{5.5}$$

for any $\omega \in \Omega$, where $\hat{T} = \hat{T}(B_0, \omega) \geq \tau$ is the pullback absorbing time in Lemma13, then it holds $B_1(\omega) \subseteq B_0(\omega)$ that

$$\begin{aligned} \Phi(t, \theta_{-t}\omega; B_1(\theta_{-t}\omega)) &= y(t, \theta_{-t}\omega; B_1(\theta_{-t}\omega)) \\ &\subseteq B_1(\omega) \subseteq B_0(\omega),\ \forall\, t \geq \hat{T}. \end{aligned} \tag{5.6}$$

Lemma 14. *Let* $(\mathbf{A}_1)$-$(\mathbf{A}_6)$ *and* $g(x) \in L^2(U)$ *hold. For any* $\tau \in \mathbb{R}, \omega \in \Omega, t \geq 0$ *and* $y_0 \in E$*, there exists* $M_1(\omega) > 0$ *such that* $\tilde{y}(r) = \tilde{y}(r, \omega; y_0)$ *is solution of the system(5.3) satisfies*

$$\|\tilde{y}(r, \omega; y_0)\|_E^2 \leq M_1(\omega),\ r \geq \tau \tag{5.7}$$

Proof. Taking the inner product of (5.3) in $L^2(U)$ with $\tilde{y}$ in E, we show that

$$\frac{1}{2}\frac{d}{dt}\|\tilde{y}\|_E^2 + \left(\tilde{H}(\tilde{y}), \tilde{y}\right) + \left(\tilde{F}_1(\tilde{y}), \tilde{y}\right) = 0, \tag{5.8}$$

by Lemma4.1, we have

$$(\tilde{H}(\tilde{y}), \tilde{y}) \geq \frac{\varepsilon}{2}(\|\tilde{u}\|_1^2 + \|\tilde{v}\|^2) + \frac{\delta}{2}\|\tilde{v}\|^2 + \frac{\varepsilon}{4}\|\tilde{\eta}\|_{\mu,1}^2, \tag{5.9}$$

where ε satisfy (4.6). Now, we estimate the third term of (5.8), such that

$$\begin{aligned} (\tilde{F}(\tilde{y}), \tilde{y}) &= \begin{pmatrix} 0 \\ f_0(\tilde{u}) \\ 0 \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{\eta} \end{pmatrix} \\ &= (f_0(\tilde{u}), \tilde{u}_t + \varepsilon\tilde{u}) \\ &= \tfrac{d}{dt}\tilde{F}_0(\tilde{u}) + \varepsilon\int_U f_0(\tilde{u})\tilde{u}dx. \end{aligned} \tag{5.10}$$

In view of $(5.1)_2$ and $(5.1)_3$, we can get

$$\begin{aligned} & f_0(\tilde{u})\tilde{u} \geq F_0(\tilde{u}) \geq 0 \\ & \tfrac{d}{dt}\tilde{F}_0(\tilde{u}) + \varepsilon\int_U f_1(\tilde{u})\tilde{u}dx \geq \tfrac{d}{dt}\tilde{F}_0(\tilde{u}) + \varepsilon(\beta_0\tilde{F}_0(\tilde{u}) + \vartheta\|\tilde{u}\|^2 - c_\vartheta) \\ & \qquad \geq \tfrac{d}{dt}\tilde{F}_0(\tilde{u}) + \beta_0\varepsilon F_0(\tilde{u}) + \varepsilon\vartheta\|\tilde{u}\|^2 - \varepsilon c_\vartheta. \end{aligned} \tag{5.11}$$

Thus, combining (5.9)-(5.11) and (5.8) yield

$$\frac{d}{dt}\left(\|\tilde{y}\|_E^2 + 2\tilde{F}_0(\tilde{u})\right) + 2\tilde{\sigma}\left(\|\tilde{y}\|_E^2 + 2\tilde{F}_0(\tilde{u})\right) \le \rho, \tag{5.12}$$

where $\rho = \varepsilon c_\vartheta$ and $\tilde{\sigma} = \min(\frac{\varepsilon}{2}, \frac{\alpha}{2}, \frac{\varepsilon}{4}, \beta_0\varepsilon)$

$$\|\tilde{y}\|_E^2 + 2\tilde{F}_0(\tilde{u}) \ge \|\tilde{y}\|_E^2 \ge 0, \tag{5.13}$$

and hence $y_0(\theta_{-t}\omega) \in M_0(\omega)$. It follows that

$$\begin{aligned}\tilde{y}_0 &= (y_0(\theta_{-t}\omega) + cz(\theta_{-t}\omega))^\top \\ &\le (M_0(\omega) + cz(\theta_{-t}\omega)) \in B_0(\theta_{-t}\omega).\end{aligned}$$

By definition of $B_0(\omega)$ and Lemma13, satisfying the below inequality

$$\|\tilde{y}(r, \omega; \tilde{y}_0)\|_E \le M_1(\omega). \tag{5.14}$$

□

Lemma 15. *Let* $(\mathbf{A}_1)$-$(\mathbf{A}_6)$ *and* $g(x) \in L^2(U)$ *hold. For any* $\tau \in \mathbb{R}, \omega \in \Omega$ *and* $t \ge 0$, *there exists a random variable radius* $M_2(\omega) > 0$, *such that the solution of a system (5.4) satisfy the following estimates* $\bar{y}(r) = \bar{y}(r, \theta_{-t}\omega; \bar{y}_0)$, *for all* $t \ge r,\ r \ge \tau$.

$$\|A^{\frac{\nu}{2}}\bar{y}(t, \theta_{-t}\omega; \bar{y}_0)\|_E^2 \le M_2(\omega), \tag{5.15}$$

we define that

$$\nu = \min\{\frac{1}{3}, \frac{4-\gamma}{2}\},\ \forall\ 0 \le \gamma \le 4. \tag{5.16}$$

Proof. Taking inner product in E of (5.4) with $A^\nu\bar{y}(r)$, then positively

$$(\bar{y}', A^\nu\bar{y}) + \left(\bar{H}(\bar{y}), A^\nu\bar{y}\right) = (\bar{F}_2(\bar{y}, \omega), A^\nu\bar{y}), \tag{5.17}$$

inserting (5.16) and Lemma12, we obtain

$$\bar{H}(\bar{y}, A^\nu\bar{y}) = \frac{\varepsilon}{2}(\|A^{\frac{1+\nu}{2}}\bar{u}\|^2 + \|A^{\frac{\nu}{2}}\bar{v}\|^2) + \frac{\delta}{2}\|A^{\frac{\nu}{2}}\bar{v}\|^2 + \frac{\varepsilon}{4}\|A^{\frac{\nu}{2}}\bar{\eta}\|_{\mu,1}^2, \tag{5.18}$$

next, we will estimate the right-hand side of (5.17) we get

$$\begin{aligned}(\bar{F}_2(\bar{y}, \omega), A^\nu\bar{y}) = &((cz(\theta_t\omega), A^\nu\bar{u})) \\ &+ (-(f(u) - f_0(\tilde{u}) + g(x) + cz(\theta_t\omega)(2\varepsilon - b(x)), A^\nu\bar{v}) \\ &+ (cz(\theta_t\omega), A^\nu\bar{\eta})_{\mu,1}.\end{aligned} \tag{5.19}$$

By using Hölder's inequality, Young's inequality and Poincaré's inequality, we get

$$((cz(\theta_t\omega), A^\nu u)) \le |c|\left\|A^{\frac{\nu}{2}}z(\theta_t\omega)\right\|\left\|A^{\frac{\nu}{2}}\nabla u\right\|^2 \le |c|\left\|A^{\frac{\nu}{2}}z(\theta_t\omega)\right\|\left\|A^{\frac{\nu}{2}}u\right\|_1^2, \tag{5.20}$$

$$(2\varepsilon cz(\theta_t\omega), A^\nu v) \leq |\varepsilon|^2 |c|^3 \left\| A^{\frac{\nu}{2}} z(\theta_t\omega) \right\|^3 + |c| \left\| A^{\frac{\nu}{2}} z(\theta_t\omega) \right\| \|A^{\frac{\nu}{2}} v\|^2, \tag{5.21}$$

$$\frac{\delta}{2}(cz(\theta_t\omega), A^\nu v) \leq \frac{\delta}{8}|c|^2 \left\| A^{\frac{\nu}{2}} z(\theta_t\omega) \right\|^2 + \frac{\delta}{2}\|A^{\frac{\nu}{2}} v\|^2, \tag{5.22}$$

$$(cz(\theta_t\omega), A^\nu \eta)_{\mu,1} \leq \frac{1}{4}|c|^3 \left\| A^{\frac{\nu}{2}} z(\theta_t\omega) \right\|^3 + |c| \left\| A^{\frac{\nu}{2}} z(\theta_t\omega) \right\| \left\| A^{\frac{\nu}{2}} \eta \right\|^2_{\mu,1}, \tag{5.23}$$

$$(q(x), A^\nu v) \leq \left\| A^{\frac{\nu}{2}} q(x) \right\| \; \left\| A^{\frac{\nu}{2}} v \right\| \leq \frac{1}{(4\delta+\varepsilon)} \left\| A^{\frac{\nu}{2}} q(x) \right\|^2 + \frac{(4\delta+\varepsilon)}{4} \left\| A^{\frac{\nu}{2}} v \right\|^2, \tag{5.24}$$

for nonlinear term, we write

$$\begin{aligned}(f(u) - f_0(\tilde{u}), A^\nu \bar{v}) &= (f(u) - f_0(\tilde{u}), A^\nu(\bar{u}_t + \varepsilon\bar{u} - cz(\theta_t\omega))) \\ &\leq \tfrac{d}{dt}\textstyle\int_U (f(u) - f_0(\tilde{u}))A^\nu \bar{u}dx + \int_U (f(u) - f_0(\tilde{u}))A^\nu \bar{u}dx \\ &\quad - \textstyle\int_U (f'(u)u_t - f_0'(\tilde{u})\tilde{u}_t)A^\nu \bar{u}dx \\ &\quad - \textstyle\int_U (f(u) - f_0(\tilde{u}))A^\nu z(\theta_t\omega)dx.\end{aligned}$$

Next, infer to assumption $\mathbf{A}_4$, (5.1)-(5.2), thanks to the Cauchy-Schwartz, the Young's inequality and using the embedding theorem $H_{1+\nu} \subset L^{\frac{6}{1-2\nu}}$, $H_{1-\nu} \subset L^{\frac{6}{1+2\nu}}$ and $H_1 \hookrightarrow L^6$, we obtain

$$\begin{aligned}\int_U (f(u) - f_0(\tilde{u}))A^\nu |z(\theta_t\omega)|dx &\leq \frac{\varepsilon}{4}\|f(u) - f_0(\tilde{u})\| + \frac{1}{\varepsilon}\|A^\nu z(\theta_t\omega)\|^2 \\ &\leq \tfrac{C\varepsilon}{4}\left(1 + \|u\|^5 + \|\tilde{u}\|^5\right) + \tfrac{1}{\varepsilon}\|A^\nu z(\theta_t\omega)\|^2 \\ &\leq R_1(\omega) + \tfrac{1}{\varepsilon}\|A^\nu z(\theta_t\omega)\|^2,\end{aligned} \tag{5.25}$$

$$\int_U (f'(u)u_t - f_0'(\tilde{u})\tilde{u}_t)A^\nu \bar{u}dx = \int_U ((f_0'(u) - f_0'(\tilde{u}))u_t + f_0'(\tilde{u})\bar{u}_t + f_1'(u)u_t)A^\nu \bar{u}dx,$$

estimate the above inequality, we get

$$\begin{aligned}\int_U (f_0'(u) - f_0'(\tilde{u}))u_t A^\nu \bar{u}dx &\leq C\int_U (f_0''(u + \theta(u-\tilde{u}))|u - \tilde{u}||u_t||A^\nu \bar{u}|dx \\ &\leq C\textstyle\int_U \left(1 + |u|^3 + |\tilde{u}|^3\right)|\bar{u}||A^\nu \bar{u}||u_t|dx \\ &\leq C\left(1 + \|u\|^3_{L^6} + \|\tilde{u}\|^3_{L^6}\right)\|\bar{u}\|_{L^{\frac{6}{1-2\nu}}}\|A^\nu \bar{u}\|_{L^{\frac{6}{1+2\nu}}}\|u_t\|_{L^2} \\ &\leq R_2(\omega)\|A^{\frac{1+\nu}{2}}\bar{u}\| \\ &\leq 4\varepsilon R_2^2(\omega) + \tfrac{\varepsilon}{16}\|A^{\frac{1+\nu}{2}}\bar{u}\|^2,\end{aligned} \tag{5.26}$$

$$\begin{aligned}\int_U f_0'(\tilde{u})\bar{u}_t A^\nu \bar{u}dx &\leq C(1 + \|\tilde{u}\|^4_{L^6})\|A^{\frac{1+\nu}{2}}\bar{u}\|_{L^{\frac{6}{1+2\nu}}}\|A^\nu \bar{u}_t\|_{L^{\frac{6}{3+2\nu}}} \\ &\leq C(1 + \|\tilde{u}\|^4_{L^6})\|A^{\frac{1+\nu}{2}}\bar{u}\|_{L^{\frac{6}{1+2\nu}}}\|A^\nu \bar{u}_t\|_{L^{\frac{6}{3+2\nu}}} \\ &\leq 4\varepsilon R_3(\omega)(\|A^{\frac{\nu}{2}}\bar{u}\|^2 + |\varepsilon|^2) + \tfrac{\varepsilon}{16}\|A^{\frac{1+\nu}{2}}\bar{u}\|^2_{L^{\frac{6}{1+2\nu}}},\end{aligned} \tag{5.27}$$

furthermore, note that $\nu \leq \frac{5-\gamma}{2}$

$$\begin{aligned}\int_U f_1'(u)u_t A^\nu \bar{u}dx &\leq C\int_U (1+|u|^\gamma)|u_t||A^\nu \bar{u}|dx\\ &\leq C(1+\|u\|^\gamma_{L^{\frac{6p}{5-2\nu}}})\|A^\nu \bar{u}\|_{L^{\frac{6}{1+2\nu}}}\|u_t\|_{L^2}\\ &\leq C(1+\|\nabla u\|^\gamma)\|A^\nu \bar{u}\|_{L^{\frac{6}{1+2\nu}}}\|u_t\|_{L^6}\\ &\leq 2\varepsilon R_4^2(\omega)+\tfrac{\varepsilon}{8}\|A^{\frac{1+\nu}{2}}\bar{u}\|^2,\end{aligned} \tag{5.28}$$

Exploiting the decay estimate together (5.18)-(5.28) with (5.17), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|A^{\frac{\nu}{2}}\bar{y}\|^2_E+2(f(u)-f_0(\tilde{u}))\right)+\frac{\varepsilon}{4}\|A^{\frac{\nu}{2}}\bar{y}\|^2_E+\frac{\beta\varepsilon}{2}(f(u)-f_0(\tilde{u}))\\ &\quad\leq |c||z(\theta_t\omega)|\|A^{\frac{\nu}{2}}\bar{y}\|^2_E+\alpha_1[1+R_1^2(\omega)+R_2^2(\omega)\\ &\qquad+R_3^2(\omega)+R_4^2(\omega)+\|A^\nu z(\theta_t\omega)\|^2+\|A^\nu z(\theta_t\omega)\|^3+\|A^{\frac{\nu}{2}}q(x)\|^2].\end{aligned} \tag{5.29}$$

According to Gronwall's inequality in (5.29) on $[0,r]$ and replacing ω to $\theta_{-t}\omega$, can be rewriting as

$$\begin{aligned}&\|A^{\frac{\nu}{2}}\bar{y}(t,\theta_{-t}\omega;y_0)\|^2_E\\ &\qquad\leq \left(\|A^{\frac{\nu}{2}}\bar{y}(r,\theta_{-t}\omega;y_0)\|^2_E+2(f(u(r,\theta_{-t}\omega;y_0))-f_0(\tilde{u}(r,\theta_{-t}\omega;y_0)))\right)\\ &\qquad\leq \left(\|A^{\frac{\nu}{2}}\bar{y}\|^2_E+(f(u)-f_0(\tilde{u}))\right)exp^{2\int_r^0(\sigma-|c||z(\theta_s\omega)|)(s,\omega)ds}\\ &\qquad\quad+\int_0^r\rho_1(\theta_s\omega)exp^{2\int_r^s(\sigma-|c||z(\theta_\varsigma\omega)|)(\varsigma,\omega)d\varsigma}ds.\end{aligned} \tag{5.30}$$

We can define $\rho_1(\theta_t\omega)$, as follows

$$\begin{aligned}\rho_1(\theta_t\omega)&=\alpha_1[1+R_1^2(\omega)+R_2^2(\omega)+R_3^2(\omega)+R_4^2(\omega)\\ &\quad+\|A^\nu z(\theta_t\omega)\|^2+\|A^\nu z(\theta_t\omega)\|^3+\|A^{\frac{\nu}{2}}q(x)\|^2].\end{aligned} \tag{5.31}$$

Observe that

$$\begin{aligned}\int_U((f(u)-f_0(\tilde{u}))A^\nu\bar{u}dx &\leq \int_{\bar{U}}((f_0(u)+f_1(u)-f_0(\tilde{u}))\,A^\nu\bar{u}dx\\ &\leq C\int_U\left(\left(1+|u|^4+|\tilde{u}|^4\right)|\bar{u}|+(1+|u|^\gamma)\right)|A^\nu\bar{u}|\,dx,\end{aligned}$$

thus, by the Sobolev embedding

$$\begin{aligned}&\int_U\left(\left(1+|u|^4+|\tilde{u}|^4\right)|\bar{u}|+(1+|u|^\gamma)\right)|A^\nu\bar{u}|\,dx\\ &\qquad\leq C(1+\|u\|^4_{L^6}+\|\tilde{u}\|^4_{L^6})\|\bar{u}\|_{L^{\frac{6}{1-2\nu}}}\|A^\nu\bar{u}\|_{L^{\frac{6}{1+2\nu}}}\\ &\qquad\quad+(1+|u|^\gamma)\,\|A^\nu\bar{u}\|_{L^{\frac{6}{1+2\nu}}}\\ &\qquad\leq (R_5(\omega)\,\|A^\nu\bar{u}\|+R_6(\omega))\left\|A^{\frac{1+\nu}{2}}\bar{u}\right\|.\end{aligned} \tag{5.32}$$

Apply the inequality (5.30) and (5.32), we can get

$$\|A^\nu\bar{y}(t,\theta_{-t}\omega;g_0)\|^2_E\leq M_2^2(\omega).$$

Then the proof is complete □. □

6 Random Attractors

In this section, we establish the existence of a $\mathcal{D}$-random attractor for the random dynamical system Φ associated with system (3.1)-(3.2) on $\mathbb{R}^p$. By Lemma4.1 that Φ has a closed random absorbing set in $\mathcal{D}$. Which, along with the $\mathcal{D}$-pullback asymptotic compactness will imply the existence of a unique $\mathcal{D}$-random attractor. Next due to decompose of solutions, we shall prove the $\mathcal{D}$-pullback asymptotic compactness of Φ. Since that $\tau \in \mathbb{R}, \omega \in \Omega, t \geq 0$, it follows that

$$\bar{\eta}(\tau,\tau-t,\theta_{-\tau}\omega;y_{\tau-t}(\theta_{-\tau}\omega),s) = \begin{cases} \bar{u}(\tau,\tau-t,\theta_{-\tau}\omega;y_{\tau-t}(\theta_{-\tau}\omega))- \\ \quad \bar{u}(\tau-s,\tau-t,\theta_{-\tau+s}\omega;y_{\tau-t}(\theta_{-\tau+s}\omega)), \ t \leq s \\ \bar{u}(\tau,\tau-t,\theta_{-\tau}\omega;y_{\tau-t}(\theta_{-\tau}\omega), \ t \geq s, \end{cases} \tag{6.1}$$

and

$$\bar{\eta}_s(\tau,\tau-t,\theta_{-\tau}\omega;y_{\tau-t}(\theta_{-\tau}\omega)) = \begin{cases} \bar{u}_t(\tau,\tau-t,\theta_{-\tau+s}\omega;y_{\tau-t}(\theta_{-\tau+s}\omega)), t \leq s \\ 0, \quad t \geq s, \end{cases} \tag{6.2}$$

Thus, by (6.1),(6.2) and Lemma15, we conclude that

$$\max\{\|\bar{\eta}_s(\tau,\tau-t,\theta_{-\tau}\omega;y_{\tau-t}(\theta_{-\tau}\omega),s)\|_{\mu,2\nu}, \|\bar{\eta}_s(\tau,\tau-t,\theta_{-\tau}\omega;y_{\tau-t}(\theta_{-\tau}\omega),s)\|_{\mu,2\nu+1}\} \\ \leq M_2(\omega). \tag{6.3}$$

Lemma 16. *.[5, 9, 18] Let $E_\nu = H_{2\nu+1} \times H_{2\nu} \times L^2_\mu(\mathbb{R}^+, H_{2\nu+1}) \hookrightarrow L^2_\mu(\mathbb{R}^+, H_{2\nu+1})$ is projection operator, setting $Y = y(\tau, B_\nu(\tau,\omega))$ is a random bounded absorbing set. Where $y(\tau)$ is the solution of system (3.17), and by Lemma13, there is a positive random radius $M_0(\omega)$ depend on τ, such that*

$$\begin{cases} 1 - Y \text{ is bounded in } L^2_\mu(\mathbb{R}^+, H_{1+2\nu}) \bigcap H^1_\mu(\mathbb{R}^+, H_{2\nu}), \\ 2 - \sup\limits_{\eta \in \tilde{B}_\nu(\tau,\omega), s \in \mathbb{R}^+} \|\eta(s)\|^2_{\mu,1} = \\ \sup\limits_{t>0} \sup\limits_{y_{\tau-t}(\theta_{-\tau}\omega) \in B_1(\theta_{-t}\omega), s \in \mathbb{R}^+} \|\eta(\tau,\tau-t,\theta_{-\tau}\omega;y_{\tau-t}(\theta_{-\tau}\omega),s)\|^2_{\mu,1} \leq M_0(\omega). \end{cases} \tag{6.4}$$

Denote by B_ν the closed ball of $H_{1+2\nu} \times H_{2\nu}$ of random variable radius $M_0(\omega)$. We apply on a finite domain B_ν is compact subset of $H_{1+2\nu} \times H_{2\nu}$. Thus we chose that a set $\tilde{B}_\nu(\tau,\omega)$

$$\tilde{B}_\nu(\tau,\omega) = \overline{\bigcup\nolimits_{y_{\tau-t}(\theta_{-\tau}\omega) \in B_1(\theta_{-t})} \bigcup\nolimits_{t \geq 0} \bar{\eta}(\tau,\tau-t,\theta_{-\tau}\omega;y_{\tau-t}(\theta_{-\tau}\omega),s)} \ \tau \leq t, \tag{6.5}$$

hence, ν is as in (5.16). From (1.3) and (6.3), we find

$$\|\eta(s)\|^2_{\mu,1} = \int_0^{+\infty} \mu(s)\|\nabla\eta(s)\|^2 ds \leq M_0(\omega) \int_0^{+\infty} e^{\delta s} ds \leq \frac{M_0(\omega)}{\delta}. \tag{6.6}$$

The next Lemma, we establish the main result about the existence of a random attractor for RDS Φ.

Lemma 17. *.[9] Assume that $y(t,\omega)$ be solution of the system (3.17) and the condition of Lemma15 hold. For any $\tau \in \mathbb{R}, \omega \in \Omega$ and $t \geq 0$, there exists a random set $\tilde{B}_\nu(\tau,\omega) \in D(E_\nu)$ with*

$$\|\tilde{B}_\nu(\tau,\omega)\|_{E_\nu} = \sup_{\tilde{y} \in \tilde{B}_\nu(\tau,\omega)} \|\tilde{y}\|_E \leq \tilde{M}_\nu(\omega),$$

is relatively compact in E. The positive number σ and $\tilde{M}_\nu(\omega) \geq 0$, so as for each $\omega \in \Omega$ it satisfies

$$d_H(\Phi(t,\theta_{-t}\omega, B_1(\theta_{-t}\omega)), \tilde{B}_\nu(\tau,\omega)) \leq \tilde{M}_\nu(\omega)e^{-\sigma t} \ \to 0 \ as \ t \to +\infty. \tag{6.7}$$

Proof. Let $y_0(\theta_{-t}\omega) \in B_1(\theta_{-t}\omega)$, by (6.1)-(6.3) and Lemma15 it conclude that $\tilde{B}_\nu(\tau,\omega)$ is relatively compact in $L^2_\mu(\mathbb{R}^+, H_1)$ and consequently in $L^2_\mu(\mathbb{R}^+, H_1)$, let $B_\nu(\tau,\omega) \subset E_\nu \subset E$ be the ball of E_ν of radius $M(\omega)$ defined by (4.22)-(4.23), where ν is as in (5.16). Lastly, we get compact set $\Upsilon(\tau,\omega) = \tilde{B}_\nu(\tau,\omega) \times B_\nu(\tau,\omega) \subset E$ then, we have the following summaries.
Since Lemma13 and $y_0(\theta_{-t}\omega) \in B_0(\theta_{-t}\omega)$ there exists a random set $M_0(\omega) \in B_0 \subseteq B(\omega) \in \mathcal{D}(E)$, we get that

$$d_H(\Phi(t,\theta_{-t}\omega, B(\theta_{-t}\omega)), B_0) \leq M_0(\omega)e^{-\sigma t} \ \to 0 \ as \ t \to +\infty, \tag{6.8}$$

next, follows from Lemma14 to $y_0(\theta_{-t}\omega) \in B_1(\theta_{-t}\omega)$, there exists a positive random variable $M_1(\omega) \in B_1(\tau,\omega) \in \mathcal{D}(E)$ and $M_1(\omega) \in B_1(\tau,\omega) \in D(E)$, such that

$$d_H(\Phi(t,\theta_{-t}\omega, B(\theta_{-t}\omega)), B_1(\tau,\omega)) \leq M_1(\omega)e^{-\sigma_1 t} \ \to 0 \ as \ t \to +\infty, \tag{6.9}$$

by Lemma15, let $y_0(\theta_{-t}\omega) \in B_1(\theta_{-t}\omega)$, there exists a positive random variable $M_2(\omega) \in B_1(\tau,\omega) \in \mathcal{D}(E)$, such that

$$d_H(\Phi(t,\theta_{-t}\omega, B(\theta_{-t}\omega)), B_1(\tau,\omega)) \leq M_2(\omega)e^{-\sigma_2 t} \ \to 0 \ as \ t \to +\infty, \tag{6.10}$$

let $\nu \geq 0$ is fixed, by above recursion of finite steps at most $\frac{1}{\nu} + 2$, there exists a random set $M_\nu \in B_\nu(\tau,\omega) \in \mathcal{D}(E_\nu)$ as for as

$$d_H(\Phi(t,\theta_{-t}\omega, B_1(\theta_{-t}\omega)), B_\nu(\tau,\omega)) \leq M_\nu(\omega)e^{-\sigma_\nu t} \ \to 0 \ as \ t \to +\infty, \tag{6.11}$$

due to (6.1)-(6.3),(6.5) and Lemma16, there exists random radius $\tilde{M}_\nu(\omega) \in \tilde{B}_\nu(\tau,\omega) \in \mathcal{D}(\tilde{E}_\nu) \subset \mathcal{D}(E_\nu) \subset \mathcal{D}(E)$, we have

$$d_H(\Phi(t,\theta_{-t}\omega, B_\nu(\theta_{-t}\omega)), \tilde{B}_\nu(\tau,\omega)) \leq \tilde{M}_\nu(\omega)e^{-\tilde{\sigma}_\nu t} \ \to 0 \ as \ t \to +\infty, \tag{6.12}$$

it holds that

$$d_H(\Phi(t,\theta_{-t}\omega,B_1(\theta_{-t}\omega)),\tilde{B}_\nu(\tau,\omega))\leq \tilde{M}_\nu(\omega)e^{-\sigma t}\ \rightarrow 0\ as\ t\rightarrow +\infty.$$

From Lemma16, we get

$$\Upsilon(\tau,\omega)=B_\nu(\tau,\omega)\times\tilde{B}_\nu(\tau,\omega), \tag{6.13}$$

from(5.5)-(5.6) and (6.12), we conclude the following attractive property of inequality(6.11). Thus, by Lemma13 there exists $T_1 = T_1(\omega,B)\geq\tau$, such that $y(t,\theta_{-t}\omega,B(\theta_{-t}\omega)\subseteq B_0(\omega))$ $\forall t\geq T_1$, let $t\geq T$ and $T_2=t-T_1\geq T(\omega,B,M_0)\geq 0$, using cocycle property (iii) of Φ, we show that

$$\begin{aligned}
&y(\tau,\tau-t,\theta_{-\tau}\omega,B(\tau-t,\theta_{-\tau}\omega))\\
&\qquad=y(\tau,\tau-T_2-T_1,\theta_{-\tau}\omega,B(\tau-T_2-T_1,\theta_{-\tau}\omega))\\
&\qquad=y(\tau,\tau-T_2,(\theta_{-\tau}\omega),y(\tau-T_2,\tau-T_2-T_1,(\theta_{-\tau}\omega),B(\tau-T_2-T_1,\theta_{-\tau}\omega))\\
&\qquad\subseteq y(\tau,\tau-T_2,\theta_{-\tau}\omega,B_0(\theta_{-T_2}\omega))\subseteq B_1(\tau,\theta_{-\tau}\omega),
\end{aligned} \tag{6.14}$$

for each $y(\tau,\tau-t,\theta_{-\tau}\omega,y_{\tau-t}(\theta_{-\tau}\omega))\in y(\tau,\tau-t,\theta_{-\tau}\omega,B(\tau-t,\theta_{-t}\omega))$, for $t\geq T_1+T(\omega,B_0)$ where $y_{\tau-t}(\theta_{-\tau}\omega)\in B(\tau-t,\theta_{-t}\omega)$. By(6.14) and Lemma15, it follows that:

$$\begin{aligned}
&\tilde{y}(\tau,\tau-t,\theta_{-\tau}\omega,y_{\tau-t}(\theta_{-\tau}\omega))\\
&\quad=y(\tau,\tau-t,\theta_{-\tau}\omega,y_{\tau-t}(\theta_{-\tau}\omega))-\bar{y}(\tau,\tau-t,\theta_{-\tau}\omega,y_{\tau-t}(\theta_{-\tau}\omega))\in\Upsilon(\tau,\omega).
\end{aligned} \tag{6.15}$$

Thus, from Lemma14 we conclude that

$$\begin{aligned}
\inf_{\hat{y}\in\Upsilon(\omega)}&\|y(\tau,\tau-t,\theta_{-\tau}\omega,y_{\tau-t}(\theta_{-\tau}\omega))-\hat{y}(\tau,\tau-t,\theta_{-\tau}\omega,y_{\tau-t}(\theta_{-\tau}\omega))\|_E^2\\
&\leq\|\tilde{y}(\tau,\tau-t,\theta_{-\tau}\omega,y_{\tau-t}(\theta_{-\tau}\omega))\|_E^2\\
&\leq M_0^2(\omega)e^{-\sigma t}\ ,\ \forall t>T_1+T(\omega,B_0),
\end{aligned} \tag{6.16}$$

such that

$$d_H(\Phi(t,\theta_{-t}\omega,B(\theta_{-t}\omega)),\Upsilon(\tau,\omega))\leq M_0(\omega)e^{-\sigma_1 t}\ \rightarrow 0\ as\ t\rightarrow+\infty\ . \tag{6.17}$$

The proof is completed. □

Theorem 18. *Assume that* $(\mathbf{A}_1)$-$(\mathbf{A}_6)$ *and* $g(x)\in L^2(U)$ *hold. Then the continuous cocycle* Φ *associated with problem (3.17) has a unique* $\mathcal{D}$*-pullback attractor* $\mathcal{A}\subseteq\Upsilon(\tau,\omega)\bigcap B_0(\omega)$, $A=\{A(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$ *in* $\mathbb{R}^n$.

Proof. Hence that the continuous cocycle Φ has a closed random absorbing set $\{A(\tau,\omega)\}_{\tau\in\mathbb{R},\omega\in\Omega}$ in $\mathcal{D}$, by Lemma13 Lemma16, and Lemma17, the continuous cocycle Φ is a $\mathcal{D}$-pullback asymptotically compact in $\mathbb{R}^n$. Since that the existence of a unique $\mathcal{D}$-random attractor for Φ follows from Lemma8 immediately. □

Acknowledgement. The author is extremely grateful to the reviewers for their comments, and suggestions, which leading to prepare this paper.

References

[1] L.Arnold, *Random dynamical systems*. Springer-Verlag, New York and Berlin,1998. MR1723992 .

[2] R. Temam, *Infinite-dimensional dynamical systems in mechanics and physics* , Applied Mathematical Sciences, vol. 68, Springer-Verlag, New York, 1988. MR1441312 .

[3] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences, vol. 44, Springer-Verlag, New York, 1983. MR710486

[4] G.D.Prato, J. Zabczyk, *Stochastic equations in infinite dimensions*, Cambridge University Press, Cambridge, 1992. MR2244975.

[5] S. Borini, V.Pata, *Uniform attractors for a strongly damped wave equation with linear memory*, Asymptot. Anal, **20**(1999),263-277. MR1715336. Zbl0936.35037.

[6] H Crauel, F. Flandoli, *Attractors for random dynamical systems*, (1994), Probab. Theory Relat. Fields,**100**(1994)365-393. MR1305587. Zbl0819.58023.

[7] V. Pata, A. Zucchi, *Attractors for a damped hyperbolic equation with linear memory* , Adv. Math. Sci. Appl., **11**(2001), 505-529. MR1907454.

[8] H. Crauel, A. Debussche, F. Flandoli, *Random attractors*, J. Dynamics and Differential equations, **9**(1997), 307-341. MR1451294 . Zbl 0884.58064 .

[9] S. F. Zhou, M. Zhao, *Random attractors for damped non-autonomous wave equations with memory and white noise*, J. nonlinear Analysis, **120**(2015), 202-226. MR3348055. Zbl 1332.37060.

[10] C. Sun, D. Cao, *Non-autonomous wave dynamics with memory-asymptotic regularity and uniform attractor*, Discrete Contin. Dyn. Syst. Ser. B, **9**(2008), 743-761. MR2379435.

[11] Q.Zh. Ma, C. Zhong, *Existence of strong global attractors for hyperbolic equation with linear memory*, J. Appl. Math. Comput, **157**(2004), 745-758. MR2094665. Zbl 1068.45018.

[12] L. Xu, Q.Zh. Ma, *Existence of random attractors for the floating beam equation with strong damping and white noise*, Bound. Value Probl, (2015) 2015:126. MR3375640. Zbl 1338.35376.

[13] M.M. Cavalcanti, L.H Fatori, T.F. Ma, *Attractors for wave eqautions with degenerate memory*, J. Differ. Equations, **260**(2016), 56-80. MR3411664. Zbl 1323.35106.

[14] J.L. Shomberg, *Regular global attractors for wave equations with degenerate memory*, Ural Math. J., **5**(2019), 59-82. MR3995656.

[15] E. Dibenedetto, *Degenerate parabolic equations*, Universitext, (1993).

[16] C.T. Anh, T.Q. Bao, L.T. Thuy, *Regularity and fractal dimension of pullback attractors for a non-autonomous semilinear degenerate parabolic equation*, Glasgow Math. J., **55**(2013), 431-448. MR3040873. Zbl 1379.37133.

[17] X. Li, C. Sun, N.Zhang. *Dynamics for a non-autonomous degenerate parabolic equation in* $D_0^1(\Omega,\sigma)$. Discrete Contin. Dyn. Syst -A, **36**(2016), no (12), 7063-7079. MR3567833. Zbl1351.35086.

[18] A.D.Abdelmajid, E.M.Ahmed, Q.zh.Ma, *Random attractors for Stochastic strongly damped non-autonomous wave equations with memory and multiplicative noise*, Open J. Math. Anal. **3**(2019), 50-70.

[19] S. Zhou. *Kernel sections for damped non-autonomous wave equations with linear memory and critical exponent*, Quart. Appl. Math. **61** (2003), 731-757. MR2019621. Zbl1057.35074.

Abdelmajid A. D.
Faculty of Petroleum and Hydrology Engineering, Alsalam University, Almugled, Sudan.
majid_dafallah@yahoo.com

Fadlallah M. M.
Department of Mathematics and physics, Faculty of Education University of Kassala Kassala, Sudan.
e-mail: fadlmushan@hotmail.com

Mohamed Y. A. Bakhet
Department of Mathematics, College of Education Rumbek University of Science and Technology Rumbek, South Sudan.
e-mail: myabakhet@hotmail.com

Ahmed E. M.
Department of Mathematics, Faculty of Pure and Applied Sciences, International University of Africa, Khartoum, Sudan.
e-mail: ahmedesag@gmail.com

License