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COUPLED FIXED POINT SETS WITH DATA-DEPENDENCE AND STABILITY

Binayak S. Choudhury, Nikhilesh Metiya and Sunirmal Kundu

Abstract. In this paper we establish a coupled fixed point theorem of a coupled multivalued mapping defined on a complete metric space. In our result we use a new contractive inequality. There are rational terms in the expression of the inequality. The contractive condition on the nonlinear map is assumed only to hold on the elements that are related by a binary relation. The main theorem has several corollaries and is illustrated with example. The main result is deduced in metric spaces. Its consequences are discussed for α-admissible mapping as well as in metric spaces with partial order and graph respectively.

1 Mathematical background and preliminaries

Our consideration in this paper is a study related to fixed points of some coupled multivalued operators on metric spaces equipped with an appropriate binary relation. A coupled contraction condition is supposed to be satisfied by the multivalued coupled operator for those points which are related by the binary relation. As a consequence of it, the inequality here is weaker than the usual case in metric fixed point theory which assumes the inequality condition for arbitrarily chosen points from the space. Such weakening of conditions have occupied recent interests in fixed point theory to a large extent. Works of this category have come to be known as relation-theoretic fixed point results. Some instances of these works are in [2, 19, 23, 34].

We use rational terms in our inequality. The use of rational terms in contraction inequalities in the domain of metric fixed point theory was initiated by Dass et al. in their work [18] in which they extended the Banach's contraction mapping principle by using a contractive rational inequality. After that the rational inequalities have been used in fixed point and related problems in several works as for instances in [5, 10, 11, 22].

Our results are for coupled multivalued mappings. Coupled fixed point results

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constitute a chapter in metric fixed point theory which has been in focus in recent times. Although the concept was introduced some time back in 1987 by Guo et al. [20], it was after publication of the work of Bhaskar et al. [7] that a large number of papers have been written on this topic and on topics related to it. Some instances of these works are [4, 12, 15, 28].

Fixed points of multivalued mappings have been treated extensively in its various aspects. An early reference in this direction is due to Nadler [27] in which the Banach contraction mapping principle is extended to the domain of setvalued analysis. Some references of fixed point results of multivalued mappings are noted in [1, 6, 16, 17, 36].

In the present paper we consider a coupled mapping on a complete metric space with a binary relation. The coupled mapping is assumed to satisfy a contraction inequality with rational terms on pairs of points which are connected by the binary relation in a specific way. We show that the coupled fixed point set of this mapping is nonempty. As special cases of the coupled fixed point result we obtain several corollaries which are coupled multivalued extensions of certain results in metric fixed point theory of ordinary functions. We consider the problem of data-dependence and stability associated with the coupled fixed point sets of these mappings. We have versions our results in metric spaces with partial orderings and with graphs also for functions satisfying α-dominated conditions. To the best of our knowledge we believe that of data dependence and stability of fixed point sets in metric spaces with binary relations are being discussed for the first time in this paper.

A data dependence problem is to estimate the distance between the fixed point sets of these two mappings. The above is only meaningful if we have an assurance of nonempty fixed point sets of these two operators. There are also some variants of the problem.

The data dependence problem is mostly dealt within the domain of setvalued analysis since multivalued mappings often have larger fixed point sets than their singlevalued counterparts. Several research papers on data dependence have been published in recent literatures of which we mention a few in references [13, 14, 31, 32]. It has important applications to differential and integral equations [13, 31].

Our problem of data dependence is with coupled mappings and their coupled fixed point sets. Such problems for coupled fixed point sets have already appeared in work of Chifu et al. [13]. We formulate a version of the problem suitable to our needs.

Stability is related limiting behavior of a system which, in this case, is the relation of the fixed point sets associated with a sequence of multivalued mappings with the limit function to which the sequence converges. There are several studies related to stabilities of fixed point sets, some of which are noted in [8, 16, 24, 26, 27]. In this paper we discuss the stability of coupled fixed point sets of the class of mappings we consider.

In the last three sections we discuss some consequences of our main result. Precisely we obtain some results in partially ordered metric spaces and in metric spaces having a graph defined on it. In another section we obtain a fixed point result for α-dominated mappings.

In the following we discuss the necessary mathematics for the discussion on the topics in the following sections. Let X and Y be two nonempty sets and R be a relation from X to Y, that is, $R \subseteq X \times Y$. We write $(x, y) \in R$ or xRy to mean x is R related to y. The set $P = \{x \in X : (x, y) \in R \text{ for some } y \in Y\}$ is called the domain of R and the set $Q = \{y \in Y : (x, \ y) \in R \text{ for some } x \in X\}$ is called the range of R. By R^{-1} we mean the set $\{(y, \ x) : (x, \ y) \in R\}$ which is called the inverse of R. A relation R from X to X is called a relation on X. Suppose R is a relation from X to Y and S is a relation from Y to Z. Then the composition $S \circ R$ is the relation from X to Z defined by $S \circ R = \{(x, z) : (x, y) \in R, \ (y, \ z) \in S, \text{ for some } y \in Y\}$.

A relation R on X is said to be

(i) reflexive if $(x, x) \in R$, for each $x \in X$,

(ii) symmetric if $(x, y) \in R \Longrightarrow (y, x) \in R$,

(iii) transitive if $(x, y) \in R$ and $(y, z) \in R \Longrightarrow (x, z) \in R$,

(iv) anti-symmetric if $(x, y) \in R$ and $(y, x) \in R \Longrightarrow x = y$,

(v) directed if given $x, \ y \in X$, there exists $z \in X$ such that $(x, z) \in R$ and $(y, z) \in R$.

Also, a relation R on a set X is said to be

(i) an equivalence relation if it is reflexive, symmetric and transitive,

(ii) a partial order relation if it is reflexive, anti-symmetric and transitive,

(iii) a quasi order relation if it is reflexive and transitive,

(iv) a linear order relation if it is a partial order such that for any $x, \ y \in X$ either $(x, y) \in R$ or $(y, x) \in R$.

Let $(X, \ d)$ be a metric space. Then $X \times X$ is also a metric space under the metric ρ defined by $\rho((x, y), \ (u, v)) = \max \{d(x, u), \ d(y, v)\}$ for all (x, y) and $(u, v) \in X \times X$. If X is complete then $(X \times X, \rho)$ is also complete.

Let $N(X) :=$ the collection of all nonempty subsets of X, $CB(X) :=$ the collection of all nonempty closed and bounded subsets of X and $K(X) :=$ the collection of all nonempty compact subsets of X. We use the following notations and definitions

$$D(x, B) = \inf \{d(x, y) : y \in B\}, \text{ where } x \in X \text{ and } B \in CB(X),$$

$$D(A, B) = \inf \{d(a, b) : a \in A, \ b \in B\}, \text{ where } A, B \in CB(X),$$

$$H(A, B) = \max \{\sup_{x \in A} D(x, B), \ \sup_{y \in B} D(y, A)\}, \text{ where } A, B \in CB(X).$$

H is known as the Hausdorff metric on $CB(X)$ [27]. Further, if $(X, \ d)$ is complete then $(CB(X), \ H)$ is also complete. Let H_ρ be the Hausdorff metric induced by ρ.

Lemma 1 ([16]). Let $B \in K(X)$, where $(X,\ d)$ is a metric space. Then for every $x \in X$ there exists $y \in B$ such that $d(x,\ y) = D(x,\ B)$.

Definition 2. An element $x \in X$ is called a fixed point of $T : X \to N(X)$ if $x \in Tx$.

Definition 3 ([20]). An element $(x, y) \in X \times X$ is called a coupled fixed point of $F : X \times X \to X$ if $x = F(x, y)$ and $y = F(y, x)$.

Definition 4 ([33]). An element $(x, y) \in X \times X$ is called a coupled fixed point of a multivalued mapping $F : X \times X \to N(X)$ if $x \in F(x, y)$ and $y \in F(y, x)$.

The set of coupled fixed points of F is denoted by $Fix(F)$.
We introduce here the notion of R-dominated mappings.

Definition 5. Let X be a nonempty set with a binary relation R on it. A mapping $F : X \times X \to X$ is said to be R-dominated if $(x,\ F(x, y)) \in R$ and $(y,\ F(y, x)) \in R$, for any $(x, y) \in X \times X$.

Definition 6. Let X be a nonempty set with a binary relation R on it. A multivalued mapping $F : X \times X \to N(X)$ is said to be R-dominated if $(x, u) \in R$ and $(y, v) \in R$, whenever $(x, y) \in X \times X$, $u \in F(x, y)$ and $v \in F(y, x)$.

Now we introduce the notion of R- regularity of a metric spaces.

Definition 7. Let (X, d) be a metric space and R be a binary relation on X. Let $\{x_n\}$ be any convergent sequence in X with limit $x \in X$. X is said to have R-regular property if $(x_n,\ x_{n+1}) \in R$, for all n implies $(x_n,\ x) \in R$, for all n.

Example 8. Let $X = [0,\ 1]$ and d be the usual metric on X. Let us define a binary relation R on X as $(x,\ y) \in R$ if and only if $x \in [0,\ 1]$ and $y \in [0,\ \frac{1}{64}]$. Let $F : X \times X \to N(X)$ be defined as follows:

$$F(x,\ y) = \left[0,\ \frac{x^2 + y^2}{64(1 + x^2 + y^2)}\right], \quad \text{for } (x,\ y) \in X \times X.$$

As $u \in F(x,\ y) \subseteq [0,\ \frac{1}{64}]$ and $v \in F(y,\ x) \subseteq [0,\ \frac{1}{64}]$, for any $(x,\ y) \in X \times X$, we have $(x, u) \in R$ and $(y, v) \in R$, for any $u \in F(x, y)$ and for any $v \in F(y,\ x)$. Therefore, F is R- dominated.

Let $\{x_n\}$ be a convergent sequence in X with limit $x \in X$ and $(x_n,\ x_{n+1}) \in R$, for all n. Then $x_n \in [0,\ \frac{1}{64}]$ for all $n \geq 2$ and hence $x \in [0,\ \frac{1}{64}]$. Therefore, $(x_n,\ x) \in R$ for all n, which implies that X has R- regular property.

Let Ψ be the collection of all functions $\psi : [0, \infty)^{10} \to [0, \infty)$ having the properties (i) ψ is continuous and nondecreasing in each coordinate; (ii) $\sum_{n=1}^{\infty} \Phi^n(t) < \infty$ for all t, where $\Phi(t) = \psi(t, t, t, t, t, t, t, t, t, t)$ and (iii) $\psi(0, 0, 0, 0, 0, 0, 0, 0, 0, 0) = 0$.

Definition 9. *Let $(X,\ d)$ be a metric space with a binary relation R on it. A multivalued mapping $F : X \times X \to K(X)$ is called generalized coupled contraction if there exist $p,\ q,\ r,\ s,\ e,\ f,\ g,\ h \geq 0$ and $\psi \in \Psi$ such that for all (x, y) and $(u, v) \in X \times X$ with $(x, u) \in R$ and $(y, v) \in R$ the following inequality holds:*

$$H(F(x,\ y),\ F(u,\ v)) \leq M(x,\ y,\ u,\ v), \tag{1.1}$$

where

$$\begin{aligned} M(x, y, u, v) = \psi\Big(d(x, u), d(y, v),\ & \frac{D(x, F(x, y))[1 + D(x, F(x, y))^p]}{1 + d(x, u)^p}, \\ & \frac{D(y, F(y, x))[1 + D(y, F(y, x))^e]}{1 + d(y, v)^e}, \frac{D(u, F(x, y))[1 + D(x, F(x, y))^q]}{1 + d(x, u)^q}, \\ & \frac{D(v, F(y, x))[1 + D(y, F(y, x))^f]}{1 + d(y, v)^f}, \frac{D(u, F(x, y))[1 + D(u, F(x, y))^r]}{1 + d(x, u)^r}, \\ & \frac{D(v, F(y, x))[1 + D(v, F(y, x))^g]}{1 + d(y, v)^g}, \frac{D(u, F(x, y))[1 + D(x, F(u, v))^s]}{1 + d(x, u)^s}, \\ & \frac{D(v, F(y, x))[1 + D(y, F(v, u))^h]}{1 + d(y, v)^h}\Big). \end{aligned}$$

We propose the following definition of stability of coupled fixed point sets.

Definition 10. Let (X, d) be a metric space. Let $\{F_n \colon X \times X \to CB(X)\}$ be a sequence of multivalued mappings with limit function $F \colon X \times X \to CB(X)$, that is, $F = \lim_{n\to\infty} F_n$. Suppose that $\{Fix(F_n)\}$ is the sequence of coupled fixed point sets of the sequence of mappings $\{F_n\}$ and $Fix(F)$ is the coupled fixed point set of F. We say that the coupled fixed point sets of $\{F_n\}$ are stable if $\lim_{n\to\infty} H_\rho(Fix(F_n),\ Fix(F)) = 0$.

2 Existence of coupled fixed point set

Theorem 11. *Let (X, d) be a complete metric space with a binary relation R on it such that X has R- regular property. Suppose that $F : X \times X \to K(X)$ is a R-dominated mapping and there exist $p,\ q,\ r,\ s,\ e,\ f,\ g,\ h \geq 0$ and $\psi \in \Psi$ such that F is a generalized coupled contraction. Then F has a coupled fixed point in $X \times X$.*

Proof. Let $(x_0, y_0) \in X \times X$. Since $F(x_0,\ y_0),\ F(y_0,\ x_0) \in K(X)$, by Lemma 1, there exist $x_1 \in F(x_0, y_0)$ and $y_1 \in F(y_0, x_0)$ such that $d(x_0, x_1) = D(x_0,\ F(x_0, y_0))$ and $d(y_0, y_1) = D(y_0,\ F(y_0, x_0))$. By R-dominated property of F, we have $(x_0, x_1) \in R$ and $(y_0, y_1) \in R$. Again as $F(x_1, y_1) \in K(X)$ and $F(y_1, x_1) \in K(X)$, by Lemma 1, there exist $x_2 \in F(x_1, y_1)$ and $y_2 \in F(y_1, x_1)$ such that $d(x_1, x_2) = D(x_1, F(x_1, y_1))$ and $d(y_1, y_2) = D(y_1, F(y_1, x_1))$. By R-dominated property of F, we have $(x_1, x_2) \in$

R and $(y_1, y_2) \in R$. Continuing in this way we obtain two sequences $\{x_n\}$ and $\{y_n\}$ such that for all $n \geq 0$,

$$x_{n+1} \in F(x_n, y_n) \ \text{ and } \ y_{n+1} \in F(y_n, x_n) \tag{2.1}$$

$$\text{with } \ (x_n,\ x_{n+1}) \in R \text{ and } \ (y_n,\ y_{n+1}) \in R \tag{2.2}$$

and

$$d(x_n, x_{n+1}) = D(x_n,\ F(x_n, y_n)) \ \text{ and } \ d(y_n,\ y_{n+1}) = D(y_n,\ F(y_n,\ x_n)). \tag{2.3}$$

Let

$$r_n = \max\ \{d(x_n,\ x_{n+1}),\ d(y_n,\ y_{n+1})\}, \ \text{ for all } \ n\ \geq 0. \tag{2.4}$$

As F is a generalized coupled contraction, using (2.1), (2.2), (2.3), (2.4) and a property of ψ, we have

$$\begin{aligned}
d(x_{n+1},\ x_{n+2}) &= D(x_{n+1},\ F(x_{n+1},\ y_{n+1})) \leq H(F(x_n,\ y_n),\ F(x_{n+1},\ y_{n+1})) \\
&\leq M(x_n,\ y_n,\ x_{n+1},\ y_{n+1}) \\
&\leq \psi\Big(d(x_n, x_{n+1}),\ d(y_n, y_{n+1}),\ \frac{D(x_n, F(x_n, y_n))[1 + D(x_n, F(x_n, y_n))^p]}{1 + d(x_n, x_{n+1})^p}, \\
&\qquad \frac{D(y_n, F(y_n, x_n))[1 + D(y_n, F(y_n, x_n))^e]}{1 + d(y_n, y_{n+1})^e}, \\
&\qquad \frac{D(x_{n+1}, F(x_n, y_n))[1 + D(x_n, F(x_n, y_n))^q]}{1 + d(x_n, x_{n+1})^q}, \\
&\qquad \frac{D(y_{n+1}, F(y_n, x_n))[1 + D(y_n, F(y_n, x_n))^f]}{1 + d(y_n, y_{n+1})^f}, \\
&\qquad \frac{D(x_{n+1}, F(x_n, y_n))[1 + D(x_{n+1}, F(x_n, y_n))^r]}{1 + d(x_n, x_{n+1})^r}, \\
&\qquad \frac{D(y_{n+1}, F(y_n, x_n))[1 + D(y_{n+1}, F(y_n, x_n))^g]}{1 + d(y_n, y_{n+1})^g}, \\
&\qquad \frac{D(x_{n+1}, F(x_n, y_n))[1 + D(x_n, F(x_{n+1}, y_{n+1}))^s]}{1 + d(x_n, x_{n+1})^s}, \\
&\qquad \frac{D(y_{n+1}, F(y_n, x_n))[1 + D(y_n, F(y_{n+1}, x_{n+1}))^h]}{1 + d(y_n, y_{n+1})^h}\Big)
\end{aligned}$$

$$
\begin{aligned}
&\leq \psi\Big(d(x_n,x_{n+1}),\ d(y_n,y_{n+1}),\ \frac{d(x_n,x_{n+1})[1+d(x_n,x_{n+1})^p]}{1+d(x_n,x_{n+1})^p},\\
&\quad \frac{d(y_n,y_{n+1})[1+d(y_n,y_{n+1})^e]}{1+d(y_n,y_{n+1})^e},\ \frac{d(x_{n+1},x_{n+1})[1+d(x_n,x_{n+1})^q]}{1+d(x_n,x_{n+1})^q},\\
&\quad \frac{d(y_{n+1},y_{n+1})[1+d(y_n,y_{n+1})^f]}{1+d(y_n,y_{n+1})^f},\ \frac{d(x_{n+1},x_{n+1})[1+d(x_{n+1},x_{n+1})^r]}{1+d(x_n,x_{n+1})^r},\\
&\quad \frac{d(y_{n+1},y_{n+1})[1+d(y_{n+1},y_{n+1})^g]}{1+d(y_n,y_{n+1})^g},\ \frac{d(x_{n+1},x_{n+1})[1+d(x_n,x_{n+2})^s]}{1+d(x_n,x_{n+1})^s},\\
&\quad \frac{d(y_{n+1},y_{n+1})[1+d(y_n,y_{n+2})^h]}{1+d(y_n,y_{n+1})^h}\Big)\\
&\leq \psi\Big(d(x_n,x_{n+1}),\ d(y_n,y_{n+1}),\ d(x_n,x_{n+1}),\ d(y_n,y_{n+1}),0,0,0,0,0,0\Big)\\
&\leq \psi\Big(\max\{d(x_n,x_{n+1}),\ d(y_n,y_{n+1})\},\ \max\{d(x_n,x_{n+1}),\ d(y_n,y_{n+1})\},\\
&\quad \max\{d(x_n,x_{n+1}),d(y_n,y_{n+1})\},\ \max\{d(x_n,x_{n+1}),d(y_n,y_{n+1})\},\\
&\quad \max\{d(x_n,x_{n+1}),d(y_n,y_{n+1})\},\ \max\{d(x_n,x_{n+1}),d(y_n,y_{n+1})\},\\
&\quad \max\{d(x_n,x_{n+1}),d(y_n,y_{n+1})\},\ \max\{d(x_n,x_{n+1}),d(y_n,y_{n+1})\}\\
&\quad \max\{d(x_n,x_{n+1}),d(y_n,y_{n+1})\},\ \max\{d(x_n,x_{n+1}),d(y_n,y_{n+1})\}\Big)\\
&= \psi\Big(r_n,\ r_n,\ r_n,\ r_n,\ r_n,\ r_n,\ r_n,\ r_n,\ r_n,\ r_n\Big) = \Phi(r_n). \qquad (2.5)
\end{aligned}
$$

Similarly, we can prove that

$$d(y_{n+1},\ y_{n+2}) \leq \Phi(r_n). \qquad (2.6)$$

Combining (2.5) and (2.6), we have

$$r_{n+1} = \max\{d(x_{n+1},x_{n+2}),\ d(y_{n+1},y_{n+2})\} \leq \Phi(r_n). \qquad (2.7)$$

Applying (2.7) repeatedly and using a property of ψ, we have

$$r_{n+1} \leq \Phi(r_n) \leq \Phi^2(r_{n-1}) \leq ... \leq \Phi^{n+1}(r_0). \qquad (2.8)$$

Now we prove that both $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Using the triangular inequality and a property of ψ, we have

$$\sum_{n=0}^{\infty} d(x_n,x_{n+1}) \leq \sum_{n=0}^{\infty} r_n \leq \sum_{n=0}^{\infty} \Phi^n(r_0) < \infty$$

and

$$\sum_{n=0}^{\infty} d(y_n,y_{n+1}) \leq \sum_{n=0}^{\infty} r_n \leq \sum_{n=0}^{\infty} \Phi^n(r_0) < \infty,$$

which imply that both $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X. As $(X,\ d)$ is complete, there exist $u,\ v \in X$ such that

$$\lim_{n\to\infty} x_n = u \text{ and } \lim_{n\to\infty} y_n = v. \tag{2.9}$$

Using (2.2), (2.9) and the R-regularity property of the space, we have $(x_n, u) \in R$ and $(y_n, v) \in R$, for all n. Owing to the contractive condition, we have

$$\begin{aligned}
D(x_{n+1},\ F(u,v)) &\le H(F(x_n,y_n),\ F(u,v)) \le M(x_n,y_n,u,v)\\
&\le \psi\Big(d(x_n,u),\ d(y_n,v),\ \frac{D(x_n,F(x_n,y_n))[1+D(x_n,F(x_n,y_n))^p]}{1+d(x_n,u)^p},\\
&\qquad \frac{D(y_n,F(y_n,x_n))[1+D(y_n,F(y_n,x_n))^e]}{1+d(y_n,v)^e},\\
&\qquad \frac{D(u,F(x_n,y_n))[1+D(x_n,F(x_n,y_n))^q]}{1+d(x_n,u)^q},\\
&\qquad \frac{D(v,F(y_n,x_n))[1+D(y_n,F(y_n,x_n))^f]}{1+d(y_n,v)^f},\\
&\qquad \frac{D(u,F(x_n,y_n))[1+D(u,F(x_n,y_n))^r]}{1+d(x_n,u)^r},\\
&\qquad \frac{D(v,F(y_n,x_n))[1+D(v,F(y_n,x_n))^g]}{1+d(y_n,v)^g},\\
&\qquad \frac{D(u,F(x_n,y_n))[1+D(x_n,F(u,v))^s]}{1+d(x_n,u)^s},\\
&\qquad \frac{D(v,F(y_n,x_n))[1+D(y_n,F(v,u))^h]}{1+d(y_n,v)^h}\Big)\\
&\le \psi\Big(d(x_n,u),\ d(y_n,v),\ \frac{d(x_n,x_{n+1})[1+d(x_n,x_{n+1})^p]}{1+d(x_n,u)^p},\\
&\qquad \frac{d(y_n,y_{n+1})[1+d(y_n,y_{n+1})^e]}{1+d(y_n,v)^e}, \frac{d(u,x_{n+1})[1+d(x_n,x_{n+1})^q]}{1+d(x_n,u)^q},\\
&\qquad \frac{d(v,y_{n+1})[1+d(y_n,y_{n+1})^f]}{1+d(y_n,v)^f}, \frac{d(u,x_{n+1})[1+d(u,x_{n+1})^r]}{1+d(x_n,u)^r},\\
&\qquad \frac{d(v,y_{n+1})[1+d(v,y_{n+1})^g]}{1+d(y_n,v)^g}, \frac{d(u,x_{n+1})[1+D(x_n,F(u,v))^s]}{1+d(x_n,u)^s},\\
&\qquad \frac{d(v,y_{n+1})[1+D(y_n,F(v,u))^h]}{1+d(y_n,v)^h}\Big).
\end{aligned} \tag{2.10}$$

Taking limit as $n \to \infty$ in (2.10) and using (2.9) and a property of ψ, we have $D(u,\ F(u,v)) \le \psi(0,0,0,0,0,0,0,0,0,0) = 0$, which implies that $D(u,\ F(u,v)) = 0$, that is, $u \in \overline{F(u,v)} = F(u,v)$, where $\overline{F(u,\ v)}$ is the closure of $F(u,\ v)$. Similarly, we prove that $v \in F(v,\ u)$. Therefore, $(u,\ v)$ is a coupled fixed point of F.

□

Now we present a few special cases illustrating the applicability of Theorem 11.

Remark 12. *we have the following corollaries from Theorem 11 by taking R to be the universal relation and choosing*

(i) $\psi(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}) = \frac{k}{2}\ [t_1 + t_2]$,

(ii) $\psi(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}) = \frac{k}{2}\ [t_3 + t_4]$ *and* $p = e = 0$,

(iii) $\psi(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}) = \frac{k}{2}\ [t_4 + t_5]$ *and* $r = f = 0$,

(iv) $\psi(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}) = \frac{k}{2}\ max\ \{t_1 + t_2, t_3 + t_4, t_5 + t_6\}$ *and* $p = e = q = f = 0$,

respectively, where $0 \leq k < 1$.

Corollary 13. A multivalued mapping $F : X \times X \to K(X)$, where (X, d) is a complete metric space, has a coupled fixed point if for all $(x, y), (u, v) \in X \times X$ one of the following inequality holds:

(i) $H(F(x, y), F(u, v)) \leq \frac{k}{2}\ [d(x, u) + d(y, v)]$,

(ii) $H(F(x, y),\ F(u, v)) \leq \frac{k}{2}\ [D(x, F(x, y)) + D(y, F(y, x))]$,

(iii) $H(F(x, y),\ F(u, v)) \leq \frac{k}{2}\ [D(u, F(x, y)) + D(v, F(y, x))]$,

(iv) $H(F(x, y),\ F(u, v)) \leq \frac{k}{2} \max\{d(x, u)+d(y, v),\ D(x, F(x, y))+D(y, F(y, x)),$
$D(u, F(x, y)) + D(v, F(y, x))\}$.

Example 14. Take the metric space $(X,\ d)$ with binary relation R as considered in Example 8. Let $F : X \times X \to K(X)$ be defined as in Example 8. Then F is R-dominated and X has R-regular property. Let $\psi : [0,\ \infty)^{10} \to [0,\ \infty)$ be defined as $\psi(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}) = \frac{1}{4} \max\{t_1, t_2\}$. Let $(x, y),\ (u, v) \in X \times X$ with $(x, u) \in R,\ (y, v) \in R$. Then $x, y \in [0, 1]$ and $u, v \in [0, \frac{1}{64}]$. Now

$$\begin{aligned} H(F(x, y),\ F(u, v)) &= \left| \frac{x^2+y^2}{64(1+x^2+y^2)} - \frac{u^2+v^2}{64(1+u^2+v^2)} \right| = \left| \frac{x^2+y^2-u^2-v^2}{64(1+x^2+y^2)\ (1+u^2+v^2)} \right| \\ &\leq \left| \frac{x^2+y^2-u^2-v^2}{64} \right| \leq \left| \frac{(x^2-u^2)+(y^2-v^2)}{64} \right| = \left| \frac{(x-u)\ (x+u)+(y-v)\ (y+v)}{64} \right| \\ &\leq \left| \frac{(x-u)+(y-v)}{32} \right| \leq \frac{\max\ \{d(x,\ u),\ d(y,\ v)\}}{4}, \end{aligned}$$

which guarantees that the inequality of Theorem 11 is satisfied. Hence all the conditions of Theorem 11 are satisfied and here $(0,\ 0)$ is a coupled fixed point of F.

3 Data dependence of coupled fixed point

Theorem 15. Let $(X,\ d)$ be a complete metric space with a binary relation R on it such that X has R-regular property. Let $F_l : X \times X \to K(X)$ $(l = 1,\ 2)$ be two R-dominated mappings. Suppose that there exist $p,\ q,\ r,\ s,\ e,\ f,\ g,\ h \geq 0$ and $\psi \in \Psi$ such that F_2 is a generalized coupled contraction. Also, suppose that $Fix(F_1)$ is nonempty and there exists an $\eta > 0$ such that $H(F_1(x,\ y),\ F_2(x,\ y)) \leq \eta$ for all $(x, y) \in X \times X$. Then $\sup\limits_{z \in Fix(F_1)} D_\rho(z, Fix(F_2)) \leq \Theta(\eta)$, where $\Theta(t) = \sum_{n=1}^{\infty} \Phi^n(t)$.

Proof. By Theorem 11, the coupled fixed point set of F_2, that is, $Fix(F_2)$ is nonempty. Let $(x_0,\ y_0)$ be a coupled fixed point of F_1, that is,

$$x_0 \in F_1(x_0,\ y_0) \ \text{ and } \ y_0 \in F_1(y_0,\ x_0). \tag{3.1}$$

By Lemma 1, there exist $x_1 \in F_2(x_0,\ y_0)$ and $y_1 \in F_2(y_0,\ x_0)$ such that $d(x_0,\ x_1) = D(x_0,\ F_2(x_0,\ y_0))$ and $d(y_0,\ y_1) = D(y_0,\ F_2(y_0,\ x_0))$. As F_2 is R-dominated, we have $(x_0,\ x_1) \in R$ and $(y_0,\ y_1) \in R$. Again by Lemma 1, there exists $x_2 \in F_2(x_1,\ y_1)$ and $y_2 \in F_2(y_1,\ x_1)$ such that $d(x_1,\ x_2) = D(x_1,\ F_2(x_1,\ y_1))$ and $d(y_1,\ y_2) = D(y_1,\ F_2(y_1,\ x_1))$. Applying the R-dominated property of F_2, we have $(x_1,\ x_2) \in R$ and $(y_1,\ y_2) \in R$. Arguing as in the proof of Theorem 11, we construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\left.\begin{array}{c} x_{n+1} \in F_2(x_n, y_n) \ \text{ and } \ y_{n+1} \in F_2(y_n, x_n), \\ d(x_n, x_{n+1}) = D(x_n,\ F_2(x_n, y_n)) \text{ and } d(y_n, y_{n+1}) = D(y_n,\ F_2(y_n, x_n)), \\ (x_n,\ x_{n+1}) \in R \ \text{ and } \ (y_n,\ y_{n+1}) \in R. \end{array}\right\} \tag{3.2}$$

Arguing similarly as in the proof of Theorem 11, we can show that

- the inequalities (2.7) and (2.8) are satisfied,
- both $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X,
- there exist $u,\ v \in X$ such that (2.9) is satisfied, that is, $\lim_{n\to\infty} x_n = u$ and $\lim_{n\to\infty} y_n = v$,
- $(u,\ v)$ is a coupled fixed point of F_2, that is, $u \in F_2(u,\ v)$ and $v \in F_2(v,\ u)$.

Using (3.1), (3.2) and the assumption of the theorem, we have

$$d(x_0, x_1) = D(x_0, F_2(x_0, y_0)) \le H(F_1(x_0, y_0), F_2(x_0, y_0)) \le \eta \tag{3.3}$$

and

$$d(y_0, y_1) = D(y_0, F_2(y_0, x_0)) \le H(F_1(y_0, x_0), F_2(y_0, x_0)) \le \eta. \tag{3.4}$$

Using (2.8), (3.3), (3.4) and a property of Φ, we have

$$\begin{aligned} d(x_0,\ u) &\le \sum_{i=0}^{n} d(x_i,\ x_{i+1}) + d(x_{n+1},\ u) \le \sum_{i=0}^{n} r_i + d(x_{n+1},\ u) \\ &\le \sum_{i=0}^{n} \Phi^i(r_0) + d(x_{n+1},\ u) \\ &= \sum_{i=0}^{n} \Phi^i(\max\ \{d(x_0,\ x_1),\ d(y_0,\ y_1)\}) + d(x_{n+1},\ u) \\ &\le \sum_{i=0}^{n} \Phi^i(\max\ \{\eta,\ \eta\}) + d(x_{n+1}, u) = \sum_{i=0}^{n} \Phi^i(\eta) + d(x_{n+1}, u). \end{aligned}$$

Taking limit as $n \to \infty$ in above inequality, we have

$$d(x_0,\ u) \leq \sum_{i=0}^{\infty} \Phi^i(\eta) = \Theta(\eta). \tag{3.5}$$

Similarly, we have

$$d(y_0,\ v) \leq \Theta(\eta). \tag{3.6}$$

Combining (3.5) and (3.6), we have

$$\rho((x_0, y_0),\ (u, v)) = \max\ \{d(x_0, u),\ d(y_0, v)\} \leq \Theta(\eta). \tag{3.7}$$

Then it follows that $D_\rho((x_0, y_0), Fix(F_2)) \leq \Theta(\eta)$. As $(x_0, y_0) \in Fix(F_1)$ is arbitrary, we have $\sup\limits_{z \in Fix(F_1)} D_\rho(z, Fix(F_2)) \leq \Theta(\eta)$. □

4 Stability of coupled fixed point sets

Lemma 16. Let $(X,\ d)$ be a metric space with a binary relation R on it. Let $\{F_n : X \times X \to K(X) : n = 1,\ 2,\ 3,\ ...\}$ be a sequence of R- dominated mappings converging to a mapping $F : X \times X \to K(X)$. Suppose that there exist $p,\ q,\ r,\ s,\ e,\ f,\ g,\ h \geq 0$ and $\psi \in \Psi$ such that each F_n, $(n = 1,\ 2,\ 3,\ ...)$ is a generalized coupled contraction. Also, suppose that for any convergent sequence $\{x_n\}$ in X with limit $x \in X$,

$$(u,\ x_n) \in R, \text{ for all } n \text{ implies } \ (u,\ x) \in R. \tag{4.1}$$

Then F is a R-dominated mapping and a generalized coupled contraction.

Proof. First we prove that F is R-dominated. Let $(x,\ y) \in X \times X$ with $u \in F(x,\ y)$ and $v \in F(y,\ x)$. Since $F_n \to F$ as $n \to \infty$, there exist two sequences $\{u_n\}$ in $\{F_n(x,\ y)\}$ and $\{v_n\}$ in $\{F_n(y,\ x)\}$ such that $u_n \to u$ and $v_n \to v$ as $n \to \infty$. As each F_n is R-dominated, we have $(x,\ u_n) \in R$ and $(y,\ v_n) \in R$ for every $n \in \mathbb{N}$. Then by (4.1), it follows that $(x,\ u) \in R$ and $(y,\ v) \in R$. Hence F is R- dominated.

Next we prove that F is a generalized coupled contraction. Let $(x, y), (u, v) \in X \times X$ with $(x, u) \in R$ and $(y, v) \in R$. Since each F_n $(n \in \mathbb{N})$ is a generalized coupled contraction, we have

$$\begin{aligned}H(F_n(x,\ y),\ F_n(u,\ v)) &\leq M(x,\ y,\ u,\ v)\\ &= \psi\Big(d(x,u),\ d(y,v),\ \frac{D(x,F_n(x,y))[1+D(x,F_n(x,y))^p]}{1+d(x,u)^p},\\ &\qquad \frac{D(y,F_n(y,x))[1+D(y,F_n(y,x))^e]}{1+d(y,v)^e},\\ &\qquad \frac{D(u,F_n(x,y))[1+D(x,F_n(x,y))^q]}{1+d(x,u)^q},\\ &\qquad \frac{D(v,F_n(y,x))[1+D(y,F_n(y,x))^f]}{1+d(y,v)^f},\\ &\qquad \frac{D(u,F_n(x,y))[1+D(u,F_n(x,y))^r]}{1+d(x,u)^r},\\ &\qquad \frac{D(v,F_n(y,x))[1+D(v,F_n(y,x))^g]}{1+d(y,v)^g},\\ &\qquad \frac{D(u,F_n(x,y))[1+D(x,F_n(u,v))^s]}{1+d(x,u)^s},\\ &\qquad \frac{D(v,F_n(y,x))[1+D(y,F_n(v,u))^h]}{1+d(y,v)^h}\Big). \qquad (4.2)\end{aligned}$$

Since F_n converges to F, taking limit as $n \to \infty$ in (4.2) and using the continuity of ψ, we have

$$\begin{aligned}H(F(x,\ y),\ F(u,\ v)) &\leq \psi\Big(d(x,u),\ d(y,v),\ \frac{D(x,F(x,y))[1+D(x,F(x,y))^p]}{1+d(x,u)^p},\\ &\qquad \frac{D(y,F(y,x))[1+D(y,F(y,x))^e]}{1+d(y,v)^e},\\ &\qquad \frac{D(u,F(x,y))[1+D(x,F(x,y))^q]}{1+d(x,u)^q},\\ &\qquad \frac{D(v,F(y,x))[1+D(y,F(y,x))^f]}{1+d(y,v)^f},\\ &\qquad \frac{D(u,F(x,y))[1+D(u,F(x,y))^r]}{1+d(x,u)^r},\\ &\qquad \frac{D(v,F(y,x))[1+D(v,F(y,x))^g]}{1+d(y,v)^g},\\ &\qquad \frac{D(u,F(x,y))[1+D(x,F(u,v))^s]}{1+d(x,u)^s},\\ &\qquad \frac{D(v,F(y,x))[1+D(y,F(v,u))^h]}{1+d(y,v)^h}\Big)\\ &= M(x,\ y,\ u,\ v).\end{aligned}$$

Therefore, F is a generalized coupled contraction. □

Theorem 17. Let (X,d) be a complete metric space with a binary relation R on it such that X has R-regular property and (4.1) holds. Let $\{F_n : X \times X \to K(X) : n = 1, 2, 3, ...\}$ be a sequence of R-dominated mappings converging uniformly to $F : X \times X \to K(X)$. Suppose that there exist $p,\ q,\ r,\ s,\ e,\ f,\ g,\ h \geq 0$ and $\psi \in \Psi$ such that each F_n $(n \in \mathbb{N})$ is a generalized coupled contraction. Then $Fix(F_n) \neq \emptyset$, for every $n \in \mathbb{N}$ and $Fix(F) \neq \emptyset$. Also, the sets of coupled fixed points of F_n are stable if $\Theta(t) \to 0$ as $t \to 0$, where $\Theta(t) = \sum_{n=1}^{\infty} \Phi^n(t)$.

Proof. By Lemma 16 and Theorem 11, $Fix(F_n) \neq \emptyset$, for every $n \in \mathbb{N}$ and $Fix(F) \neq \emptyset$. Let $\Delta_n = \sup_{(x,\ y) \in X \times X} H(F_n(x,y),\ F(x,y))$. By Theorem 15, we have

$$\sup_{z \in Fix(F)} D_\rho(z, Fix(F_n)) \leq \Theta(\Delta_n) \ \text{ and } \ \sup_{z_n \in Fix(F_n)} D_\rho(z_n, Fix(F)) \leq \Theta(\Delta_n)$$

. Then it follows that

$$H_\rho(Fix(F_n),\ Fix(F)) \leq \Theta(\Delta_n). \tag{4.3}$$

Since $F_n \to F$ uniformly, we have $\Delta_n \to 0$ as $n \to \infty$. Taking limit as $n \to \infty$ in (4.3) and using the continuity of ψ, we have $\lim_{n\to\infty} H_\rho(Fix(F_n),\ Fix(F)) = 0$. Therefore, the sets of coupled fixed points of F_n are stable. □

5 Some consequences in partially ordered metric spaces

Our next results are on partially ordered metric spaces in which there is a large literature of fixed point theory [5, 9, 29, 30].

Definition 18. Let $(X,\ \preceq)$ be a nonempty partially ordered set. A mapping $F : X \times X \to X$ is said to be dominated if $x \preceq F(x,y)$ and $F(y,x) \preceq y$, whenever $(x,\ y) \in X \times X$. A multivalued mapping $F : X \times X \to N(X)$ is said to be dominated if $x \preceq u$ and $v \preceq y$, whenever $(x,\ y) \in X \times X$ and $u \in F(x,y)$, $v \in F(y,x)$.

Definition 19. Let $(X,\ d)$ be a metric space with a partial order $\preceq$ on it. Then X is said to have regular property if for every increasing (or decreasing) sequence $\{x_n\}$ in X, $x_n \to x$ as $n \to \infty$ implies $x_n \preceq x$ (or $x \preceq x_n$), for all n.

Definition 20. Let $(X,\ d)$ be a metric space with a partial order $\preceq$ on it. A multivalued mapping $F : X \times X \to K(X)$ is said to be a ordered generalized coupled contraction if there exist $p,\ q,\ r,\ s,\ e,\ f,\ g,\ h \geq 0$ and $\psi \in \Psi$ such that the inequality (1.1) in Definition 9 is satisfied for all $(x,\ y)$ and $(u,\ v) \in X \times X$ with either $[x \preceq u$ and $v \preceq y]$ or $[u \preceq x$ and $y \preceq v]$.

Theorem 21. Let $(X,\ d)$ be a complete metric space with a partial order $\preceq$ on it such that X has regular property. Suppose that $F : X \times X \to K(X)$ is a dominated mapping and there exist $p,\ q,\ r,\ s,\ e,\ f,\ g,\ h \geq 0$ and $\psi \in \Psi$ such that F is a ordered generalized coupled contraction. Then F has a coupled fixed point in $X \times X$.

Proof. Define a binary relation R on X as $(x,\ y) \in R$ if and only if $x \preceq y$ or $y \preceq x$, for $x,\ y \in X$. Then (i) either $[x \preceq u$ and $v \preceq y]$ or $[u \preceq x$ and $y \preceq v]$ imply $(x,u) \in R$ and $(y,v) \in R$, (ii) $x \preceq u$ and $v \preceq y$ imply $(x,\ u) \in R$ and $(y,\ v) \in R$, (iii) $x_n \preceq x_{n+1}$, $x_n \preceq x$ imply $(x_n,\ x_{n+1}) \in R$, $(x_n,\ x) \in R$ and also $x_{n+1} \preceq x_n$, $x \preceq x_n$ imply $(x_n,\ x_{n+1}) \in R$, $(x_n,\ x) \in R$. Consequently, all the assumptions reduce to the assumptions of Theorem 11. By an application of Theorem 11, we conclude that F has a coupled fixed point in $X \times X$. □

6 Some consequences for α-dominated mappings

Our next results for an α-dominated mapping. This concept is defined here and is conceptual extension of the admissibility condition. Various types of admissibility conditions have been used in fixed point theory in works like [16, 17, 35].

Definition 22. Let X be a nonempty set and $\alpha : X \times X \to [0,\ \infty)$. A mapping $F : X \times X \to X$ is said to be α-dominated if $\alpha(x,\ F(x,y)) \geq 1$ and $\alpha(y, F(y,x)) \geq 1$, whenever $(x,y) \in X \times X$. A multivalued mapping $F : X \times X \to N(X)$ is said to be α-dominated if $\alpha(x,u) \geq 1$ and $\alpha(y,v) \geq 1$, whenever $(x,y) \in X \times X$ and $u \in F(x,y)$, $v \in F(y,x)$.

Definition 23. Let (X,d) be a metric space and $\alpha : X \times X \to [0,\ \infty)$. Then X is said to have α-regular property if for every convergent sequence $\{x_n\}$ with limit $x \in X$, $\alpha(x_n,\ x_{n+1}) \geq 1$, for all n implies $\alpha(x_n,\ x) \geq 1$, for all n.

Definition 24. Let $(X,\ d)$ be a metric space and $\alpha : X \times X \to [0,\ \infty)$. A multivalued mapping $F : X \times X \to K(X)$ is called α-generalized coupled contraction if there exist $p,\ q,\ r,\ s,\ e,\ f,\ g,\ h \geq 0$ and $\psi \in \Psi$ such that the inequality (1.1) in Definition 9 is satisfied for all $(x,\ y)$ and $(u,\ v) \in X \times X$ with $\alpha(x,\ u) \geq 1$ and $\alpha(y,\ v) \geq 1$.

Theorem 25. Let (X,d) be a complete metric space and $\alpha : X \times X \to [0,\ \infty)$ be a mapping such that X has α-regular property. Suppose that $F : X \times X \to K(X)$ is an α-dominated mapping and there exist $p,\ q,\ r,\ s,\ e,\ f,\ g,\ h \geq 0$ and $\psi \in \Psi$ such that F is an α-generalized coupled contraction. Then F has a coupled fixed point in $X \times X$.

Proof. Define a binary relation R on X as $(x,\ y) \in R$ if and only if $\alpha(x,\ y) \geq 1$, for $x,\ y \in X$. Then all the assumptions reduce to the assumptions of Theorem 11. Therefore, by an application of Theorem 11, we have that F has a coupled fixed point in $X \times X$. □

7 Some consequences on graphic contractions

Our final section is on graphic contraction. Fixed point problem on the structures of metric spaces with a graph have appeared in works like [3, 6, 21, 25].

Let X be a nonempty set and $\Delta := \{(x,\ x) : x \in X\}$. Let G be a directed graph such that its vertex set $V(G)$ coincides with X, that is, $V(G) = X$ and the edge set $E(G)$ contains all loops, that is, $\Delta \subseteq E(G)$. Assume that G has no parallel edges. By G^{-1} we denote the conversion of a graph G, that is, the graph obtained from G by reversing the directions of the edges. Thus we have $V(G^{-1}) = V(G)$ and $E(G^{-1}) = \{(x,\ y) \in X \times X :\ (y,\ x) \in E(G)\}$. A nonempty set X is said to be endowed with a directed graph $G(V, E)$ if $V(G) = X$ and $\Delta \subseteq E(G)$.

Definition 26. Let X be a nonempty set endowed with a graph $G(V, E)$. A mapping $F : X \times X \to X$ is said to be G-dominated if $(x, F(x, y)) \in E$ and $(y, F(y, x)) \in E$, whenever $(x, y) \in X \times X$. A multivalued mapping $F : X \times X \to N(X)$ is said to be G-dominated if $(x, u) \in E$ and $(y, v) \in E$, whenever $(x,\ y) \in X \times X$ and $u \in F(x, y)$, $v \in F(y, x)$.

Definition 27. Let $(X,\ d)$ be a metric space endowed with a directed graph $G(V, E)$. Then X is said to have G-regular property if for every convergent sequence $\{x_n\}$ with limit $x \in X$, $(x_n,\ x_{n+1}) \in E$, for all n implies $(x_n,\ x) \in E$, for all n.

Definition 28. Let (X, d) be a metric space endowed with a directed graph $G(V, E)$. A multivalued mapping $F : X \times X \to K(X)$ is called graphic generalized coupled contraction if there exist $p,\ q,\ r,\ s,\ e,\ f,\ g,\ h \geq 0$ and $\psi \in \Psi$ such that the inequality (1.1) in Definition 9 is satisfied for all $(x,\ y)$ and $(u,\ v) \in X \times X$ with $(x,\ u) \in E$ and $(y,\ v) \in E$.

Theorem 29. Let $(X,\ d)$ be a complete metric space endowed with a directed graph $G(V, E)$ such that X has G-regular property. Suppose that $F : X \times X \to K(X)$ is a G-dominated mapping and there exist $p,\ q,\ r,\ s,\ e,\ f,\ g,\ h \geq 0$ and $\psi \in \Psi$ such that F is a graphic generalized coupled contraction. Then F has a coupled fixed point in $X \times X$.

Proof. Define a binary relation R on X as $(x,\ y) \in R$ if and only if $(x,\ y) \in E$, for $x,\ y \in X$. Then all the assumptions reduce to the assumptions of Theorem 11. Therefore, by an application of Theorem 11, we conclude that F has a coupled fixed point in $X \times X$. □

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Binayak S. Choudhury
Department of Mathematics, Indian Institute of Engineering Science and Technology,
Shibpur, Howrah - 711103, West Bengal, India
E-mail: binayak12@yahoo.co.in

Nikhilesh Metiya
Department of Mathematics, Sovarani Memorial College,
Jagatballavpur, Howrah-711408, West Bengal, India
E-mail: metiya.nikhilesh@gmail.com

Sunirmal Kundu
Department of Mathematics, Government General Degree College, Salboni,
Paschim Mednipur - 721516, West Bengal, India
E-mail: sunirmalkundu2009@rediffmail.com

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