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EXISTENCE RESULTS FOR LANGEVIN EQUATION WITH RIESZ-CAPUTO FRACTIONAL DERIVATIVE

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Abstract. In this paper, we examine existence and uniqueness of solutions for nonlinear Langevin equation involving Riesz-Caputo fractional derivatives, with a class of anti-periodic boundary conditions. By applying a variety of fixed point theorems as Banach, Schaefer and Krasnoselskii fixed point theorems. Three examples are given to illustrate main results.

1 Introduction

The fractional calculus is a well established domain in mathematical analysis, it is of a great applicability some disciplines such as the theory of control, signal, image processing, chemistry and economics [11, 15, 1, 14].

Fractional Langevin differential equations have been one of the important subject in physics, chemistry and electrical engineering. The Langevin equation (first formulated by Langevin in 1908) is found to be an effective tool to describe the evolution of physical phenomena in fluctuating environments [12]. In physics the Langevin equation is a stochastic differential equation and it is used in modeling physics phenomena, analyzing the stock market, photo electron and complex systems, protein dynamics and anomalous transport [3, 16]. Therefore the generalized Langevin equation can be used to formulate a large number of various problems. Many researchers studied fractional nonlinear Langevin equation with fractional derivatives [8, 2, 7, 10, 18].
In this paper we investigated the existence of solution of the following nonlinear Langevin equation with Riesz-Caputo derivative:

$$\begin{cases} {}_{0}^{RC}D_T^{\alpha}({}_{0}^{RC}D_T^{\beta}+\chi)x(t)=f(t,x(t)) \ ,0<t<T, \\ x(0)+x(T)=0 \ , \ x'(0)+x'(T)=0, \end{cases} \tag{1.1}$$

where ${}^{RC}D^{\alpha}$ and ${}^{RC}D^{\beta}$ are the Riesz-Caputo fractional derivatives of order $1<\alpha\leq 2$ and $0<\beta\leq 1$, $\chi\in\mathbb{R}$ and $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ is a continuous function with

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respected to its both variables, t and x.
The paper is organized as the follows: In section 2, we presented some preliminaries and lemmas witch will be used through the paper. section 3 is devoted to the main results, concerning existence and uniqueness solution of 1.1. In section 4 three examples are treated, illustrating our main results.

2 Preliminaries

In this section, we introduce basic concepts, definitions, properties of fractional calculus and some auxiliary lemmas which we need later.
Let $\alpha > 0$, and $n-1 < \alpha \leq n$, $n \in \mathbb{N}$ and $n = \lceil \nu \rceil$, where $\lceil . \rceil$ is the ceiling of a number.

Definition 1. *([5]) Riesz-Caputo derivative of order α, of a function $f \in C^n([0,T])$ is defined by*

$$\begin{aligned} {}_0^{RC}D_T^{\alpha} f(t) &= \frac{1}{\Gamma(n-\alpha)} \int_0^T |t-u|^{n-\alpha-1} f^{(n)}(u)du \\ &= \frac{1}{2}({}_0^C D_t^{\alpha} + (-1)^n {}_t^C D_T^{\alpha}) f(t), \end{aligned}$$

where ${}_0^C D_t^{\alpha}$ is the left Caputo derivative and ${}_t^C D_T^{\alpha}$ is right Caputo derivative,

$$\begin{aligned} {}_0^C D_t^{\alpha} f(t) &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-u)^{n-\alpha-1} f^{(n)}(u)du, \\ {}_t^C D_T^{\alpha} f(t) &= \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^T (u-t)^{n-\alpha-1} f^{(n)}(u)du. \end{aligned}$$

Remark 2. *In particular if $f \in C^2([0,T])$ and $0 < \alpha \leq 1$, then*

$$ {}_0^{RC}D_T^{\alpha} f(t) = \frac{1}{2}({}_0^C D_t^{\alpha} - {}_t^C D_T^{\alpha}) f(t).$$

If $f \in C^2([0,1])$ and if $1 < \alpha \leq 2$, then

$$ {}_0^{RC}D_T^{\alpha} f(t) = \frac{1}{2}({}_0^C D_t^{\alpha} + {}_t^C D_T^{\alpha}) f(t).$$

Definition 3. *([5]) The left, right fractional and Riemann-Liouville integrals of order α are defined by:*

$$ {}_0I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} f(u)du,$$

$$ {}_tI_T^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_t^T (u-t)^{\alpha-1}f(u)du, $$

$$ {}_0I_T^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^T |u-t|^{\alpha-1}f(u)du. $$

Lemma 4. *([5]) If $f(t) \in C^n([0,T])$, Then*

$$ {}_0I_{t0}^{\alpha C}D_t^{\alpha}f(t) = f(t) - \sum_{k=0}^{n-1}\frac{f^{(k)}(0)(t-0)^k}{k!}(t-0)^k, $$

and

$$ {}_tI_{Tt}^{\alpha C}D_T^{\alpha}f(t) = (-1)^n\left[f(t) - \sum_{k=0}^{n-1}\frac{(-1)^k f^{(k)}(T)}{k!}(T-t)^k\right]. $$

From the above definitions and lemmas, we have

$$ \begin{aligned} {}_0I_{T0}^{\alpha RC}D_T^{\alpha}f(t) &= \frac{1}{2}({}_0I_{t0}^{\alpha C}D_t^{\alpha} +_t I_{T0}^{\alpha C}D_t^{\alpha})f(t) \\ &\quad + (-1)^n\frac{1}{2}({}_0I_{tt}^{\alpha C}D_T^{\alpha} +_t I_{Tt}^{\alpha C}D_T^{\alpha})f(t) \\ &= \frac{1}{2}({}_0I_{t0}^{\alpha C}D_t^{\alpha} + (-1)_t^n I_{Tt}^{\alpha C}D_T^{\alpha})f(t). \end{aligned} $$

In particular if $1 < \alpha \leq 2$ and $f \in C^2([0,T])$, then

$$ {}_0I_{T0}^{\alpha RC}D_T^{\alpha}f(t) = f(t) - \frac{1}{2}(f(0)+f(T)) - \frac{1}{2}f'(0)t + \frac{1}{2}f'(T)(T-t). \tag{2.1} $$

3 Main Results

Lemma 5. *Let $g \in C([0,T],\mathbb{R})$ and $x \in C^2([0,T],\mathbb{R})$. Then the problem*

$$ \begin{cases} {}_0^{RC}D_T^{\alpha}({}_0^{RC}D_T^{\beta} + \chi)x(t) = g(t),\ 0<t<T, \\ x(0)+x(T) = 0\ ,\ x'(0)+x'(T) = 0, \end{cases} \tag{3.1} $$

is equivalent to the integral equation given by

$$ \begin{aligned} x(t) &= \frac{-\chi}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}x(s)ds - \frac{\chi}{\Gamma(\beta)}\int_t^T (s-t)^{\beta-1}x(s)ds \\ &+ \frac{\chi^2 T(t^{\beta}-(T-t)^{\beta})}{\beta\Gamma(\beta-1)(2\Gamma(\beta)+\chi T^{\beta})}\int_0^T (T-s)^{\beta-2}x(s)ds \\ &- \frac{\chi T(t^{\beta}-(T-t)^{\beta})}{\beta\Gamma(\alpha+\beta-1)(2\Gamma(\beta)+\chi T^{\beta})}\int_0^T (T-s)^{\alpha+\beta-2}g(s)ds \\ &+ \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}g(s)ds \\ &+ \frac{1}{\Gamma(\alpha+\beta)}\int_t^T (s-t)^{\alpha+\beta-1}g(s)ds \end{aligned} \tag{4} $$

Also, we define the notations:

$$\phi_1 = \frac{2|\chi|T^{\beta}}{\Gamma(\beta+1)} + \frac{\chi^2 T^{2\beta}}{\Gamma(\beta+1)|2\Gamma(\beta)+\chi T^{\beta}|} \quad (5)$$
$$+ \frac{|\chi| L_1 T^{\alpha+2\beta}}{\beta\Gamma(\alpha+\beta)|2\Gamma(\beta)+\chi T^{\beta}|} + \frac{2L_1 T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)},$$
$$k_1 = \left(\frac{2|\chi|T^{\beta}}{\Gamma(\beta+1)} + \frac{\chi^2 T^{2\beta}}{\Gamma(\beta+1)|2\Gamma(\beta)+\chi T^{\beta}|} \right), \quad (6)$$
$$k_2 = \left[\frac{|\chi| T^{\alpha+2\beta}}{\beta\Gamma(\alpha+\beta)|2\Gamma(\beta)+\chi T^{\beta}|} + \frac{2T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \right], \quad (7)$$

Proof. Applying the integral operator ${}_0I_T^{\alpha}$ to both sides of (3.1)and by using Lemma(4), we get

$$ {}_0^{RC}D_T^{\beta} x(t) + \chi x(t) + \frac{{}_0^{RC}D_T^{\beta} x'(T)T}{2} + \frac{1}{2}\chi x'(t)T ={}_0 I_T^{\alpha} g(t). \quad (8)$$

Applying the integral operator ${}_0I_T^{\beta}$ to the both sided of (8)and using the Lemma (4), we obtain

$$x(t) = \frac{1}{2}(x(0)+x(T)) - \chi {}_0I_T^{\beta} x(t) - \frac{1}{2} {}_0I_T^{\beta} x'(T)T +{}_0 I_T^{\alpha+\beta} g(t). \quad (9)$$

Rewriting equation (9) under the form

$$x(t) = \frac{-\chi}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}x(s)ds - \frac{\chi}{\Gamma(\beta)}\int_t^T (s-t)^{\beta-1}x(s)ds$$
$$- \frac{\chi T x'(T)}{2\Gamma(\beta+1)}t^{\beta} + \frac{\chi T x'(T)}{2\Gamma(\beta+1)}(T-t)^{\beta} + \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}g(s)ds$$
$$+ \frac{1}{\Gamma(\alpha+\beta)}\int_t^T (s-t)^{\alpha+\beta-1}g(s)ds. \quad (10)$$

Then taking the derivative of (10), we get

$$x'(t) = \frac{-\chi}{\Gamma(\beta-1)}\int_0^t (t-s)^{\beta-2}x(s)ds + \frac{\chi}{\Gamma(\beta-1)}\int_t^T (s-t)^{\beta-2}x(s)ds$$
$$- \frac{\chi T x'(T)}{2\Gamma(\beta)}t^{\beta-1} - \frac{\chi T x'(T)}{2\Gamma(\beta)}(T-t)^{\beta-1}$$
$$+ \frac{\alpha+\beta-1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-2}g(s)ds - \frac{\alpha+\beta-1}{\Gamma(\alpha+\beta)}\int_t^T (s-t)^{\alpha+\beta-2}g(s)ds.$$

Using the boundary conditions of (1.1), we deduce

$$x'(T) = \frac{-2\chi(\beta-1)}{2\Gamma(\beta)+\chi T^{\beta}} \int_0^T (T-s)^{\beta-2} x(s)ds + \frac{2\Gamma(\beta)}{\Gamma(\alpha+\beta-1)(2\Gamma(\beta)+\chi T^{\beta})} \int_0^T (T-s)^{\alpha+\beta-2} g(s)ds. \tag{11}$$

Substituting the value of (11) in (10), we obtain (4). The proof is complete. □

Let us denote by $\mathbb{E} = C([0,T],\mathbb{R})$ the Banach space of all continuous function from $[0,T] \to \mathbb{R}$ endowed with the norm defined by

$$\|x\| = \sup\{|x(t)| : t \in [0,T]\}.$$

By lemma(5), we define an operator $H : \mathbb{E} \to \mathbb{E}$, associated to (1.1)

$$\begin{aligned}(Hx)(t) = {} & \frac{-\chi}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} x(s)ds - \frac{\chi}{\Gamma(\beta)} \int_t^T (s-t)^{\beta-1} x(s)ds \\ & + \frac{\chi^2 T(t^{\beta}-(T-t)^{\beta})}{\beta\Gamma(\beta-1)(2\Gamma(\beta)+\chi T^{\beta})} \int_0^T (T-s)^{\beta-2} x(s)ds \\ & - \frac{\chi T(t^{\beta}-(T-t)^{\beta})}{\beta\Gamma(\alpha+\beta-1)(2\Gamma(\beta)+\chi T^{\beta})} \int_0^T (T-s)^{\alpha+\beta-2} f(s,x(s))ds \\ & + \frac{1}{\Gamma(\alpha+\beta)}{}_0^{t} (t-s)^{\alpha+\beta-1} f(s,x(s))ds \\ & + \frac{1}{\Gamma(\alpha+\beta)} \int_t^T (s-t)^{\alpha+\beta-1} f(s,x(s))ds \end{aligned} \tag{12}$$

In the following subsections, we prove existence, existence and uniqueness results for the boundary value problem(1.1), by using Banach, Scheafer fixed and Krasnoselskii fixed point theorems.

3.1 Existence and uniqueness result via Banach fixed point theorem

Theorem 6. *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that:*

(H_1) *There exists a constant $L_1 > 0$ such that*

$$|f(t,x) - f(t,y)| \le L_1 \|x-y\|,$$

for each $t \in [0,T]$ and $x, y \in \mathbb{R}$.
Then the boundary value problem (1.1) has a unique solution on $[0,T]$ if

$$\phi_1 < 1,$$

where ϕ_1 is defined by (5).

Proof. By using operator H, which is defined by (12), we have

$$
\begin{aligned}
|(Hx)(t)-(Hy)(t)| &\leq \frac{\chi}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}|x(s)-y(s)|ds + \frac{\chi}{\Gamma(\beta)}\int_t^T (s-t)^{\beta-1}|x(s)-y(s)|ds \\
&+ \frac{\chi^2 T|t^\beta-(T-t)^\beta|}{\beta\Gamma(\beta-1)(|2\Gamma(\beta)+\chi T^\beta|}\int_0^T (T-s)^{\beta-2}|x(s)-y(s)|ds \\
&+ \frac{|\chi|T|t^\beta-(T-t)^\beta|}{\beta\Gamma(\alpha+\beta-1)|2\Gamma(\beta)+\chi T^\beta|}\int_0^T (T-s)^{\alpha+\beta-2}|f(s,x(s))-f(s,y(s))|ds \\
&+ \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}|f(s,x(s))-f(s,y(s))|ds \\
&+ \frac{1}{\Gamma(\alpha+\beta)}\int_t^T (s-t)^{\alpha+\beta-1}|f(s,x(s)-f(s,y(s))|ds, \\
&\leq \frac{|\chi|T^\beta}{\Gamma(\beta+1)}\|x-y\| + \frac{|\chi|T^\beta}{\Gamma(\beta+1)}\|x-y\| + \frac{\chi^2T^{2\beta}}{\Gamma(\beta-1)(|2\Gamma(\beta)+\chi T^\beta|}\|x-y\| \\
&+ \frac{L_1|\chi|T^{\alpha+2\beta}}{\beta\Gamma(\alpha+\beta-1)|2\Gamma(\beta)+\chi T^\beta|}\|x-y\| + \frac{L_1T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\|x-y\| + \frac{LT^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\|x-y\| \\
&\leq \left\{\frac{2|\chi|T^\beta}{\Gamma(\beta+1)} + \frac{\chi^2T^{2\beta}}{\Gamma(\beta+1)|2\Gamma(\beta)+\chi|T^\beta} + \frac{|\chi|L_1T^{\alpha+2\beta}}{\beta\Gamma(\alpha+\beta)|2\Gamma(\beta)+\chi T^\beta|} + \frac{2L_1T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right\}\|x-y\|, \\
&\leq \phi_1\|x-y\|,
\end{aligned}
$$

for any $x, y \in \mathbb{E}$, and for each $t \in [0,T]$. Thus implies that $\|Hx-Hy\| \leq \phi_1\|x-y\|$. As $\phi_1 < 1$, the operator $H : \mathbb{E} \to \mathbb{E}$ is a contraction mapping. As result, the boundary value problem (1.1) has a unique solution on $[0,T]$. □

3.2 Existence result via Scheafer fixed point theorem

Lemma 7. *Let $\mathbb{E}$ be a Banach space. Assume that the operator $H : \mathbb{E} \to \mathbb{E}$ is a completely continuous and assume also that the set*

$$V = \{x \in \mathbb{E}/x = \mu Tx, 0 < \mu < 1\},$$

is bounded. Then H has a fixed point in $\mathbb{E}$.

Theorem 8. *Assume that there exists a positive constant $L_2 > 0$, such that $|f(t,x)| \leq L_2$, for $t \in [0,T], x \in \mathbb{R}$. Then the boundary value problem (1.1) has at least one solution on $[0,T]$.*

Proof. **Step 1**: In the first step, we show that the operator H defined by (12) is completely continuous. Observe that the continuity of H follows from the continuity

of f.
For a positive constant d, let

$$B_d = \{x \in \mathbb{E} : \|x\| \leq d\},$$

be a closed bounded subset in $\mathbb{E}$.
Step 2: H maps bounded sets into bounded sets in $\mathbb{E}$.
For each $x \in B_d$ and $t \in [0,T]$, we have

$$\begin{aligned}
|(Hx)(t)| \leq & \frac{|\chi|}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}|x(s)|ds + \frac{|\chi|}{\Gamma(\beta)} \int_t^T (t-s)^{\beta-1}|x(s)|ds \\
& + \frac{\chi^2 T|t^\beta - (T-t)^\beta|}{\beta\Gamma(\beta-1)|2\Gamma(\beta)+\chi T^\beta|} \int_0^T (T-s)^{\beta-2}|x(s)|ds \\
& + \frac{|\chi|T|t^\beta - (T-t)^\beta|}{\beta\Gamma(\alpha+\beta-1)|2\Gamma(\beta)+\chi T^\beta|} \int_0^T (T-s)^{\alpha+\beta-2}|f(s,x(s))|ds \\
& + \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1}|f(s,x(s))|ds \\
& + \frac{1}{\Gamma(\alpha+\beta)} \int_t^T (s-t)^{\alpha+\beta-1}|f(s,x(s))|ds \\
& \leq \frac{2|\chi|T^\beta}{\Gamma(\beta+1)}\|x(s)\| + \frac{\chi^2 T^{2\beta}}{\Gamma(\beta+1)|2\Gamma(\beta)+\chi T^\beta|}\|x(s)\| \\
& + \frac{|\chi|T^{\alpha+2\beta}}{\beta\Gamma(\alpha+\beta)|2\Gamma(\beta)+\chi T^\beta|}L_2 + \frac{2T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}L_2 := K.
\end{aligned}$$

Then K is a constant and $\|H(x)\| \leq K$, which implies that H maps bounded sets into bounded sets in $\mathbb{E}$.
Step 3: H maps bounded sets into equicontinuous sets of $\mathbb{E}$(H is completely continuous).

Let $t_1, t_2 \in [0,T]$, with $t_1 < t_2$, and $x \in B_d$. Then we have

$$
\begin{aligned}
&|(Hx)(t_2) - (Hx)(t_1)| \le \frac{|\chi|}{\Gamma(\beta)} \int_0^{t_1} \left|(t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1}\right| |x(s)| ds \\
&+ \frac{|\chi|}{\Gamma(\beta)} \int_{t_1}^{t_2} |(t_2 - s)^{\beta-1} - (s - t_1)^{\beta-1}||x(s)| ds \\
&+ \frac{|\chi|}{\Gamma(\beta)} \int_{t_2}^{T} \left|(s - t_2)^{\beta-1} - (s - t_1)^{\beta-1}\right| |x(s)| ds \\
&+ \frac{1}{\Gamma(\alpha+\beta)} \int_0^{t_1} \left|(t_2 - s)^{\alpha+\beta-1} - (t_1 - s)^{\alpha+\beta-1}\right| |f(s, x(s))| ds \\
&+ \frac{1}{\Gamma(\alpha+\beta)} \int_{t_1}^{t_2} \left|(t_2 - s)^{\alpha+\beta-1} - (s - t_1)^{\alpha+\beta-1}\right| |f(s, x(s))| ds \\
&+ \frac{1}{\Gamma(\alpha+\beta)} \int_{t_2}^{T} \left|(s - t_2)^{\alpha+\beta-1} - (s - t_1)^{\alpha+\beta-1}\right| f(s, x(s))| ds \\
&+ \frac{\chi^2 T \left(|(T - t_2)^\beta - (T - t_1)^\beta| + |t_2^\beta - t_1^\beta|\right)}{\beta\Gamma(\beta - 1)|2\Gamma(\beta) + \chi T^\beta|} \int_0^T (T - s)^{\beta-1} |x(s)| ds \\
&+ \frac{|\chi| T \left(|(T - t_2)^\beta - (T - t_1)^\beta| + |t_2^\beta - t_1^\beta|\right)}{\beta\Gamma(\alpha + \beta - 1)|2\Gamma(\beta) + \chi T^\beta|} \int_0^T (T - s)^{\alpha+\beta-2} |f(s, x(s))| ds \\
&\le \frac{|\chi|\rho}{\Gamma(\beta+1)} \left|(t_2 - t_1)^\beta + (t_2^\beta - t_1^\beta)\right| + \frac{|\chi|\rho}{\Gamma(\beta+1)} \left|(t_2 - t_1)^\beta + (t_2 - t_1)^\beta\right| \\
&+ \frac{|\chi|\rho}{\Gamma(\beta+1)} \left|(t_2 - t_1)^\beta - (t_1 - t_2)^\beta\right| \\
&+ \frac{L_2}{\Gamma(\alpha+\beta+1)} \left|(t_2 - t_1)^{\alpha+\beta} + (t_2 - t_1)^{\alpha+\beta}\right| \\
&+ \frac{L_2}{\Gamma(\alpha+\beta+1)} \left|(t_2 - t_1)^{\alpha+\beta} - (t_1 - _2)^{\alpha+\beta}\right| \\
&+ \frac{L_2}{\Gamma(\alpha+\beta+1)} \left|(T - t_2)^{\alpha+\beta} - (T - t_1)^{\alpha+\beta} - (t_2 - t_1)^{\alpha+\beta}\right| \\
&+ \frac{\chi^2 T^\beta \left|(T - t_1)^\beta - (T - t_2)^\beta| + (t_2^\beta - t_1^\beta)\right|}{\Gamma(\beta+1)|2\Gamma(\beta) + \chi T^\beta|} \\
&+ \frac{|\chi| T^{\alpha+\beta} \left|(T - t_2)^\beta - (T - t_1)^\beta + (t_2^\beta - t_1^\beta)\right|}{\beta\Gamma(\alpha+\beta)|2\Gamma(\beta) + \chi T^\beta|}
\end{aligned}
$$

As $t_2 - t_1 \to 0$, the right-hind side of the above inequality tends to zeros independently of $x \in B_d$. That means H is equicontinuous and by Arzela-Ascoli theorem the operator $H : \mathbb{E} \to \mathbb{E}$ is completely continuous.

Step 4: Finally, we consider the set V defined by:

$$V = \{x \in \mathbb{E} / x = \mu H x, 0 < \mu < 1\},$$

and show that V is bounded.
For $x \in V$and $t \in [0,T]$, we have

$$\begin{aligned} |x(t)| &\leq \left(\frac{2|\chi|T^{\beta}}{\Gamma(\beta+1)} + \frac{\chi^2 T^{2\beta}}{\Gamma(\beta+1)|2\Gamma(\beta) + \chi T^{\beta}|} \right) \|x(s)\| \\ &+ \frac{|\chi| T^{\alpha+2\beta}}{\beta\Gamma(\alpha+\beta)|2\Gamma(\beta)+\chi T^{\beta}|} L_2 + \frac{2T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} L_2 \\ &\leq \left(\frac{2|\chi|T^{\beta}}{\Gamma(\beta+1)} + \frac{\chi^2 T^{2\beta}}{\Gamma(\beta+1)|2\Gamma(\beta) + \chi T^{\beta}|} \right) d \\ &+ \left(\frac{|\chi| T^{\alpha+2\beta}}{\beta\Gamma(\alpha+\beta)|2\Gamma(\beta)+\chi T^{\beta}|} + \frac{2T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \right) L_2. \end{aligned}$$

Consequently

$$\|x(t)\| \leq (k_1 d + k_2 L_2) = G.$$

Then

$$\|x\| \leq G.$$

Therefore, V is bounded. Hence, by lemma (7), the boundary value problem (1.1) has at least one solution on $[0,T]$. □

3.3 Existence result via Krasnoselskii's fixed point theorem

Lemma 9. *Let M be a closed, bounded, convex and nonempty subset of a Banach space $\mathbb{E}$,let A, B be two operators, such that,*

(a) *$Ax + By \in M$, whenever ,$x, y \in M$,*

(b) *A is compact and continuous,*

(c) *B is a contraction mapping.*
Then there exists $z \in M$ such that $z = Az + Bz$.

Theorem 10. *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function, and suppose that condition (H_1) holds.*
In addition, we assume that the function f satisfies the assumptions:

(H_3) *There exists a nonnegative function $\Omega \in (C[0,T], \mathbb{R}^+)$ such that $|f(t,x(t))| \leq \Omega(t)$ for any $(t,x) \in [0,T] \times \mathbb{R}$.*

(H_4) *$L_1 k_2 < 1$, where k_2 is defined by (7).*

Then the boundary value problem (1.1) has a least one solution in $[0,T]$.

Proof. We first define two new operators H_1 and H_2 by:

$$\begin{aligned}(H_1x)(t) &= \frac{-\chi}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}x(s)ds - \frac{\chi}{\Gamma(\beta)}\int_t^T (s-t)^{\beta-1}x(s)ds \\ &+ \frac{\chi^2 T(t^\beta-(T-t)^\beta)}{\beta\Gamma(\beta-1)(2\Gamma(\beta)+\chi T^\beta)}\int_0^T (T-s)^{\beta-2}x(s)ds, \quad (13)\\ (H_2x)(t) &= -\frac{\chi T(t^\beta-(T-t)^\beta)}{\beta\Gamma(\alpha+\beta-1)(2\Gamma(\beta)+\chi T^\beta)}\int_0^T (T-s)^{\alpha+\beta-2}f(s,x(s))ds \\ &+ \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}f(s,x(s))ds \\ &+ \frac{1}{\Gamma(\alpha+\beta)}\int_t^T (s-t)^{\alpha+\beta-1}f(s,x(s))ds, , t\in[0,T]. \quad (14)\end{aligned}$$

We consider a closed, bounded, convex and nonempty subset of Banach space defined by $\mathbb{E}$ as $B_\rho=\{x\in\mathbb{E}, \|x\|\leq\rho\}$,with $sup_{t\in[0,T]}|\Omega(t)|=\|\Omega\|$. We take

$$\rho \geq \frac{k_2\|\Omega\|}{(1-k_1)},$$

where k_1,k_2 are given by (6) and (7) respectively.
Now, we show that $H_1x+H_2y\in B_\rho$, indeed for any $x,y\in B_\rho$, where H_1 and H_2 are denoted by (13) and (14), respectively, we have

$$\begin{aligned}|H_1x(t)+H_2y(t)| &\leq \frac{|\chi|}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}|x(s)|ds + \frac{|\chi}{\Gamma(\beta)}\int_t^T (s-t)^{\beta-1}|x(s)|ds \\ &+ \frac{\chi^2 T(t^\beta-(T-t)^\beta)}{\beta\Gamma(\beta-1)(2\Gamma(\beta)+\chi T^\beta)}\int_0^T (T-s)^{\beta-2}|x(s)|ds \\ &+ \frac{\chi T(t^\beta-(T-t)^\beta)}{\beta\Gamma(\alpha+\beta-1)(2\Gamma(\beta)+\chi T^\beta)}\int_0^T (T-s)^{\alpha+\beta-2}|f(s,y(s))|ds \\ &+ \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}|f(s,y(s))|ds \\ &+ \frac{1}{\Gamma(\alpha+\beta)}\int_t^T (s-t)^{\alpha+\beta-1}|f(s,y(s))|ds \\ &\leq \left(\frac{2|\chi|T^\beta}{\Gamma(\beta+1)} + \frac{\chi^2T^{2\beta}}{\Gamma(\beta+1)|2\Gamma(\beta)+\chi T^\beta|}\right)\|x(s)\| \\ &+ \left(\frac{|\chi|T^{\alpha+2\beta}}{\beta\Gamma(\alpha+\beta)|2\Gamma(\beta)+\chi T^\beta|} + \frac{2T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right)\|\Omega\| \\ &= k_1\rho + k_2\|\Omega\| \leq \rho.\end{aligned}$$

which implies that $\|H_1x + H_2y\| \leq \rho$. This shows that $H_1x + H_2y \in B_\rho$.
The next step is related to the compactness and continuity of the operator H_1.
Continuity of f implies that the operator H_1 is continuous, also H_1 is uniformly bounded on B_ρ as

$$\begin{aligned}\|(H_1x)(t)\| &\leq \left(\frac{2|\chi|T^\beta}{\Gamma(\beta+1)} + \frac{\chi^2T^{2\beta}}{\Gamma(\beta+1)|2\Gamma(\beta)+\chi T^\beta|}\right)\|x(s)\| \\ &\leq \left(\frac{2|\chi|T^\beta}{\Gamma(\beta+1)} + \frac{\chi^2T^{2\beta}}{\Gamma(\beta+1)|2\Gamma(\beta)+\chi T^\beta|}\right)\rho \\ &= k_1\rho.\end{aligned}$$

Now we will prove the compactness of the operator H_1, for $t_1, t_2 \in [0,T]$,$t_1 < t_2$ we have

$$\begin{aligned}|(H_1x)(t_2) - (H_1x)(t_1)| &\leq \frac{|\chi|}{\Gamma(\beta)}\int_0^{t_1}|(t_2-s)^{\beta-1} - (t_1-s)^{\beta-1}||x(s)|ds \\ &+ \frac{|\chi|}{\Gamma(\beta)}\int_{t_2}^{T}|(s-t_2)^{\beta-1} - (s-t_1)^{\beta-1}||x(s)|ds \\ &\frac{|\chi|}{\Gamma(\beta)}\int_{t_1}^{t_2}|(t_2-s)^{\beta-1} - (t_1-s)^{\beta-1}||x(s)|ds \\ &+ \frac{\chi^2T\left(|(T-t_2)^\beta - (T-t_1)^\beta| + |t_2^\beta - t_1^\beta|\right)}{\beta\Gamma(\beta-1)|2\Gamma(\beta)+\chi T^\beta|}\int_0^T(T-s)^{\beta-2}|x(s)|ds \\ &\leq \frac{|\chi|}{\Gamma(\beta+1)}|(t_2^\beta - t_1^\beta) - (t_2-t_1)^\beta|\|x(s)\| \\ &+ \frac{|\chi|}{\Gamma(\beta+1)}|(T-t_2)^\beta - (T-t_1)^\beta + (t_2-t_1)^\beta|\|x(s)\| \\ &\frac{|\chi|}{\Gamma(\beta+1)}|(t_2-t_1)^\beta - (t_1-t_2)^\beta|\|x(s)\| \\ &+ \frac{\chi^2T\left(|(T-t_1)^\beta - (T-t_2)^\beta| + (t_2^\beta - t_1^\beta)\right)|}{\beta\Gamma(\beta+1)|2\Gamma(\beta)+\chi T^\beta|}\|x(s)\|,\end{aligned}$$

we see that the right-hand side of the above inequality tends to zero independently of $x \in B_\rho$, as $t_2 - t_1$. Thus H_1 is equicontinuous, so H_1 is relatively compact on B_ρ. Therefore, by the conclusion of the Arzela-Ascoli theorem, the operator H_1 is continuous and compact on B_ρ.
Now, we prove that H_2 is contraction mapping.

Let $x, y \in \mathbb{E}$, and for each $t \in [0, T]$, we have

$$\begin{aligned}
|H_2x(t) - H_2y(t)| \leq & \frac{|\chi|T|t^{\beta} - (T-t)^{\beta}|}{\beta\Gamma(\alpha+\beta-1)(2\Gamma(\beta)+\chi T^{\beta})} \int_0^T (T-s)^{\alpha+\beta-2}|f(s,x(s)) - f(s,y(s))|ds \\
& + \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1}|f(s,x(s)) - f(s,y(s))|ds \\
& + \frac{1}{\Gamma(\alpha+\beta)} \int_t^T (s-t)^{\alpha+\beta-1}|f(s,x(s)) - f(s,y(s))|ds \\
& \leq \left(\frac{|\chi|T^{\alpha+2\beta}}{\beta\Gamma(\alpha+\beta)|2\Gamma(\beta)+\chi T^{\beta}|} + \frac{2T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \right) L_1\|x-y\| \\
& = k_2L_1\|x-y\|.
\end{aligned}$$

Which implies that $\|H_2x(t) - H_2y(t)\| \leq k_2L_1\|x-y\|$. As $k_2L_1 < 1$. Thus all the assumptions of Lemma (9) are satisfied. So the conclusion of Lemma (9) implies that the boundary value problem (1.1) has at least one solution on $[0, T]$. □

4 Examples

Example 11. *Consider the following nonlinear Langevin equation with Riesz-Caputo fractional derivative:*

$$\begin{cases} {}_0^{RC}D_T^{\frac{3}{2}}({}_0^{RC}D_T^{\frac{1}{3}} + \frac{1}{7})x(t) = \frac{\pi|x(t)|}{|x(t)|+5}\frac{\cos^2 t}{(\pi+t^2)} \ ,0 < t < 1, \\ x(0) + x(T) = 0 \ , \ x'(0) + x'(T) = 0, \end{cases} \tag{15}$$

Here, $\alpha = \frac{3}{2}, \beta = \frac{1}{3}, \chi = \frac{1}{7}, L_1 = \frac{1}{5\pi}$. *Moreover,*

$$f(t,x) = \frac{\pi|x(t)|}{|x(t)|+5}\frac{\cos^2 t}{(\pi+t^2)}.$$

Hence, we have

$$|f(t,x) - f(t,y)| \leq \frac{1}{5\pi}\|x-y\|.$$

The condition (H_1) *is satisfied with* $L_1 = \frac{1}{5\pi}, \phi = 0,4032$. *So* $\phi_1 < 1$.
Then by using theorem (6) the boundary value problem (15)has a unique solution on $[0, 1]$.

Example 12. *Consider the following nonlinear Langevin equation with Riesz-Caputo fractional derivative:*

$$\begin{cases} {}_0^{RC}D_T^{\frac{11}{6}}({}_0^{RC}D_T^{\frac{1}{5}} + \frac{1}{9})x(t) = \frac{\cos^2 t+1}{3}\sin^{-1}\left(\frac{|x|}{|x|+1}\right) \ ,0 < t < \frac{1}{2}, \\ x(0) + x(T) = 0 \ , \ x'(0) + x'(T) = 0, \end{cases} \tag{16}$$

Here, $\alpha = \frac{11}{6}, \beta = \frac{1}{5}, \chi = \frac{1}{9}, L_1 = \frac{2}{3}, \phi_1 = 0,3781$*. Also we have* $|f(t,x)| \leq \frac{\pi}{3}$*, with* $L_2 = \frac{\pi}{3}$*. Clearly the hypothesis of theorem (8) is satisfied. Thus boundary value problem (16) admits at least a solution on* $[0, \frac{1}{2}]$ *.*

Example 13. *Consider the following nonlinear Langevin equation with Riesz-Caputo fractional derivative:*

$$\begin{cases} {}^{RC}_{0}D^{\frac{9}{5}}_{T}({}^{RC}_{0}D^{\frac{1}{2}}_{T} + \frac{1}{12})x(t) = \frac{1}{\sqrt{t^2+121}}\left(\frac{|x|}{(1+|x|)}\right) + \frac{e^t}{2} \text{ },0 < t < 2, \\ x(0) + x(T) = 0 \text{ , } x'(0) + x'(T) = 0, \end{cases} \tag{17}$$

Here, $\alpha = \frac{9}{5}, \beta = \frac{1}{2}, \chi = \frac{1}{12}$*,*

$$f(t,x) = \frac{1}{\sqrt{t^2+121}} \frac{|x|}{(1+|x|)} + \frac{e^t}{2}.$$

Moreover,

$$|f(t,x)| \leq \frac{1}{\sqrt{t^2+121}} + \frac{e^t}{2}.$$

Hence, we have

$$|f(t,x) - f(t,y)| \leq \frac{1}{11}\|x-y\|,$$

with $L_1 = \frac{1}{11}, k_2 = 0,3584, L_1k_2 = 0,0326$*. So* $L_1k_2 < 1$*.*
Then by using theorem (10) the boundary value problem (17) has at least a solution on $[0, 2]$*.*

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