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A DYNAMIC CONTACT PROBLEM FOR THERMO-ELECTRO-VISOPLASTIC MATERIALS WITH DAMAGE AND INTERNAL STATE VARIABLE

Laid Maiza, Tedjani Hadj Ammar and Mohamed Laid Gossa

Abstract. This work studies a mathematical model involving a dynamic contact between two thermo-elasto-viscoplastic piezoelectric bodies with internal state variables and damage. The contact is modelled with normal compliance condition and adhesion effect of contact surfaces. We derive variational formulation of the problem and we prove an existence and uniqueness result of the weak solution. The proof is based on classical existence and uniqueness result on parabolic inequalities, differential equations and fixed-point arguments.

1 Introduction

Important progress has been made in recent years in the modeling and mathematical study of the different processes involved in contact between deformable bodies. When there is an interaction between the mechanical, electrical and thermal properties of the considered material, contact problems involving thermo-piezoelectricity arise. This type of materials has many applications in sensors and actuators as magnetic probes, electric packing, microphones, hydrophones, ultrasonic image processing due to the transition of energy in thermo-electromechanical conversion. Mindlin [27] was the first to introduce the thermo-piezoelectric theory using motion equations in pyroelectric and piezoelectric media to model reflection and refraction phenomena. Nowacki [22, 29] discussed the physical laws of thermo-piezoelectric materials and Chandrasekharaiah [5, 6] extended the thermo-piezoelectricity theory of Mindlin to a particular model. Then, the propagation of waves in bodies made of thermo-piezoelectric materials [19, 25, 33, 32, 34, 31] and its references have been studied by several researchers. Contact problems involving thermo-elastic materials can be found in [1, 11, 14], the study of an electro-thermo-viscoelastic bodies is considered

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in [12, 3]. In [4, 26], the mathematical model that describes the frictional contact between a thermo-piezoelectric body and a conductive base is already discussed. In [13], Essoufi et al. considered the modeling of quasistatic thermo-electro-viscoelastic body behavior and the contact with nonfrictional and nonconductive foundation by Signorini condition, they demonstrated the existence and uniqueness of the weak solution and derived error estimates on the approximate solutions.

In this article, we study a dynamic contact problem between two a thermo-electro viscoplastic bodies with damage and internal state variable. To this purpose we introduce the constitutive law:

$$\begin{aligned}\boldsymbol{\sigma} &= \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}) - (\mathcal{E})^* E(\xi) \; + \\ &\int_0^t \mathcal{G}\Big(\boldsymbol{\sigma}(s) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(s)) - (\mathcal{E})^*\nabla\xi(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \varsigma(s), \boldsymbol{k}(s), \tau(s)\Big) ds, \end{aligned} \tag{1.1}$$

$$\dot{\boldsymbol{k}} = \Theta\big(\boldsymbol{\sigma} - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) - (\mathcal{E})^*\nabla\xi, \boldsymbol{\varepsilon}(\boldsymbol{u}), \varsigma, \boldsymbol{k}, \tau\big), \tag{1.2}$$

$$\boldsymbol{D} = \mathcal{E}\varepsilon(\boldsymbol{u}) + \boldsymbol{\beta}E(\xi), \tag{1.3}$$

where $\boldsymbol{u}$ denotes the displacement field, $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}(\boldsymbol{u})$ represent the stress and the linearized strain tensor, respectively, $\boldsymbol{D}$ is the electric displacement field. Here $\mathcal{A}$ and $\mathcal{B}$ are nonlinear operators describing the purely viscous and the elastic properties of the material, respectively, $\mathcal{G}$ is a nonlinear constitutive function describing the viscoplastic behaviour of the material. $\boldsymbol{k}$ denotes the internal state variable, ς and τ represent the damage and the temperature field, respectively, Θ is also a nonlinear constitutive function. There is a variety of choices for the internal state variables, for reference in the field see [9, 10]. Some commonly used internal state variables are the plastic strain and a number of tensor variables that take into account the spatial display of dislocations and the work-hardening of the material. $E(\xi)$ is the electric field that satisfies $E(\xi) = -\nabla\xi$, where ξ is the electric potential. Also, $\mathcal{E}$ represents the third order piezoelectric tensor, $(\mathcal{E})^*$ is its transposition and $\boldsymbol{\beta}$ denotes the electric permittivity tensor. It follows from (1.1) that at each time moment, the stress tensor $\boldsymbol{\sigma}$ is split into three parts: $\boldsymbol{\sigma} = \boldsymbol{\sigma}_V + \boldsymbol{\sigma}_E + \boldsymbol{\sigma}_R$, where $\boldsymbol{\sigma}_V = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}})$ represents the purely viscous part of the stress, $\boldsymbol{\sigma}_E = -(\mathcal{E})^*E(\xi)$ represents the electric part of the stress and $\boldsymbol{\sigma}_R$ is the elastoplastic part of the stress which satisfies

$$\boldsymbol{\sigma}_R = \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \int_0^t \mathcal{G}\Big(\boldsymbol{\sigma}_R(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \varsigma(s), \boldsymbol{k}(s), \tau(s)\Big) ds. \tag{1.4}$$

Note also that when $\mathcal{G} = 0$ the constitutive law (1.1) becomes the Kelvin-Voigt electro-viscoelastic constitutive relation,

$$\boldsymbol{\sigma} = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}) - (\mathcal{E})^* E(\xi). \tag{1.5}$$

Contact problems with Kelvin-Voigt materials of the form (1.5) can be found in [2, 26, 35]. The paper is organized as follows. There are some preliminary principles related to our problem in the section 2. We present the mechanical model of the problem in the section 3. We add the section 4 with the assumptions on the problem data. We also derive the problem's variational formulation, and the main result is stated in the Theorem 2. Section 5, we proof of Theorem 2 based on nonlinear evolution equation with monotone operator, parabolic inequalities, differential equations and fixed point arguments.

2 Notation and preliminaries

In this section, we present some basic notations and preliminary material, which will be used throughout this paper. For more details, we refer the reader to [28, 36].
Let $\mathbb{S}^d$ be the space of second order symmetric tensors on $\mathbb{R}^d$. The canonical inner products and norms on $\mathbb{S}^d$ and $\mathbb{R}^d$ are given by

$$\boldsymbol{u}.\boldsymbol{v} = u_i.v_i, \quad |\boldsymbol{v}| = (\boldsymbol{v}.\boldsymbol{v})^{\frac{1}{2}}, \quad \forall \boldsymbol{u},\boldsymbol{v} \in \mathbb{R}^d,$$
$$\boldsymbol{\sigma}.\boldsymbol{\tau} = \sigma_{ij}.\tau_{ij}, \quad |\boldsymbol{\tau}| = (\boldsymbol{\tau}.\boldsymbol{\tau})^{\frac{1}{2}}, \quad \forall \boldsymbol{\sigma},\boldsymbol{\tau} \in \mathbb{S}^d.$$

Everywhere, the indices i and j run between 1 and d the summation convention over repeated indices is adopted.

Let Ω^1 and Ω^2 be two bounded domains in $\mathbb{R}^d$. Everywhere in this paper, we use a superscript α to indicate that a quantity is related to the domain Ω^α, $\alpha = 1, 2$. For each domain Ω^α we assume that its boundary Γ^α is Lipschitz continuous and let Γ_1^α be a measurable part of Γ^α such that $meas(\Gamma_1^\alpha) > 0$. We denote by $\nu^\alpha = (\nu_i^\alpha)$ the outward unit normal at Γ^α. Also, an index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable; for example, $\boldsymbol{u}_{i,j}^\alpha = \partial \boldsymbol{u}_i^\alpha / \partial x_j$.

We introduce the spaces and the corresponding inner productsas follows:

$$H^\alpha = \{\boldsymbol{v}^\alpha = (v_i^\alpha)_{1\le i\le d};\ v_i^\alpha \in L^2(\Omega^\alpha)\},$$
$$\mathcal{H}^\alpha = \{\boldsymbol{\tau}^\alpha = (\tau_{ij}^\alpha)_{1\le i,j\le d};\ \tau_{ij}^\alpha = \tau_{ji}^\alpha \in L^2(\Omega^\alpha)\},$$
$$H_1^\alpha = \{\boldsymbol{v}^\alpha = (v_i^\alpha)_{1\le i\le d};\ \varepsilon(\boldsymbol{v}^\alpha) \in \mathcal{H}^\alpha\},$$
$$\mathcal{H}_1^\alpha = \{\boldsymbol{\tau}^\alpha = (\tau_{ij}^\alpha)_{1\le i\le d};\ \boldsymbol{\tau}^\alpha \in \mathcal{H}^\alpha,\ \ \mathrm{Div}\boldsymbol{\tau}^\alpha \in H^\alpha\},$$
$$\boldsymbol{Y}^\alpha = \left\{\boldsymbol{\lambda}^\alpha = (\lambda_i^\alpha)_{1\le i\le m};\ \lambda_i^\alpha \in L^2(\Omega^\alpha)\right\}, \quad \boldsymbol{V}^\alpha = \left\{\boldsymbol{v}^\alpha \in H^1(\Omega^\alpha)^d;\ \boldsymbol{v}^\alpha = 0 \text{ on } \Gamma_1^\alpha\right\}.$$

It is easy to check that the spaces $H^\alpha, \mathcal{H}^\alpha, H_1^\alpha, \mathcal{H}_1^\alpha, \boldsymbol{Y}^\alpha$, and $\boldsymbol{V}^\alpha$ are all Hilbert spaces equipped with the inner products

$$(\boldsymbol{u}^\alpha, \boldsymbol{v}^\alpha)_{H^\alpha} = \int_{\Omega^\alpha} \boldsymbol{u}^\alpha.\boldsymbol{v}^\alpha dx, \quad (\boldsymbol{\sigma}^\alpha, \boldsymbol{\tau}^\alpha)_{\mathcal{H}^\alpha} = \int_{\Omega^\alpha} \boldsymbol{\sigma}^\alpha.\boldsymbol{\tau}^\alpha dx,$$

$$(\boldsymbol{u}^\alpha, \boldsymbol{v}^\alpha)_{H_1^\alpha} = \int_{\Omega^\alpha} \boldsymbol{u}^\alpha.\boldsymbol{v}^\alpha dx + \int_{\Omega^\alpha} \nabla \boldsymbol{u}^\alpha.\nabla \boldsymbol{v}^\alpha dx,$$
$$(\boldsymbol{\sigma}^\alpha, \boldsymbol{\tau}^\alpha)_{\mathcal{H}_1^\alpha} = \int_{\Omega^\alpha} \boldsymbol{\sigma}^\alpha.\boldsymbol{\tau}^\alpha dx + \int_{\Omega^\alpha} \operatorname{Div} \boldsymbol{\sigma}^\alpha. \operatorname{Div} \boldsymbol{\tau}^\alpha dx,$$
$$(\boldsymbol{\lambda}^\alpha, \boldsymbol{\mu}^\alpha)_{\boldsymbol{Y}^\alpha} = \int_{\Omega^\alpha} \boldsymbol{\lambda}^\alpha.\boldsymbol{\mu}^\alpha dx, \quad (\boldsymbol{u}^\alpha, \boldsymbol{v}^\alpha)_{\boldsymbol{V}^\alpha} = (\varepsilon(\boldsymbol{u}^\alpha), \varepsilon(\boldsymbol{v}^\alpha))_{\mathcal{H}^\alpha}$$

and the associated norms $\|.\|_{H^\alpha}$, $\|.\|_{\mathcal{H}^\alpha}$, $\|.\|_{H_1^\alpha}$, $\|.\|_{\mathcal{H}_1^\alpha}$, $\|.\|_{\boldsymbol{Y}_1^\alpha}$ and $\|.\|_{\boldsymbol{V}^\alpha}$ and respectively. Here and below we use the notation

$$\nabla \boldsymbol{u}^\alpha = (u_{i,j}^\alpha), \ \varepsilon(\boldsymbol{u}^\alpha) = (\varepsilon_{ij}(\boldsymbol{u}^\alpha)), \quad \varepsilon_{ij}(\boldsymbol{u}^\alpha) = \frac{1}{2}(u_{i,j}^\alpha + u_{j,i}^\alpha), \quad \forall u^\alpha \in H_1^\alpha,$$
$$\operatorname{Div} \boldsymbol{\sigma}^\alpha = (\sigma_{ij,j}^\alpha), \quad \forall \boldsymbol{\sigma}^\alpha \in \mathcal{H}_1^\alpha.$$

Completeness of the space $(V^\alpha, \|.\|_{V^\alpha})$ follows from the assumption $meas(\Gamma_1^\alpha > 0)$, which allows the use of Korn's inequality.

We denote $\boldsymbol{v}^\alpha$ as the trace of an element $\boldsymbol{v}^\alpha \in H_1^\alpha$ on Γ^α. For every element $\boldsymbol{v}^\alpha \in \boldsymbol{V}^\alpha$, we denote by v_ν^α and $\boldsymbol{v}_\tau^\alpha$ the normal and the tangential components of $\boldsymbol{v}^\alpha$ on the boundary Γ^α given by $v_\nu^\alpha = \boldsymbol{v}^\alpha.\nu^\alpha$, $\boldsymbol{v}_\tau^\alpha = \boldsymbol{v}^\alpha - v_\nu^\alpha \boldsymbol{\nu}^\alpha$. Also, for an element $\boldsymbol{\sigma}^\alpha \in \mathcal{H}_1^\alpha$ we denote by $\boldsymbol{\sigma\nu}^\alpha$, σ_ν^α and $\boldsymbol{\sigma}_\tau^\alpha$ the trace, the normal trace and the tangential trace of $\boldsymbol{\sigma}^\alpha$ to Γ^α, respectively. In addition, the Sobolev trace theorem, there exists a constant $c_{tr} > 0$, depending only on Ω^α, Γ_1^α and Γ_3 such that

$$\|\boldsymbol{v}^\alpha\|_{L^2(\Gamma_3)^d} \leq c_{tr} \|\boldsymbol{v}^\alpha\|_{V^\alpha} \quad \forall \boldsymbol{v}^\alpha \in V^\alpha. \tag{2.1}$$

Denote $L_0^\alpha = L^2(\Omega^\alpha)$, $L_1^\alpha = H^1(\Omega^\alpha)$, $(.,.)_{L_0^\alpha} = (.,.)_{L^2(\Omega^\alpha)}$, $(.,.)_{L_1^\alpha} = (.,.)_{H^1(\Omega^\alpha)}$, $\|.\|_{L_0^\alpha} = \|.\|_{L^2(\Omega^\alpha)}$ and $\|.\|_{L_1^\alpha} = \|.\|_{H^1(\Omega^\alpha)}$. For the electric unknowns ξ^α and $\boldsymbol{D}^\alpha$ we use the spaces

$$W^\alpha = \{\xi\alpha \in L_1^\alpha; \ \xi^\alpha = 0 \text{ on } \Gamma_a^\alpha\},$$
$$\mathcal{W}^\alpha = \left\{\boldsymbol{D}^\alpha = (D_i^\alpha); \ D_i^\alpha \in L^2(\Omega^\alpha), \ \operatorname{div} \boldsymbol{D}^\alpha \in L^2(\Omega^\alpha)\right\}.$$

These are real Hilbert spaces with inner products

$$(\xi^\alpha, \psi^\alpha)_{W^\alpha} = \int_{\Omega^\alpha} \nabla\xi^\alpha.\nabla\psi^\alpha dx,$$
$$(\boldsymbol{D}^\alpha, \boldsymbol{\Psi}^\alpha)_{\mathcal{W}^\alpha} = \int_{\Omega^\alpha} \boldsymbol{D}^\alpha.\boldsymbol{\Psi}^\alpha dx + \int_{\Omega^\alpha} \operatorname{div} \boldsymbol{D}^\alpha. \operatorname{div} \boldsymbol{\Psi}^\alpha dx,$$

where $\operatorname{div} \boldsymbol{D}^\alpha = (\boldsymbol{D}_{i,i}^\alpha)$, and the associated norms are denoted by $\|.\|_{W^\alpha}$ and $\|.\|_{\mathcal{W}^\alpha}$, respectively. Completeness of the space $(W^\alpha, \|.\|_{W^\alpha})$ is a consequence of the assumption $meas(\Gamma_a^\alpha) > 0$ which allows the use of Friedrichs-Poincaré inequality.

In order to simplify the notations, we define the product spaces

$$\boldsymbol{V}=\boldsymbol{V}^1\times\boldsymbol{V}^2,\quad H=H^1\times H^2,\quad H_1=H_1^1\times H_1^2,\quad \mathcal{H}=\mathcal{H}^1\times\mathcal{H}^2,$$
$$\mathcal{H}_1=\mathcal{H}_1^1\times\mathcal{H}_1^2,\quad \boldsymbol{Y}=\boldsymbol{Y}^1\times\boldsymbol{Y}^2,\quad L_0=L_0^1\times L_0^2,\quad L_1=L_1^1\times L_1^2,$$
$$W=W^1\times W^2,\quad \mathcal{W}=\mathcal{W}^1\times\mathcal{W}^2.$$

The spaces $\boldsymbol{V}$, H, $\mathcal{H}$, $\boldsymbol{Y}$, L_0, L_1, W and $\mathcal{W}$ are real Hilbert spaces endowed with the canonical inner products denoted by $(.,.)_{\boldsymbol{V}}$, $(.,.)_H$, $(.,.)_{\mathcal{H}}$, $(.,.)_{\boldsymbol{Y}}$, $(.,.)_{L_0}$, $(.,.)_{L_1}$, $(.,.)_W$, and $(.,.)_{\mathcal{W}}$. The associate norms will be denoted by $\|.\|_{\boldsymbol{V}}$, $\|.\|_H$, $\|.\|_{\mathcal{H}}$, $\|.\|_{\boldsymbol{Y}}$, $\|.\|_{L_0}$, $\|.\|_{L_1}$, $\|.\|_W$, and $\|.\|_{\mathcal{W}}$, respectively.

Finally, for any real Hilbert space $\mathbb{H}$, we use the classical notation for the spaces $L^p(0,T;\mathbb{H})$, $W^{k,p}(0,T;\mathbb{H})$, where $1\le p\le\infty$, $k\ge 1$. We denote by $C(0,T;\mathbb{H})$ and $C^1(0,T;\mathbb{H})$ the space of continuous and continuously differentiable functions from $[0,T]$ to $\mathbb{H}$, respectively, with the norms

$$\|\pi\|_{C(0,T;\mathbb{H})}=\max_{t\in[0,T]}\|\pi(t)\|_{\mathbb{H}},$$
$$\|\pi\|_{C^1(0,T;\mathbb{H})}=\max_{t\in[0,T]}\|\pi(t)\|_{\mathbb{H}}+\max_{\pi\in[0,T]}\|\dot{\pi}(t)\|_{\mathbb{H}},$$

respectively. Moreover, we use the dot above to indicate the derivative with respect to the time variable.

Moreover, if $\mathbb{H}_1$ and $\mathbb{H}_2$ are real Hilbert spaces then $\mathbb{H}_1\times\mathbb{H}_2$ denotes the product Hilbert space endowed with the canonical inner product $(.,.)_{\mathbb{H}_1\times\mathbb{H}_2}$.

3 Model of the Problem

We consider two thermo-piezoelectric bodies, occupying two bounded domains Ω^1, Ω^2 of the space $\mathbb{R}^d$ ($d=2,3$ in applications). For each domain Ω^α, the boundary Γ^α is assumed to be Lipschitz continuous, and is partitioned into three disjoint measurable parts Γ_1^α, Γ_2^α and Γ_3^α, on one hand, and on two measurable parts Γ_a^α and Γ_b^α, on the other hand, such that $meas(\Gamma_1^\alpha)>0$, $meas(\Gamma_a^\alpha)>0$. Let $T>0$ and let $[0,T]$ be the time interval of interest. The Ω^α body is submitted to $\boldsymbol{f}_0^\alpha$ forces and volume electric charges of density q_0^α. The bodies are assumed to be clamped on $\Gamma_1^\alpha\times(0,T)$, so the displacement field vanishes there. The surface tractions $\boldsymbol{f}_2^\alpha$ act on $\Gamma_2^\alpha\times(0,T)$. We also assume that the electrical potential vanishes on $\Gamma_a^\alpha\times(0,T)$ and a surface electric charge of density q_2^α is prescribed on $\Gamma_b^\alpha\times(0,T)$. The two bodies can enter in contact along the common part $\Gamma_3^1=\Gamma_3^2=\Gamma_3$. We use an thermo-elasto-viscoplastic piezoelectric law with damage and internal state variable given by (1.1)–(1.3) where the damage of the materials caused by plastic deformations. The differential inclusion used for the evolution of the damage field is

$$\dot{\varsigma}^\alpha-\kappa^\alpha\Delta\varsigma^\alpha+\partial\psi_{K^\alpha}(\varsigma^\alpha)\ni\phi^\alpha\big(\boldsymbol{\sigma}^\alpha-\mathcal{A}^\alpha\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\alpha)-(\mathcal{E}^\alpha)^*\nabla\xi^\alpha,\boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha),\varsigma^\alpha,\boldsymbol{k}^\alpha,\tau^\alpha\big),$$

where K^α denotes the set of admissible damage functions defined by

$$K^\alpha = \{\omega \in H^1(\Omega^\alpha);\ 0 \le \omega \le 1, \text{ a.e. in } \Omega^\alpha\}, \tag{3.1}$$

κ^α is a positive coefficient, $\partial\psi_{K^\alpha}$ represents the subdifferential of the indicator function of the set K^α and Ψ^α is a given constitutive function which describes the sources of the damage in the system. When $\varsigma = 1$, there is no damage in the material, when $\varsigma = 0$, the material is completely damaged, when $0 < \varsigma < 1$ there is partial damage and the system has a reduced load carrying capacity. General novel models for damage were derived in [17, 16, 18] from the virtual power principle. With these assumptions, the classical formulation of the dynamic problem for frictionless contact problem with normal compliance and adhesion between two thermo-elasto-viscoplastic piezoelectric bodies with damage and with internal state variable is the following.

Problem P. For $\alpha = 1, 2$, find a displacement field $\boldsymbol{u}^\alpha : \Omega^\alpha \times [0,T] \longrightarrow \mathbb{R}^d$, a stress field $\boldsymbol{\sigma}^\alpha : \Omega^\alpha \times [0,T] \longrightarrow \mathbb{S}^d$, an electric potential field $\xi^\alpha : \Omega^\alpha \times [0,T] \longrightarrow \mathbb{R}$, a damage field $\varsigma^\alpha : \Omega^\alpha \times [0,T] \longrightarrow \mathbb{R}$, a bonding field $\zeta : \Gamma_3 \times [0,T] \longrightarrow \mathbb{R}$, a electric displacement field $\boldsymbol{D}^\alpha : \Omega^\alpha \times [0,T] \longrightarrow \mathbb{R}^d$, a temperature $\tau^\alpha : \Omega^\alpha \times [0,T] \longrightarrow \mathbb{R}$, and an internal state variable field $\boldsymbol{k}^\alpha : \Omega^\alpha \times [0,T] \longrightarrow \mathbb{R}^m$, such that

$$\boldsymbol{\sigma}^\alpha = \mathcal{A}^\alpha \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\alpha) + \mathcal{B}^\alpha \boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha) + (\mathcal{E}^\alpha)^* \nabla \xi^\alpha +$$
$$\int_0^t \mathcal{G}^\alpha \Big(\boldsymbol{\sigma}^\alpha(s) - \mathcal{A}^\alpha \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\alpha(s)) - (\mathcal{E}^\alpha)^* \nabla \xi^\alpha(s), \boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha(s)), \varsigma^\alpha(s), \boldsymbol{k}^\alpha(s), \tau^\alpha(s)\Big) ds$$
$$\text{in } \Omega^\alpha \times (0,T), \tag{3.2}$$

$$\dot{\boldsymbol{k}}^\alpha = \Theta^\alpha\big(\boldsymbol{\sigma}^\alpha - \mathcal{A}^\alpha \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\alpha) - (\mathcal{E}^\alpha)^* \nabla \xi^\alpha(s), \boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha), \varsigma^\alpha, \boldsymbol{k}^\alpha, \tau^\alpha\big) \text{ in} \Omega^\alpha \times (0,T), \tag{3.3}$$

$$\boldsymbol{D}^\alpha = \mathcal{E}^\alpha \varepsilon(\boldsymbol{u}^\alpha) - \boldsymbol{\beta}^\alpha \nabla \xi^\alpha \quad \text{in } \Omega^\alpha \times (0,T), \tag{3.4}$$

$$\dot{\varsigma}^\alpha - \kappa^\alpha \Delta \varsigma^\alpha + \partial\psi_{K^\alpha}(\varsigma^\alpha) \ni \phi^\alpha\big(\boldsymbol{\sigma}^\alpha - \mathcal{A}^\alpha \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\alpha) - (\mathcal{E}^\alpha)^* \nabla \xi^\alpha, \boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha), \varsigma^\alpha, \boldsymbol{k}^\alpha, \tau^\alpha\big)$$
$$\text{in } \Omega^\alpha \times (0,T), \tag{3.5}$$

$$\dot{\tau}^\alpha - \kappa_0^\alpha \Delta \tau^\alpha = \Psi^\alpha\big(\boldsymbol{\sigma}^\alpha - \mathcal{A}^\alpha \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\alpha) - (\mathcal{E}^\alpha)^* \nabla \xi^\alpha, \boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha), \varsigma^\alpha, \boldsymbol{k}^\alpha, \tau^\alpha\big) + \chi^\alpha$$
$$\text{in } \Omega^\alpha \times (0,T), \tag{3.6}$$

$$\rho^\alpha \ddot{\boldsymbol{u}}^\alpha = \operatorname{Div} \boldsymbol{\sigma}^\alpha + \boldsymbol{f}_0^\alpha \quad \text{in } \Omega^\alpha \times (0,T), \tag{3.7}$$

$$\operatorname{div} \boldsymbol{D}^\alpha - q_0^\alpha = 0 \quad \text{in } \Omega^\alpha \times (0,T), \tag{3.8}$$

$$\boldsymbol{u}^\alpha = 0 \quad \text{on } \Gamma_1^\alpha \times (0,T), \tag{3.9}$$

$$\boldsymbol{\sigma}^\alpha \boldsymbol{\nu}^\alpha = \boldsymbol{f}_2^\alpha \quad \text{on } \Gamma_2^\alpha \times (0,T), \tag{3.10}$$

$$\dot{\zeta} = H_{ad}(\zeta, R_\nu(u_\nu^1 + u_\nu^2), \boldsymbol{R}_\tau(\boldsymbol{u}_\tau^1 - \boldsymbol{u}_\tau^2)) \quad \text{on } \Gamma_3 \times (0,T), \tag{3.11}$$

$$\begin{cases} \sigma_\nu^1 = \sigma_\nu^2 \equiv \sigma_\nu, \\ \sigma_\nu = -p_\nu(u_\nu^1 + u_\nu^2) + \gamma_\nu \zeta^2 R_\nu(u_\nu^1 + u_\nu^2) \end{cases} \quad \text{on } \Gamma_3 \times (0,T), \tag{3.12}$$

$$\begin{cases} \boldsymbol{\sigma}_\tau^1 = -\boldsymbol{\sigma}_\tau^2 \equiv \boldsymbol{\sigma}_\tau, \\ \boldsymbol{\sigma}_\tau = p_\tau(\zeta)\boldsymbol{R}_\tau(\boldsymbol{u}_\tau^1 - \boldsymbol{u}_\tau^2) \end{cases} \quad \text{on } \Gamma_3 \times (0,T), \tag{3.13}$$

$$\frac{\partial \varsigma^\alpha}{\partial \nu^\alpha} = 0 \quad \text{on } \Gamma^\alpha \times (0,T), \tag{3.14}$$

$$\kappa_0^\alpha \frac{\partial^\alpha \tau^\alpha}{\partial \nu^\alpha} + \lambda_0^\alpha \tau^\alpha = 0 \quad \text{on } \Gamma^\alpha \times (0,T), \tag{3.15}$$

$$\xi^\alpha = 0 \quad \text{on } \Gamma_a^\alpha \times (0,T), \tag{3.16}$$

$$\boldsymbol{D}^\alpha.\boldsymbol{\nu}^\alpha = q_2^\alpha \quad \text{on } \Gamma_b^\alpha \times (0,T), \tag{3.17}$$

$$\boldsymbol{u}^\alpha(0) = \boldsymbol{u}_0^\alpha, \dot{\boldsymbol{u}}^\alpha(0) = \boldsymbol{v}_0^\alpha, \varsigma^\alpha(0) = \varsigma_0^\alpha, \boldsymbol{k}^\alpha(0) = \boldsymbol{k}_0^\alpha, \tau^\alpha(0) = \tau_0^\alpha \text{ in } \Omega^\alpha, \tag{3.18}$$

$$\zeta(0) = \zeta_0, \quad \text{on } \Gamma_3. \tag{3.19}$$

First, equations (3.2)–(3.4) represent the thermo-elastic-viscoplastic piezoelectric constitutive law with internal state variable and damage. The equation (3.6) is an energy conservation equation where Ψ^α is a nonlinear constitutive function describing the heat produced by the work of internal forces, and χ^α is the heat source of the given volume. The equations (3.7) and (3.8) are the equilibrium equations for the fields of stress and electric displacement. Next, the equations (3.9) and (3.10) represent the displacement and traction boundary condition, respectively. Equation (3.11) represents the ordinary differential equation which describes the evolution of the bonding field and it was already used in [7, 8, 15, 20, 21], see also [36] for details, where H_{ad} is the adhesion evolution rate function. Condition (3.12) represents the normal compliance condition with adhesion. Condition (3.13) is the tangential boundary condition on the contact surface, showing that the shear on the contact surface depends on the adhesion field and on the tangential displacement, p_ν and p_τ are given functions,

$$R_\nu(s) = \begin{cases} L & \text{if } s < -L, \\ -s & \text{if } -L \leq s \leq 0, \\ 0 & \text{if } s > 0, \end{cases} \qquad \boldsymbol{R}_\tau(\boldsymbol{v}) = \begin{cases} \boldsymbol{v} & \text{if } |\boldsymbol{v}| \leq L, \\ L\frac{\boldsymbol{v}}{|\boldsymbol{v}|} & \text{if } |\boldsymbol{v}| > L \end{cases}$$

with $L > 0$ being a characteristic length of the bond, beyond which it does not offer any additional traction (see, e.g., [30]). Boundary conditions (3.14), (3.15) represent, respectively on Γ^α, a homogeneous Neumann boundary condition for the damage field and a Fourier boundary condition for the temperature. (3.16) and (3.17) represent the electric boundary conditions. (3.18) represents the initial displacement field, the initial velocity and the initial damage. Finally (3.19) represents the initial condition in which ζ_0 is the given initial bonding field.

4 Weak formulation and main result

To derive the variational formulation for Problem P, we need to introduce the following assumptions

The viscosity operator $\mathcal{A}^\alpha : \Omega^\alpha \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies:

H(1): (a) There exist constants $C^1_{\mathcal{A}^\alpha}, C^2_{\mathcal{A}^\alpha} > 0$ such that
$|\mathcal{A}^\alpha(\boldsymbol{x}, \boldsymbol{\omega})| \leq C^1_{\mathcal{A}^\alpha}|\boldsymbol{\omega}| + C^2_{\mathcal{A}^\alpha}, \quad \forall\, \boldsymbol{\omega} \in \mathbb{S}^d,$ a.e. $\boldsymbol{x} \in \Omega^\alpha$,
(b) There exist constants $m_{\mathcal{A}^\alpha} > 0$ such that
$(\mathcal{A}^\alpha(\boldsymbol{x}, \boldsymbol{\omega}_1) - \mathcal{A}^\alpha(\boldsymbol{x}, \boldsymbol{\omega}_2)) \cdot (\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2) \geq m_{\mathcal{A}^\alpha}|\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2|^2,$
$\forall\, \boldsymbol{\omega}_1, \boldsymbol{\omega}_2 \in \mathbb{S}^d,$ a.e. $\boldsymbol{x} \in \Omega^\alpha$,
(c) $\mathcal{A}^\alpha(., \boldsymbol{\omega})$ is measurable on Ω^α, for any $\boldsymbol{\omega} \in \mathbb{S}^d$,
(d) $\mathcal{A}^\alpha(\boldsymbol{x}, .)$ is continuous on $\mathbb{S}^d$, a.e. $\boldsymbol{x} \in \Omega^\alpha$.

The elasticity operator $\mathcal{B}^\alpha : \Omega^\alpha \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies:

H(2): (a) There exists $L_{\mathcal{B}^\alpha} > 0$ such that
$|\mathcal{B}^\alpha(\boldsymbol{x}, \boldsymbol{\omega}_1) - \mathcal{B}^\alpha(\boldsymbol{x}, \boldsymbol{\omega}_2)| \leq L_{\mathcal{B}^\alpha}|\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2|, \ \forall \boldsymbol{\omega}_1, \boldsymbol{\omega}_2 \in \mathbb{S}^d,$ a.e.$\boldsymbol{x} \in \Omega^\alpha$,
(b) $\mathcal{B}^\alpha(., \boldsymbol{\omega})$ is measurable on Ω^α, $\forall \boldsymbol{\omega} \in \mathbb{S}^d$,
(c) $\mathcal{B}^\alpha(., \mathbf{0})$ belongs to $\mathcal{H}^\alpha$.

The viscoplasticity operator $\mathcal{G}^\alpha : \Omega^\alpha \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{S}^d$ satisfies:

H(3): (a) There exists a constants $L_{\mathcal{G}^\alpha} > 0$ such that
$|\mathcal{G}^\alpha(\boldsymbol{x}, \boldsymbol{\eta}_1, \boldsymbol{\omega}_1, d_1, \boldsymbol{k}_1, \tau_1) - \mathcal{G}^\alpha(\boldsymbol{x}, \boldsymbol{\eta}_2, \boldsymbol{\omega}_2, d_2, \boldsymbol{k}_2, \tau_2)| \leq L_{\mathcal{G}^\alpha}\big(|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2|$
$+|\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2| + |d_1 - d_2| + |\boldsymbol{k}_1 - \boldsymbol{k}_2| + |\tau_1 - \tau_2|\big), \quad \forall \boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\omega}_1,$
$\boldsymbol{\omega}_2 \in \mathbb{S}^d, \ \forall \boldsymbol{k_1}, \boldsymbol{k_2} \in \mathbb{R}^m, \ \forall d_1, d_2 \in \mathbb{R}, \ \forall \tau_1, \tau_2 \in \mathbb{R}$ a.e. $\boldsymbol{x} \in \Omega^\alpha$,
(b) $\mathcal{G}^\alpha(., \boldsymbol{\eta}, \boldsymbol{\omega}, d, \boldsymbol{k}, \tau)$ is measurable in Ω^α, $\forall \boldsymbol{\eta}, \boldsymbol{\omega} \in \mathbb{S}^d, d \in \mathbb{R}, \ \boldsymbol{k} \in \mathbb{R}^m$,
(c) $\mathcal{G}^\alpha(., \mathbf{0}, \mathbf{0}, 0, \mathbf{0}, 0)$ belongs to $\mathcal{H}^\alpha$.

The nonlinear constitutive functions H_{ad}, Θ^α, ϕ^α and Ψ^α are assumed to satisfy the followig:

H(4): $H_{ad} : \Gamma_3 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{d-1} \to \mathbb{R}$ is such that
(a) There exists $L_{H_{ad}} > 0$ such that
$|H_{ad}(\boldsymbol{x}, \zeta_1, r_1, \omega_1) - H_{ad}(\boldsymbol{x}, \zeta_2, r_2, \omega_2)| \leq$
$L_{H_{ad}}\big(|\zeta_1 - \zeta_2| + |r_1 - r_2| + |\omega_1 - \omega_2|\big),$
for all $\zeta_1, \zeta_2, r_1, r_2 \in \mathbb{R}, \omega_1, \omega_2 \in \mathbb{R}^{d-1}$, for a.e. $\boldsymbol{x} \in \Gamma_3$,
(b) $H_{ad}(., \zeta, r, \omega)$ is measurable on Γ_3, $\forall \zeta, r \in \mathbb{R}, \omega \in \mathbb{R}^{d-1}$,
(c) $H_{ad}(\boldsymbol{x}, ., ., .)$ is continuous on $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^{d-1}$ a.e. $\boldsymbol{x} \in \Gamma_3$,
(d) $H_{ad}(\boldsymbol{x}, 0, r, \omega) = 0, \quad \forall r \in \mathbb{R}, \ \omega \in \mathbb{R}^{d-1}$, a.e. $\boldsymbol{x} \in \Gamma_3$,
$H_{ad}(\boldsymbol{x}, \zeta, r, \omega) \geq 0, \quad \forall \zeta \leq 0, r \in \mathbb{R}, \ \omega \in \mathbb{R}^{d-1}$, a.e. $\boldsymbol{x} \in \Gamma_3$, and
$H_{ad}(\boldsymbol{x}, \zeta, r, \omega) \leq 0, \quad \forall\, \zeta \geq 1, r \in \mathbb{R}, \ \omega \in \mathbb{R}^{d-1}$, a.e. $\boldsymbol{x} \in \Gamma_3$.

H(5): $\Theta^\alpha : \Omega^\alpha \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{S}^d$ is such that
(a) There exists $L_{\Theta^\alpha} > 0$ such that
$|\Theta^\alpha(\boldsymbol{x}, \boldsymbol{\eta}_1, \boldsymbol{\omega}_1, \alpha_1, \boldsymbol{k}_1, \tau_1) - \Theta^\alpha(\boldsymbol{x}, \boldsymbol{\eta}_2, \boldsymbol{\omega}_2, \alpha_2, \boldsymbol{k_2}, \tau_2)| \leq$
$L_{\Theta^\alpha}\big(|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2| + |\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2| + |\alpha_1 - \alpha_2| + |\boldsymbol{k}_1 - \boldsymbol{k}_2| + |\tau_1 - \tau_2|\big),$
$\forall \boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\omega}_1, \boldsymbol{\omega}_2 \in \mathbb{S}^d, \boldsymbol{k}_1, \boldsymbol{k}_2 \in \mathbb{R}^m, \tau_1, \tau_2, \alpha_1, \alpha_2 \in \mathbb{R}, \quad a.e.\ \boldsymbol{x} \in \Omega^\alpha$,
(b) $\Theta^\alpha(., \boldsymbol{\eta}, \boldsymbol{\omega}, \alpha, \boldsymbol{k}, \tau)$ is measurable on Ω^α, $\forall \boldsymbol{\eta}, \boldsymbol{\omega} \in \mathbb{S}^d, \boldsymbol{k} \in \mathbb{R}^m, \alpha, \tau \in \mathbb{R}$,
(c) $\Theta^\alpha(., \mathbf{0}, \mathbf{0}, 0, \mathbf{0}, 0)$ belongs to $L^2(\Omega^\alpha)$.

H(6): $\phi^\alpha : \Omega^\alpha \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{S}^d$ is such that

(a) There exists $L_{\phi^\alpha} > 0$ such that
$|\phi^\alpha(\boldsymbol{x}, \boldsymbol{\eta}_1, \boldsymbol{\omega}_1, \alpha_1, \boldsymbol{k}_1, \tau_1) - \phi^\alpha(\boldsymbol{x}, \boldsymbol{\eta}_2, \boldsymbol{\omega}_2, \alpha_2, \boldsymbol{k}_2, \tau_2)| \leq$
$L_{\phi^\alpha}\big(|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2| + |\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2| + |\alpha_1 - \alpha_2| + |\boldsymbol{k}_1 - \boldsymbol{k}_2| + |\tau_1 - \tau_2|\big)$,
$\forall\, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\omega}_1, \boldsymbol{\omega}_2 \in \mathbb{S}^d, \boldsymbol{k}_1, \boldsymbol{k}_2 \in \mathbb{R}^m, \tau_1, \tau_2, \alpha_1, \alpha_2 \in \mathbb{R},$ *a.e.* $\boldsymbol{x} \in \Omega^\alpha$,

(b) $\phi^\alpha(., \boldsymbol{\eta}, \boldsymbol{\omega}, \alpha, \boldsymbol{k}, \tau)$ is measurable on Ω^α, $\forall \boldsymbol{\eta}, \boldsymbol{\omega} \in \mathbb{S}^d$, $\boldsymbol{k} \in \mathbb{R}^m, \alpha, \tau \in \mathbb{R}$,

(c) $\phi^\alpha(., \mathbf{0}, \mathbf{0}, 0, \mathbf{0}, 0)$ belongs to $L^2(\Omega^\alpha)$.

H(7): $\Psi^\alpha : \Omega^\alpha \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{S}^d$ is such that

(a) There exists $L_{\Psi^\alpha} > 0$ such that
$|\Psi^\alpha(\boldsymbol{x}, \boldsymbol{\eta}_1, \boldsymbol{\omega}_1, \alpha_1, \boldsymbol{k}_1, \tau_1) - \Psi^\alpha(\boldsymbol{x}, \boldsymbol{\eta}_2, \boldsymbol{\omega}_2, \alpha_2, \boldsymbol{k}_2, \tau_2)| \leq$
$L_{\Psi^\alpha}\big(|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2| + |\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2| + |\alpha_1 - \alpha_2| + |\boldsymbol{k}_1 - \boldsymbol{k}_2| + |\tau_1 - \tau_2|\big)$,
$\forall\, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\omega}_1, \boldsymbol{\omega}_2 \in \mathbb{S}^d, \boldsymbol{k}_1, \boldsymbol{k}_2 \in \mathbb{R}^m,\ \tau_1, \tau_2, \alpha_1, \alpha_2 \in \mathbb{R},$ *a.e.* $\boldsymbol{x} \in \Omega^\alpha$,

(b) $\Psi^\alpha(., \boldsymbol{\eta}, \boldsymbol{\omega}, \alpha, \boldsymbol{k}, \tau)$ is measurable on Ω^α, $\forall \boldsymbol{\eta}, \boldsymbol{\omega} \in \mathbb{S}^d$, $\boldsymbol{k} \in \mathbb{R}^m, \alpha, \tau \in \mathbb{R}$,

(c) $\Psi^\alpha(., \mathbf{0}, \mathbf{0}, 0, \mathbf{0}, 0)$ belongs to $L^2(\Omega^\alpha)$.

The piezoelectric tensor and the electric permittivity tensor satisfy the following conditions:

H(8): $\mathcal{E}^\alpha : \Omega^\alpha \times \mathbb{S}^d \to \mathbb{R}^d$ is such that

(a) $\mathcal{E}^\alpha = (e^\alpha_{ijk}),\ e^\alpha_{ijk} = e^\alpha_{ikj} \in L^\infty(\Omega^\alpha),\ 1 \leq i, j, k \leq d$,

(b) $\mathcal{E}^\alpha \sigma.\boldsymbol{v} = \sigma.(\mathcal{E}^\alpha)^* \boldsymbol{v},\quad \forall \sigma \in \mathbb{S}^d,\ \forall \boldsymbol{v} \in \mathbb{R}^d$.

H(9): $\boldsymbol{\beta}^\alpha : \Omega^\alpha \times \mathbb{R}^d \to \mathbb{R}^d$ is such that

(a) $\boldsymbol{\beta}^\alpha = (\beta^\alpha_{ij}),\ \beta^\alpha_{ij} = \beta^\alpha_{ji} \in L^\infty(\Omega^\alpha), \quad 1 \leq i, j \leq d$,

(b) There exists $m_{\boldsymbol{\beta}^\alpha} > 0$ such that
$\boldsymbol{\beta}^\alpha \mathbf{E}.\mathbf{E} \geq m_{\boldsymbol{\beta}^\alpha} |\mathbf{E}|^2,\ \forall \mathbf{E} = (E_i) \in \mathbb{R}^d,$ a.e. $\boldsymbol{x} \in \Omega^\alpha$.

The normal compliance function p_ν and the tangential function p_τ satisfy the assumptions:

H(10): (a) $p_\nu : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$

(b) There exists $L_\nu > 0$ such that
$|p_\nu(\boldsymbol{x}, r_1) - p_\nu(\boldsymbol{x}, r_2)| \leq L_\nu |r_1 - r_2|,\quad \forall r_1, r_2 \in \mathbb{R},$ a.e. $\boldsymbol{x} \in \Gamma_3$,

(c) $p_\nu(., r)$ is measurable on $\Gamma_3, \forall r \in \mathbb{R}$,

(d) $p_\nu(\boldsymbol{x}, r) = 0, \forall r \leq 0$, a.e. $\boldsymbol{x} \in \Gamma_3$.

H(11): (a) $p_\tau : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$

(b) There exists $L_\tau > 0$ such that
$|p_\tau(\boldsymbol{x}, d_1) - p_\tau(\boldsymbol{x}, d_2)| \leq L_\tau |d_1 - d_2|,\ \forall d_1, d_2 \in \mathbb{R},$ a.e. $\boldsymbol{x} \in \Gamma_3$,

(c) There exists $M_\tau > 0$ such that
$|p_\tau(\boldsymbol{x}, d)| \leq M_\tau\ \forall\, d \in \mathbb{R},$ a.e. $\boldsymbol{x} \in \Gamma_3$,

(d) $p_\tau(., d)$ is measurable on Γ_3, $\forall d \in \mathbb{R}$.

(e) $p_\tau(., 0) \in L^2(\Gamma_3)$.

We suppose that the mass density, the forces, the traction densities and the foundation's temperatures satisfy:

H(12): (a) $\rho^\alpha \in L^\infty(\Omega^\alpha)$, $\exists \rho_0 > 0$; $\rho^\alpha(x) \geq \rho_0$ a.e. $x \in \Omega^\alpha$,
(b) $\mathbf{f}_0^\alpha \in L^2(0,T;L^2(\Omega^\alpha)^d)$, $\mathbf{f}_2^\alpha \in L^2(0,T;L^2(\Gamma_2^\alpha)^d)$,
(c) $q_0^\alpha \in C(0,T;L^2(\Omega^\alpha))$, $q_2^\alpha \in C(0,T;L^2(\Gamma_b^\alpha))$,
(d) $\chi^\alpha \in L^2(0,T;L^2(\Omega^\alpha))$.

The energy coefficient, microcrack diffusion coefficient and adhesion coefficient satisfy:

H(13): $\kappa_0^\alpha, \kappa^\alpha > 0$, $\gamma_\nu \in L^\infty(\Gamma_3)$, $\gamma_\nu \geq 0$, a.e. on Γ_3.

Also, we assume that the initial values satisfy:

H(14): (a) $\boldsymbol{k}_0^\alpha \in \boldsymbol{Y}^\alpha$, $\boldsymbol{u}_0^\alpha \in \boldsymbol{V}^\alpha$, $\boldsymbol{v}_0^\alpha \in H^\alpha$, $\varsigma_0^\alpha \in K^\alpha$, $\tau_0^\alpha \in L_1^\alpha$,
(b) $\zeta_0 \in L^2(\Gamma_3)$, $0 \leq \zeta_0 \leq 1$, a.e. on Γ_3.

We will use a modified inner product on H, given by

$$(\!(\boldsymbol{u},\boldsymbol{v})\!)_H = \sum_{\alpha=1}^{2} (\rho^\alpha \boldsymbol{u}^\alpha, \boldsymbol{v}^\alpha)_{H^\alpha},$$

and let $|\!|\!|.|\!|\!|_H$ be the associated norm. It follows from assumption H(ρ^α), that $|\!|\!|.|\!|\!|_H$ and $\|.\|_H$ are equivalent norms on H, and the inclusion mapping of $(\boldsymbol{V}, \|.\|_{\boldsymbol{V}})$ into $(H, |\!|\!|.|\!|\!|_H)$ is continuous and dense. We denote by $\boldsymbol{V}'$ the dual of $\boldsymbol{V}$. Identifying H with its own dual. Then $(\boldsymbol{u},\boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}} = (\!(\boldsymbol{u},\boldsymbol{v})\!)_H$, $\forall \boldsymbol{u} \in H, \forall \boldsymbol{v} \in \boldsymbol{V}$. We define six mappings $\mathbf{f} : [0,T] \to \boldsymbol{V}'$, $q : [0,T] \to W$, $a : L_1 \times L_1 \to \mathbb{R}$, $a_0 : L_1 \times L_1 \to \mathbb{R}$, $j_{ad} : L^\infty(\Gamma_3) \times \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$ and $j_{\nu c} : \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$, respectively, by

$$(\mathbf{f}(t),\boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}} = \sum_{\alpha=1}^{2} \int_{\Omega^\alpha} \mathbf{f}_0^\alpha(t)\cdot\boldsymbol{v}^\alpha dx + \sum_{\alpha=1}^{2} \int_{\Gamma_2^\alpha} \mathbf{f}_2^\alpha(t)\cdot\boldsymbol{v}^\alpha \, da \ \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{4.1}$$

$$(q(t),\varsigma)_W = \sum_{\alpha=1}^{2} \int_{\Omega^\alpha} q_0^\alpha(t)\varsigma^\alpha \, dx - \sum_{\alpha=1}^{2} \int_{\Gamma_b^\alpha} q_2^\alpha(t)\varsigma^\alpha \, da \quad \forall \varsigma \in W, \tag{4.2}$$

$$a(\varsigma,\omega) = \sum_{\alpha=1}^{2} \kappa^\alpha \int_{\Omega^\alpha} \nabla\varsigma^\alpha.\nabla\omega^\alpha dx, \tag{4.3}$$

$$a_0(\xi,\zeta) = \sum_{\alpha=1}^{2} \kappa_0^\alpha \int_{\Omega^\alpha} \nabla\xi^\alpha.\nabla\zeta^\alpha dx + \sum_{\alpha=1}^{2} \lambda_0^\alpha \int_{\Gamma^\alpha} \xi^\alpha\zeta^\alpha da, \tag{4.4}$$

$$a_0(\xi,\zeta) = \sum_{\alpha=1}^{2} \kappa_0^\alpha \int_{\Omega^\alpha} \nabla\xi^\alpha.\nabla\zeta^\alpha dx + \sum_{\alpha=1}^{2} \lambda_0^\alpha \int_{\Gamma^\alpha} \xi^\alpha\zeta^\alpha da, \tag{4.5}$$

$$j_{ad}(\zeta,\boldsymbol{u},\boldsymbol{v})=\int_{\Gamma_3}\Big(-\gamma_\nu\zeta^2R_\nu(u_\nu^1+u_\nu^2)(v_\nu^1+v_\nu^2)\Big)da$$
$$+\int_{\Gamma_3}\big(p_\tau(\zeta)\boldsymbol{R}_\tau(\boldsymbol{u}_\tau^1-\boldsymbol{u}_\tau^2)(\boldsymbol{v}_\tau^1-\boldsymbol{v}_\tau^2)\big)da, \tag{4.6}$$

$$j_{\nu c}(\boldsymbol{u},\boldsymbol{v})=\int_{\Gamma_3}p_\nu(u_\nu^1+u_\nu^2)(v_\nu^1+v_\nu^2)da. \tag{4.7}$$

We note that conditions H(12)(b) and H(12)(c) imply

$$\mathbf{f}\in L^2(0,T;\boldsymbol{V}'),\quad q\in C(0,T;W). \tag{4.8}$$

By a standard procedure based on Green's formula, we derive the following variational formulation of the mechanical (3.2)–(3.19).

Problem PV. Find $\boldsymbol{u}=(\boldsymbol{u}^1,\boldsymbol{u}^2):[0,T]\to\boldsymbol{V}$, $\boldsymbol{\sigma}=(\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^2):[0,T]\to\mathcal{H}$, $\xi=(\xi^1,\xi^2):[0,T]\to W$, $\varsigma=(\varsigma^1,\varsigma^2):[0,T]\to L_1$, $\zeta:[0,T]\to L^\infty(\Gamma_3)$,

$\tau = (\tau^1, \tau^2) : [0, T] \to L_1$, and $\boldsymbol{k} = (\boldsymbol{k}^1, \boldsymbol{k}^2) : [0, T] \to \boldsymbol{Y}$ such that

$$\boldsymbol{\sigma}^\alpha = \mathcal{A}^\alpha \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\alpha) + \mathcal{B}^\alpha \boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha) + (\mathcal{E}^\alpha)^* \nabla \xi^\alpha + \int_0^t \mathcal{G}^\alpha\Big(\boldsymbol{\sigma}^\alpha(s) - \mathcal{A}^\alpha \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\alpha(s)) - (\mathcal{E}^\alpha)^* \nabla \xi^\alpha(s), \boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha(s)), \varsigma^\alpha(s), \boldsymbol{k}^\alpha(s), \tau^\alpha(s)\Big) ds \quad \text{in } \Omega^\alpha \times (0, T), \tag{4.9}$$

$$\dot{\boldsymbol{k}}^\alpha = \Theta^\alpha(\boldsymbol{\sigma}^\alpha - \mathcal{A}^\alpha \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\alpha) - (\mathcal{E}^\alpha)^* \nabla \xi^\alpha(s), \boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha), \varsigma^\alpha, \boldsymbol{k}^\alpha, \tau^\alpha) \text{in } \Omega^\alpha \times (0, T), \tag{4.10}$$

$$(\ddot{u}, v)_{\boldsymbol{V}' \times \boldsymbol{V}} + \sum_{\alpha=1}^{2} (\boldsymbol{\sigma}^\alpha, \ \varepsilon(\boldsymbol{v}^\alpha))_{\mathcal{H}^\alpha} + j_{ad}(\zeta(t), \boldsymbol{u}(t), \boldsymbol{v}) + j_{\nu c}(\boldsymbol{u}(t), \boldsymbol{v}) = (\mathbf{f}(t), \boldsymbol{v})_{\boldsymbol{V}' \times \boldsymbol{V}}, \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \ \text{a.e. } t \in (0, T), \tag{4.11}$$

$$\varsigma(t) \in K, \quad \sum_{\alpha=1}^{2} (\dot{\varsigma}^\alpha(t), \omega^\alpha - \varsigma^\alpha(t))_{L^2(\Omega^\alpha)} + a(\varsigma(t), \omega - \varsigma(t)) \geq \sum_{\alpha=1}^{2} \Big(\phi^\alpha\big(\boldsymbol{\sigma}^\alpha - \mathcal{A}^\alpha \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\alpha)) - (\mathcal{E}^\alpha)^* \nabla \xi^\alpha, \boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha), \varsigma^\alpha, \boldsymbol{k}^\alpha, \tau^\alpha\big)(t), \omega^\alpha - \varsigma^\alpha(t)\Big)_{L^2(\Omega^\alpha)} \quad \forall \omega \in K, \ \text{a.e. } t \in (0, T), \tag{4.12}$$

$$a_0(\tau(t), \delta) = \sum_{\alpha=1}^{2} \Big(\Psi^\alpha\big(\boldsymbol{\sigma}^\alpha - \mathcal{A}^\alpha \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\alpha) - (\mathcal{E}^\alpha)^* \nabla \xi^\alpha, \boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha), \varsigma^\alpha, \boldsymbol{k}^\alpha, \tau^\alpha\big)(t), \delta^\alpha\Big)_{L_0^\alpha} + \sum_{\alpha=1}^{2} (\dot{\tau}^\alpha(t) - \chi^\alpha(t), \delta^\alpha)_{L_0^\alpha} \quad \forall \delta \in L_1, \ \text{a.e. } t \in (0, T), \tag{4.13}$$

$$\sum_{\alpha=1}^{2} \left(\boldsymbol{\beta}^\alpha \nabla \xi^\alpha(t) - \mathcal{E}^\alpha \varepsilon(\boldsymbol{u}^\alpha(t)), \nabla \phi^\alpha\right)_{H^\alpha} = (q(t), \phi)_W \ \forall \phi \in W, \text{a.e. } t \in (0, T), \tag{4.14}$$

$$\dot{\zeta} = H_{ad}(\zeta, R_\nu(u_\nu^1 + u_\nu^2), \boldsymbol{R}_\tau(\boldsymbol{u}_\tau^1 - \boldsymbol{u}_\tau^2)) \quad \text{a.e. } t \in (0, T), \tag{4.15}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}(0) = \boldsymbol{v}_0, \quad \varsigma(0) = \varsigma_0, \quad \zeta(0) = \zeta_0, \quad \boldsymbol{k}(0) = \boldsymbol{k}_0, \quad \tau(0) = \tau_0, \tag{4.16}$$

where $K = K^1 \times K^2$.

Remark 1. *We note that, in Problem* $\boldsymbol{P}$ *and in Problem* $\boldsymbol{PV}$*, we do not need to impose explicitly the restriction* $0 \leq \zeta \leq 1$*. Indeed, equation* (4.15) *guarantees that* $\zeta(x, t) \leq \zeta_0(x)$ *and, therefore, assumption H(4) shows that* $\zeta(x, t) \leq 1$ *for* $t \geq 0$*, a.e.* $x \in \Gamma_3$*. On the other hand, if* $\zeta(x, t_0) = 0$ *at time* t_0*, then it follows from* (4.15) *that* $\dot{\zeta}(x, t) = 0$ *for all* $t \geq t_0$ *and therefore,* $\zeta(x, t) = 0$ *for all* $t \geq t_0$*, a.e.* $x \in \Gamma_3$*. We conclude that* $0 \leq \zeta(x, t) \leq 1$ *for all* $t \in [0, T]$*, a.e.* $x \in \Gamma_3$*.*

Now, we propose our existence and uniqueness result

Theorem 2 (Existence and uniqueness). *Assume that H(1)– H(14) hold. Then there exists a unique solution* $\{\boldsymbol{u}, \boldsymbol{\sigma}, \xi, \varsigma, \zeta, \boldsymbol{k}, \tau\}$ *to Problem* $\boldsymbol{PV}$*, Moreover, the solution satisfies*

$$\boldsymbol{u} \in W^{1,2}(0,T;\boldsymbol{V}) \cap C^1(0,T;H), \;\; \ddot{\boldsymbol{u}} \in L^2(0,T;\boldsymbol{V}'), \tag{4.17}$$

$$\boldsymbol{\sigma} \in L^2(0,T;\mathcal{H}), \;\; (\mathrm{Div}\,\boldsymbol{\sigma}^1, \mathrm{Div}\,\boldsymbol{\sigma}^2) \in L^2(0,T;\boldsymbol{V}'), \tag{4.18}$$

$$\xi \in C(0,T;W), \tag{4.19}$$

$$\varsigma \in W^{1,2}(0,T;L_0) \cap L^2(0,T;L_1), \tag{4.20}$$

$$\zeta \in W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap \mathcal{Z}, \tag{4.21}$$

$$\boldsymbol{k} \in W^{1,2}(0,T;\boldsymbol{Y}), \tag{4.22}$$

$$\tau \in W^{1,2}(0,T;L_0) \cap L^2(0,T;L_1). \tag{4.23}$$

The functions $\{\boldsymbol{u}, \xi, \varsigma, \boldsymbol{\sigma}, \boldsymbol{k}, \tau, \zeta, \boldsymbol{D}\}$ which satisfy (4.9)–(4.16) and (3.4) are called weak solution of the thermo-piezoelectric contact Problem **P**. We conclude by Theorem 2 that, under the assumptions H(1)–H(14), the mechanical problem (3.2)–(3.19) has a unique weak solution $\{\boldsymbol{u}, \xi, \varsigma, \boldsymbol{\sigma}, \boldsymbol{k}, \tau, \zeta, \boldsymbol{D}\}$. To precise the regularity of the weak solution, we note that the constitutive relation (3.4), the assumptions H(8)-H(9) , and the regularities (4.17), (4.19) show that $\boldsymbol{D} \in C(0,T;H)$. Moreover, using (4.14) and notation (4.2), we obtain

$$div\boldsymbol{D}^\alpha(t) = q_0^\alpha(t) \quad \forall t \in [0,T], \;\; \alpha = 1,2.$$

It follows now from the regularities H(12)(c) that div $\boldsymbol{D}^\alpha \in C(0,T;H^\alpha)$, $\alpha = 1,2$, which shows that

$$\boldsymbol{D} \in C(0,T;\mathcal{W}). \tag{4.24}$$

We conclude that the weak solution $\{\boldsymbol{u}, \xi, \varsigma, \boldsymbol{\sigma}, \boldsymbol{k}, \tau, \zeta, \boldsymbol{D}\}$ of the thermo-piezoelectric contact Problem **P** has the regularity (4.17)–(4.24).

5 Proof of Theorem 2

The proof of Theorem 2 which will be carried out in several steps and is based on arguments of nonlinear equations with monotone operators, a classical existence and uniqueness result on parabolic inequalities and fixed point arguments. We assume in what follows that assumptions of Theorem 2 hold, and we consider that C is a generic positive constant which depends on Ω^α, Γ_1^α, Γ_2^α, Γ_3, p_ν, p_τ, $\mathcal{A}^\alpha$, $\boldsymbol{\beta}^\alpha$, $\mathcal{B}^\alpha$, $\mathcal{G}^\alpha$, $\mathcal{E}^\alpha$, Θ^α, H_{ad}, γ_ν, ϕ^α, Ψ^α, κ_0^α, λ_0^α, κ^α, χ^α and T, with $\alpha = 1,2$. But does not depend on t nor of the rest of input data, and whose value may change from place to place. Let a $\eta = (\eta^1, \eta^2) \in L^2(0,T;\boldsymbol{V}')$ be given. In the first step we consider the following variational problem.

Problem $\mathbf{PV}_\eta^{u\xi}$. Find $(\boldsymbol{u}_\eta, \xi_\eta) : [0,T] \to \boldsymbol{V} \times W$ such that

$$(\ddot{u}_\eta(t), v)_{\boldsymbol{V}'\times\boldsymbol{V}} + \sum_{\alpha=1}^{2} (\mathcal{A}^\alpha \varepsilon(\dot{\boldsymbol{u}}^\alpha(t)),\ \varepsilon(\boldsymbol{v}^\alpha))_{\mathcal{H}^\alpha} = (\mathbf{f}(t) - \eta(t), \boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}},$$
$$\forall \boldsymbol{v} \in \boldsymbol{V}, \text{a.e. } t \in (0,T), \tag{5.1}$$

$$\sum_{\alpha=1}^{2} (\boldsymbol{\beta}^\alpha \nabla \xi_\eta^\alpha(t) - \mathcal{E}^\alpha \varepsilon(\boldsymbol{u}_\eta^\alpha(t)), \nabla\phi^\alpha)_{H^\alpha} = (q(t), \phi)_W,\ \forall \phi \in W \text{ a.e. } t \in (0,T), \tag{5.2}$$

$$\boldsymbol{u}_\eta^\alpha(0) = \boldsymbol{u}_0^\alpha,\quad \dot{\boldsymbol{u}}_\eta^\alpha(0) = \boldsymbol{v}_0^\alpha \quad \text{in } \Omega^\alpha. \tag{5.3}$$

We have the following result for the problem.

Lemma 3. *There exists a unique solution $(\boldsymbol{u}_\eta, \xi_\eta)$ of Problem $\boldsymbol{PV}_\eta^{u\xi}$ and it satisfies*

$$\boldsymbol{u}_\eta \in W^{1,2}(0,T;\boldsymbol{V}) \cap C^1(0,T;H),\ \ddot{\boldsymbol{u}}_\eta \in L^2(0,T;\boldsymbol{V}'), \tag{5.4}$$
$$\xi_\eta \in C(0,T;W). \tag{5.5}$$

Proof. We define the operator $A : \boldsymbol{V} \to \boldsymbol{V}'$ by

$$(A\boldsymbol{u}, \boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}} = \sum_{\alpha=1}^{2} (\mathcal{A}^\alpha \varepsilon(\boldsymbol{u}^\alpha),\ \varepsilon(\boldsymbol{v}^\alpha))_{\mathcal{H}^\alpha} \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}. \tag{5.6}$$

We use (5.6) and H(1) to find that

$$\|A\boldsymbol{u} - A\boldsymbol{v}\|_{\boldsymbol{V}'}^2 \le \sum_{\alpha=1}^{2} \|\mathcal{A}^\alpha \varepsilon(\boldsymbol{u}^\alpha) - \mathcal{A}^\alpha \varepsilon(\boldsymbol{v}^\alpha)\|_{\mathcal{H}^\alpha}^2 \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}.$$

Keeping in mind H(1) and Krasnoselski Theorem (see, for example [24, p.60]), we deduce that $A : \boldsymbol{V} \to \boldsymbol{V}'$ is a continuous, and so hemicontinuous. Now, by H(1)(c) and (5.6), it follows that

$$(A\boldsymbol{u} - A\boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}} \ge m\|\boldsymbol{u} - \boldsymbol{v}\|_{\boldsymbol{V}}^2 \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}, \tag{5.7}$$

where the positive constant $m = \min\{m_{\mathcal{A}^1}, m_{\mathcal{A}^2}\}$. Choosing $\boldsymbol{v} = 0$ in (5.7) we obtain

$$\begin{aligned}(A\boldsymbol{u}, \boldsymbol{u})_{\boldsymbol{V}'\times\boldsymbol{V}} &\ge m\|\boldsymbol{u}\|_{\boldsymbol{V}}^2 - \|Ao\|_{\boldsymbol{V}'}^2 \|\boldsymbol{u}\|_{\boldsymbol{V}} \\ &\ge \frac{1}{2} m\|\boldsymbol{u}\|_{\boldsymbol{V}}^2 - \frac{1}{2m}\|Ao\|_{\boldsymbol{V}'}^2 \quad \forall \boldsymbol{u} \in \boldsymbol{V}.\end{aligned} \tag{5.8}$$

Moreover, by (5.6) and H(1)(b) we find

$$\|A\boldsymbol{u}\|_{\boldsymbol{V}'} \le C^1 \|u\|_{\boldsymbol{V}} + C^2 \quad \forall \boldsymbol{u} \in \boldsymbol{V},$$

where $C^1 = \max\{C^1_{\mathcal{A}^1}, C^1_{\mathcal{A}^2}\}$ and $C^2 = \max\{C^2_{\mathcal{A}^1}, C^2_{\mathcal{A}^2}\}$. Finally, we recall that by (4.8) we have $\mathbf{f} - \eta \in L^2(0, T; \boldsymbol{V}')$ and $\boldsymbol{v}_0 \in H$. Therefore, using a standard for ordinary differential equations in abstract spaces (see, for example, [36, Theorem 2.29]), we know there exists a unique function ϑ_η such that

$$\vartheta_\eta \in L^2(0, T; \boldsymbol{V}) \cap C(0, T; H), \quad \dot{\vartheta}_\eta \in L^2(0, T; \boldsymbol{V}'), \tag{5.9}$$

$$\dot{\vartheta}_\eta(t) + A\vartheta_\eta(t) = \mathbf{f}(t) - \eta(t), \quad a.e.\ t \in [0, T] \tag{5.10}$$

$$\vartheta_\eta(0) = \boldsymbol{v}_0. \tag{5.11}$$

Let $\boldsymbol{u}_\eta : [0, T] \to \boldsymbol{V}$ be the function defined by

$$\boldsymbol{u}_\eta(t) = \int_0^t \vartheta_\eta(s)ds + \boldsymbol{u}_0 \quad \forall t \in [0, T]. \tag{5.12}$$

It follows from (5.6) and (5.9)– (5.12), that $\boldsymbol{u}_\eta$ is a solution to (5.1), (5.3), with the regularity (5.4). Next, we define a bilinear form: $b(.,.) : W \times W \to \mathbb{R}$ such that

$$b(\xi, \phi) = \sum_{\alpha=1}^{2} (\boldsymbol{\beta}^\alpha \nabla \xi^\alpha, \nabla \phi^\alpha)_{H^\alpha} \qquad \forall \xi, \phi \in W. \tag{5.13}$$

We use H(9) and (5.13) to show that the bilinear form $b(.,.)$ is continuous, symmetric and coercive on W. Moreover, using (4.2) and the Riesz Representation Theorem we may define an element $q_\eta : [0, T] \to W$ such that

$$(q_\eta(t), \phi)_W = (q(t), \phi)_W + \sum_{\alpha=1}^{2} (\mathcal{E}^\alpha \varepsilon(\boldsymbol{u}^\alpha_\eta(t)), \nabla \phi^\alpha)_{H^\alpha} \qquad \forall \phi \in W, t \in (0, T).$$

We apply the Lax-Milgram Theorem to deduce that there exists a unique element $\xi_\eta(t) \in W$ such that

$$b(\xi_\eta(t), \phi) = (q_\eta(t), \phi)_W \qquad \forall \phi \in W. \tag{5.14}$$

It follows from (5.14), that the pair $(\boldsymbol{u}_\eta, \xi_\eta)$ is the solution to the nonlinear variational equation (5.2). Let now $t_1, t_2 \in [0, T]$, it follows from (5.2) that

$$\|\xi_\eta(t_1) - \xi_\eta(t_2)\|_W \le C\big(\|\boldsymbol{u}_\eta(t_1) - \boldsymbol{u}_\eta(t_2)\|_{\boldsymbol{V}} + \|q(t_1) - q(t_2)\|_W\big). \tag{5.15}$$

Since $\boldsymbol{u}_\eta \in C^1(0, T; H)$ and $q \in C(0, T; W)$, inequality (5.15) implies that $\xi_\eta \in C(0, T; W)$. This completes the proof □

In the second step, we let $\theta = (\theta^1, \theta^1) \in L^2(0.T; L_0)$ be given and consider the following initial-value problem.
Problem $\mathbf{PV}^{\varsigma}_\theta$. *Find* $\varsigma_\theta = (\varsigma^1_\theta, \varsigma^2_\theta) : [0, T] \to L_1$ such that

$$\varsigma_\theta(t) \in K, \quad \sum_{\alpha=1}^{2} (\dot{\varsigma}^\alpha_\theta(t) - \theta^\alpha(t), \mu^\alpha - \varsigma^\alpha_\theta(t))_{L^2(\Omega^\alpha)} + a(\varsigma_\theta(t), \mu - \varsigma_\theta(t)) \ge 0 \tag{5.16}$$
$$\forall \mu \in K, \ \text{a.e. } t \in (0, T).$$

In the study of Problem $\mathbf{PV}^{\varsigma}_\theta$ we have the following result.

Lemma 4. *There exists a unique solution ς_θ of Problem $\boldsymbol{PV}_\theta^\varsigma$ and it satisfies*

$$\varsigma_\theta \in W^{1,2}(0,T;L_0) \cap L^2(0,T;L_1).$$

Proof. We use a standard result for parabolic variational inequalities (see, e.g., [36, p.47]). □

In the third step we use the field $\boldsymbol{u}_\eta$ obtained in Lemma 3 and we consider the following initial-value problem.
Problem $\mathbf{PV}_\eta^\zeta$. Find $\zeta_\eta : [0,T] \to L^2(\Gamma_3)$ such that

$$\dot{\zeta}_\eta(t) = H_{ad}\big(\zeta_\eta(t), R_\nu(\boldsymbol{u}^1_{\eta\boldsymbol{\nu}}(t) + \boldsymbol{u}^2_{\eta\boldsymbol{\nu}}(t)), \boldsymbol{R}_\tau(\boldsymbol{u}^1_{\eta\boldsymbol{\tau}}(t) - \boldsymbol{u}^2_{\eta\boldsymbol{\tau}}(t))\big), \tag{5.17}$$

$$\zeta_\eta(0) = \zeta_0. \tag{5.18}$$

We have the following result.

Lemma 5. *There exists a unique solution $\zeta_\eta \in W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap \mathcal{Z}$ to Problem $\boldsymbol{PV}_\eta^\zeta$.*

Proof. We consider the mapping $F_\eta : [0,T] \times L^2(\Gamma_3) \to L^2(\Gamma_3)$,

$$F_\eta(t,\zeta) = H_{ad}\big(\zeta(t), R_\nu(\boldsymbol{u}^1_{\eta\boldsymbol{\nu}}(t) + \boldsymbol{u}^2_{\eta\boldsymbol{\nu}}(t)), \boldsymbol{R}_\tau(\boldsymbol{u}^1_{\eta\boldsymbol{\tau}}(t) - \boldsymbol{u}^2_{\eta\boldsymbol{\tau}}(t))\big),$$

for all $t \in [0,T]$ and $\zeta \in L^2(\Gamma_3)$. It follows from the properties of the truncation operator R_ν and $\boldsymbol{R}_\tau$ that F_η is Lipschitz continuous with respect to the second variable, uniformly in time. Moreover, for all $\zeta \in L^2(\Gamma_3)$, the mapping $t \to F_\eta(t,\zeta)$ belongs to $L^\infty(0,T;L^2(\Gamma_3))$. Thus using the Cauchy–Lipschitz theorem given in [23, p. 60], we deduce that there exists a unique function $\zeta_\eta \in W^{1,\infty}(0,T;L^2(\Gamma_3))$ solution of the equation (5.17). Also, the arguments used in Remark 1 show that $0 \le \zeta_\eta(t) \le 1$ for all $t \in [0,T]$, a.e. on Γ_3. This completes the proof. □

In the fourth step. Let $\pi = (\pi^1, \pi^1) \in L^2(0,T;L_0)$ and consider the auxiliary problem.

Problem $\mathbf{PV}_\pi^\tau$. Find $\tau_\pi : [0,T] \to L_0$, such that

$$\textstyle\sum_{\alpha=1}^2 (\dot{\tau}^\alpha_\pi(t) - \pi^\alpha(t) - \chi^\alpha(t), \lambda^\alpha)_{L_0^\alpha} + a_0(\tau_\pi(t), \lambda) = 0,\ \forall \lambda \in L_0, \tag{5.19}$$

$$\tau_\pi(0) = \tau_0. \tag{5.20}$$

Lemma 6. *There exists a unique solution τ_π to the auxiliary problem $\boldsymbol{PV}_\pi^\tau$ satisfying (4.23).*

Proof. Furthermore, by an application of the Poincaré-Friedrichs inequality, we can find a constant $c_0 > 0$ such that

$$\int_{\Omega^\alpha} |\nabla\lambda|^2 dx + \frac{\lambda_0^\alpha}{\kappa_0^\alpha} \int_{\Gamma^\alpha} |\lambda|^2 da \ge c_0 \int_{\Omega^\alpha} |\lambda|^2 dx, \quad \forall \lambda \in L_1^\alpha,\ \alpha = 1, 2.$$

Thus, we obtain

$$a_0(\lambda,\lambda) \geq c_1\|\lambda\|^2_{L_1}, \quad \forall \lambda \in L_1,$$

where $c_1 = \kappa_0 \min(1, c_0)/2$, which implies that a_0 is L_1-elliptic. Consequently, based on classical arguments of functional analysis concerning parabolic equations, the variational equation (5.19) has a unique solution τ_π satisfying $\tau_\pi(0) = \tau_0$ and the regularity (4.23). □

In the fifth step, we let $\mu \in L^2(0,T,\boldsymbol{Y})$ be given, and define $\boldsymbol{k}_\mu \in W^{1,2}(0,T,\boldsymbol{Y})$ by

$$\boldsymbol{k}_\mu(t) = \boldsymbol{k}_0 + \int_0^t \mu(s)ds. \tag{5.21}$$

We use $(\boldsymbol{u}_\eta, \xi_\eta)$ obtained in Lemma 3, ς_θ obtained in Lemma 4, τ_π obtained in Lemma 6 and $\boldsymbol{k}_\mu$ defined in (5.21) to construct the following Cauchy problem for the stress field.

Problem $\mathbf{PV}^\sigma_{\eta\mu\theta\pi}$. Find $\boldsymbol{\sigma}_{\eta\mu\theta\pi} = (\boldsymbol{\sigma}^1_{\eta\mu\theta\pi}, \boldsymbol{\sigma}^2_{\eta\mu\theta\pi}) : [0,T] \to \mathcal{H}$ such that

$$\boldsymbol{\sigma}^\alpha_{\eta\mu\theta\pi}(t) = \mathcal{B}^\alpha \boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha_\eta(t)) + \int_0^t \mathcal{G}^\alpha(\boldsymbol{\sigma}^\alpha_{\eta\mu\theta\pi}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha_\eta(s)), \varsigma^\alpha_\theta(s), \boldsymbol{k}^\alpha_\mu(s), \tau^\alpha_\pi(s))ds, \tag{5.22}$$
$$\text{a.e. } t \in (0,T), \quad \alpha = 1,2.$$

In the study of Problem $\mathbf{PV}^\sigma_{\eta\mu\theta\pi}$ we have the following result.

Lemma 7. *There exists a unique solution of Problem $\boldsymbol{PV}^\sigma_{\eta\mu\theta\pi}$ and it satisfies $\boldsymbol{\sigma}_{\eta\mu\theta\pi} \in L^2(0,T;\mathcal{H})$.*

Proof. We introduce the operator $\Lambda_{\eta\mu\theta\pi} = (\Lambda^1_{\eta\mu\theta\pi}, \Lambda^2_{\eta\mu\theta\pi}) : L^2(0,T;\mathcal{H}) \to L^2(0,T;\mathcal{H})$ defined by

$$\Lambda^\alpha_{\eta\mu\theta\pi}\boldsymbol{\sigma}(t) = \mathcal{B}^\alpha \boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha_\eta(t)) + \int_0^t \mathcal{G}^\alpha\big(\boldsymbol{\sigma}^\alpha(s), \boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha_\eta(s)), \varsigma^\alpha_\theta(s), \boldsymbol{k}^\alpha_\mu(s), \tau^\alpha_\pi(s)\big)ds, \tag{5.23}$$

for all $\boldsymbol{\sigma} = (\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2) \in L^2(0,T;\mathcal{H})$, $t \in [0,T]$ and $\alpha = 1,2$. For $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2 \in L^2(0,T;\mathcal{H})$ we use (5.23) and H(3), to obtain

$$\|\Lambda_{\eta\mu\theta\pi}\boldsymbol{\sigma}_1(t) - \Lambda_{\eta\mu\theta\pi}\boldsymbol{\sigma}_2(t)\|_{\mathcal{H}} \leq \max(L_{\mathcal{G}^1}, L_{\mathcal{G}^2}) \int_0^t \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathcal{H}}\, ds$$

for all $t \in [0,T]$. It follows from this inequality that for p large enough, a power $\Lambda^p_{\eta\mu\theta\pi}$ of the operator $\Lambda_{\eta\mu\theta\pi}$ is a contraction on the Banach space $L^2(0,T;\mathcal{H})$ and, therefore, there exists a unique element $\boldsymbol{\sigma}_{\eta\mu\theta\pi} \in L^2(0,T;\mathcal{H})$ such that $\Lambda_{\eta\mu\theta\pi}\boldsymbol{\sigma}_{\eta\mu\theta\pi} = \boldsymbol{\sigma}_{\eta\mu\theta\pi}$. Moreover, $\boldsymbol{\sigma}_{\eta\mu\theta\pi}$ is the unique solution of Problem $\mathbf{PV}^\sigma_{\eta\mu\theta\pi}$, which concludes the proof. □

Lemma 8. *Let* $(\eta_1, \mu_1, \theta_1, \pi_1), (\eta_2, \mu_2, \theta_2, \pi_2) \in L^2(0, T; \boldsymbol{V}' \times \boldsymbol{Y} \times L_0 \times L_0)$ *and let* $\boldsymbol{\sigma}_i$ *denote the functions obtained in Lemma 7, for* $i = 1, 2$. *Then, the following inequalities hold:*

$$\|\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)\|^2_{\mathcal{H}} \leq C\Big(\|\boldsymbol{u}_{\eta_1}(t) - \boldsymbol{u}_{\eta_2}(t)\|^2_{\boldsymbol{V}} + \int_0^t \|\boldsymbol{u}_{\eta_1}(s) - \boldsymbol{u}_{\eta_2}(s)\|^2_{\boldsymbol{V}}\, ds + \int_0^t \|\varsigma_{\theta_1}(s) - \varsigma_{\theta_2}(s)\|^2_{L_0} ds + \int_0^t \|\boldsymbol{k}_{\mu_1}(s) - \boldsymbol{k}_{\mu_2}(s)\|^2_{Y} ds + \int_0^t \|\tau_{\pi_1}(s) - \tau_{\pi_2}(s)\|^2_{L_0} ds\Big), \qquad \text{a.e. } t \in (0, T). \tag{5.24}$$

Proof. Let $t \in [0, T]$. Using (5.22) and the properties H(2) - H(3) of $\mathcal{B}^\alpha$ and $\mathcal{G}^\alpha$, we find

$$\|\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)\|^2_{\mathcal{H}} \leq c\Big(\|\boldsymbol{u}_{\eta_1}(t) - \boldsymbol{u}_{\eta_2}(t)\|^2_{\boldsymbol{V}} + \int_0^t \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|^2_{\mathcal{H}}\, ds + \int_0^t \|\boldsymbol{u}_{\eta_1}(s) - \boldsymbol{u}_{\eta_2}(s)\|^2_{V}\, ds + \int_0^t \|\varsigma_{\theta_1}(s) - \varsigma_{\theta_2}(s)\|^2_{L_0}\, ds + \int_0^t \|\boldsymbol{k}_{\mu_1}(s) - \boldsymbol{k}_{\mu_2}(s)\|^2_{\boldsymbol{Y}}\, ds + \int_0^t \|\tau_{\pi_1}(s) - \tau_{\pi_2}(s)\|^2_{L_0}\, ds\Big).$$

Using the Gronwall's inequality in the previous inequality we deduce the estimate (5.24), which concludes the proof of Lemma 8. □

We now pass to the final step of the proof of Theorem 2 in which we use a fixed point argument. To this end, we consider the operator:

$$\Pi : L^2(0, T; \boldsymbol{V}' \times \boldsymbol{Y} \times L_0 \times L_0) \to L^2(0, T; \boldsymbol{V}' \times \boldsymbol{Y} \times L_0 \times L_0)$$

defined by

$$\Pi(\eta, \mu, \theta, \pi) = \big(\Pi^1(\eta, \mu, \theta, \pi), \Pi^2(\eta, \mu, \theta, \pi), \Pi^3(\eta, \mu, \theta, \pi), \Pi^4(\eta, \mu, \theta, \pi)\big) \tag{5.25}$$

with

$$(\Pi^1(\eta,\mu,\theta,\pi)(t),\boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}} = \sum_{\alpha=1}^{2}\big(\mathcal{B}^\alpha\varepsilon(\boldsymbol{u}_\eta^\alpha(t)) + (\mathcal{E}^\alpha)^*\nabla\xi_\eta^\alpha\ ,\ \varepsilon(\boldsymbol{v}^\alpha)\big)_{\mathcal{H}^\alpha}$$

$$+\sum_{\alpha=1}^{2}\Big(\int_0^t \mathcal{G}^\alpha\big(\boldsymbol{\sigma}_{\eta\mu\theta\pi}^\alpha, \varepsilon(\boldsymbol{u}_\eta^\alpha(s)), \varsigma_\theta^\alpha(s), \boldsymbol{k}_\mu^\alpha(s), \tau_\pi^\alpha(s)\big)\,ds\ ,\ \varepsilon(\boldsymbol{v}^\alpha)\Big)_{\mathcal{H}^\alpha}$$

$$+j_{ad}(\zeta_\eta(t),\boldsymbol{u}_\eta(t),\boldsymbol{v}) + j_{\nu c}(\boldsymbol{u}_\eta(t),\ \boldsymbol{v}),\quad \forall \boldsymbol{v}\in\boldsymbol{V}, \tag{5.26}$$

$$\Pi^2(\eta,\mu,\theta,\pi)(t) = \Big(\Theta^1\big(\boldsymbol{\sigma}_{\eta\mu\theta\pi}^1(t), \boldsymbol{\varepsilon}(\boldsymbol{u}_\eta^1(t)), \varsigma_\theta^1(t), \boldsymbol{k}_\mu^1(t), \tau_\pi^1(t)\big), \Theta^2\big(\boldsymbol{\sigma}_{\eta\mu\theta\pi}^2(t), \boldsymbol{\varepsilon}(\boldsymbol{u}_\eta^2(t)), \varsigma_\theta^2(t), \boldsymbol{k}_\mu^2(t), \tau_\pi^2(t)\big)\Big) \tag{5.27}$$

$$\Pi^3(\eta,\mu,\theta,\pi)(t) = \Big(\phi^1\big(\boldsymbol{\sigma}_{\eta\mu\theta\pi}^1(t), \boldsymbol{\varepsilon}(\boldsymbol{u}_\eta^1(t)), \varsigma_\theta^1(t), \boldsymbol{k}_\mu^1(t), \tau_\pi^1(t)\big), \phi^2\big(\boldsymbol{\sigma}_{\eta\mu\theta\pi}^2(t), \boldsymbol{\varepsilon}(\boldsymbol{u}_\eta^2(t)), \varsigma_\theta^2(t), \boldsymbol{k}_\mu^2(t), \tau_\pi^2(t)\big)\Big) \tag{5.28}$$

$$\Pi^4(\eta,\mu,\theta,\pi)(t) = \Big(\Psi^1\big(\boldsymbol{\sigma}_{\eta\mu\theta\pi}^1(t), \boldsymbol{\varepsilon}(\boldsymbol{u}_\eta^1(t)), \varsigma_\theta^1(t), \boldsymbol{k}_\mu^1(t), \tau_\pi^1(t)\big), \Psi^2\big(\boldsymbol{\sigma}_{\eta\mu\theta\pi}^2(t), \boldsymbol{\varepsilon}(\boldsymbol{u}_\eta^2(t)), \varsigma_\theta^2(t), \boldsymbol{k}_\mu^2(t), \tau_\pi^2(t)\big)\Big). \tag{5.29}$$

For the operator Π, we have the following result.

Lemma 9. *The operator* Π *has a unique fixed point* $(\eta^*,\mu^*,\theta^*,\pi^*) \in L^2(0,T;\boldsymbol{V}'\times \boldsymbol{Y}\times L_0\times L_0)$.

Proof. Let $(\eta_1,\mu_1,\theta_1,\pi_1)$,$(\eta_2,\mu_2,\theta_2,\pi_2)$ in $L^2(0,T;\boldsymbol{V}'\times\boldsymbol{Y}\times L_0\times L_0)$ and let $t\in[0,T]$. We use the notation $\boldsymbol{u}_i = \boldsymbol{u}_{\eta_i}$, $\boldsymbol{v}_i = \dot{\boldsymbol{u}}_{\eta_i}$, $\boldsymbol{\sigma}_i = \boldsymbol{\sigma}_{\eta_i\mu_i\theta_i\pi_i}$, $\xi_i = \xi_{\eta_i}$, $\varsigma_i = \varsigma_{\theta_i}$, $\zeta_i = \zeta_{\eta_i}$, $\boldsymbol{k}_i = \boldsymbol{k}_{\mu_i}$ and $\tau_i = \tau_{\pi_i}$ for $i = 1,2$. We use H(2), H(3), H(8), H(10), H(11), (5.24) and the definition of R_ν, $\boldsymbol{R}_\tau$, we have

$$\|\Pi^1(\eta_1,\mu_1,\theta_1,\pi_1)(t) - \Pi^1(\eta_2,\mu_2,\theta_2,\pi_2)(t)\|^2_{\boldsymbol{V}'} \le C\Big(\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|^2_{\boldsymbol{V}}$$

$$+\int_0^t \|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s))\|^2_{\boldsymbol{V}}\,ds + \int_0^t \|\varsigma_1(s)-\varsigma_2(s))\|^2_{L_0}\,ds + \|\xi_1(t)-\xi_2(t)\|^2_W$$

$$+\|\zeta_1(t)-\zeta_2(t)\|^2_{L^2(\Gamma_3)} + \int_0^t \|\boldsymbol{k}_1(s)-\boldsymbol{k}_2(s))\|^2_{\boldsymbol{Y}} + \int_0^t \|\tau_1(s)-\tau_2(s))\|^2_{L_0}ds\Big).$$

By similar arguments, from (5.24), (5.27) and H(5) it follows that

$$\|\Pi^2(\eta_1,\mu_1,\theta_1,\pi_1)(t)-\Pi^2(\eta_2,\mu_2,\theta_2,\pi_2)(t)\|^2_{\boldsymbol{Y}} \le C\Big(\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|^2_{\boldsymbol{V}}$$
$$+\|\xi_1(t)-\xi_2(t)\|^2_W+\int_0^t\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s))\|^2_{\boldsymbol{V}}\,ds+\|\varsigma_1(t)-\varsigma_2(t))\|^2_{L_0}+$$
$$\int_0^t\|\varsigma_1(s)-\varsigma_2(s))\|^2_{L_0}\,ds+\|\xi_1(t)-\xi_2(t)\|^2_W+\|\zeta_1(t)-\zeta_2(t)\|^2_{L^2(\Gamma_3)}$$
$$+\int_0^t\|\boldsymbol{k}_1(s)-\boldsymbol{k}_2(s))\|^2_{\boldsymbol{Y}}+\int_0^t\|\tau_1(s)-\tau_2(s))\|^2_{L_0}ds\Big).$$

Moreover, from (5.24), (5.28) and H(6) we obtain

$$\|\Pi^3(\eta_1,\mu_1,\theta_1,\pi_1)(t)-\Pi^3(\eta_2,\mu_2,\theta_2,\pi_2)(t)\|^2_{L_0} \le C\Big(\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|^2_{\boldsymbol{V}}$$
$$+\|\xi_1(t)-\xi_2(t)\|^2_W+\int_0^t\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s))\|^2_{\boldsymbol{V}}\,ds+\|\varsigma_1(t)-\varsigma_2(t))\|^2_{L_0}+$$
$$\int_0^t\|\varsigma_1(s)-\varsigma_2(s))\|^2_{L_0}\,ds+\|\xi_1(t)-\xi_2(t)\|^2_W+\|\zeta_1(t)-\zeta_2(t)\|^2_{L^2(\Gamma_3)}$$
$$+\int_0^t\|\boldsymbol{k}_1(s)-\boldsymbol{k}_2(s))\|^2_{\boldsymbol{Y}}+\int_0^t\|\tau_1(s)-\tau_2(s))\|^2_{L_0}ds\Big).$$

Similarly, using (5.29) and H(7), we obtain the following estimate for Π^4

$$\|\Pi^4(\eta_1,\mu_1,\theta_1,\pi_1)(t)-\Pi^4(\eta_2,\mu_2,\theta_2,\pi_2)(t)\|^2_{L_0} \le C\Big(\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|^2_{\boldsymbol{V}}$$
$$+\|\xi_1(t)-\xi_2(t)\|^2_W+\int_0^t\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s))\|^2_{\boldsymbol{V}}\,ds+\|\varsigma_1(t)-\varsigma_2(t))\|^2_{L_0}+$$
$$\int_0^t\|\varsigma_1(s)-\varsigma_2(s))\|^2_{L_0}\,ds+\|\xi_1(t)-\xi_2(t)\|^2_W+\|\zeta_1(t)-\zeta_2(t)\|^2_{L^2(\Gamma_3)}$$
$$+\int_0^t\|\boldsymbol{k}_1(s)-\boldsymbol{k}_2(s))\|^2_{\boldsymbol{Y}}+\int_0^t\|\tau_1(s)-\tau_2(s))\|^2_{L_0}ds\Big).$$

Consequently,

$$\|\Pi(\eta_1,\mu_1,\theta_1,\pi_1)(t)-\Pi(\eta_2,\mu_2,\theta_2,\pi_2)(t)\|^2_{\boldsymbol{V}'\times\boldsymbol{Y}\times L_0\times L_0} \le C\Big(\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|^2_{\boldsymbol{V}}$$
$$+\int_0^t\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s))\|^2_{\boldsymbol{V}}ds+\|\varsigma_1(t)-\varsigma_2(t))\|^2_{L_0}+\int_0^t\|\varsigma_1(s)-\varsigma_2(s))\|^2_{L_0}ds$$
$$+\|\xi_1(t)-\xi_2(t)\|^2_W+\int_0^t\|\tau_1(s)-\tau_2(s))\|^2_{Y}ds+\int_0^t\|\boldsymbol{k}_1(s)-\boldsymbol{k}_2(s))\|^2_{\boldsymbol{Y}}ds$$
$$+\|\zeta_1(t)-\zeta_2(t)\|^2_{L^2(\Gamma_3)}+\|\tau_1(t)-\tau_2(t))\|^2_{L_0}+\|\boldsymbol{k}_1(t)-\boldsymbol{k}_2(t))\|^2_{\boldsymbol{Y}}\Big). \tag{5.30}$$

Moreover, from (5.1) we obtain

$$(\dot{\boldsymbol{v}}_1-\dot{\boldsymbol{v}}_2,\boldsymbol{v}_1-\boldsymbol{v}_2)_{\boldsymbol{V}'\times\boldsymbol{V}}+\sum_{\alpha=1}^{2}(\mathcal{A}^\alpha\varepsilon(\boldsymbol{v}_1^\alpha)-\mathcal{A}^\alpha\varepsilon(\boldsymbol{v}_2^\alpha),\varepsilon(\boldsymbol{v}_1^\alpha-\boldsymbol{v}_2^\alpha))_{\mathcal{H}^\alpha}=$$
$$-(\eta_1-\eta_2,\boldsymbol{v}_1-\boldsymbol{v}_2)_{\boldsymbol{V}'\times\boldsymbol{V}}.$$

We integrate this equality with respect to time, use the initial conditions $\boldsymbol{v}_1(0)=\boldsymbol{v}_2(0)=\boldsymbol{v}_0$ and condition H(1)(c) to find

$$m\int_0^t\|\boldsymbol{v}_1(s)-\boldsymbol{v}_2(s))\|_{\boldsymbol{V}}^2ds\leq-\int_0^t(\eta_1(s)-\eta_2(s),\boldsymbol{v}_1(s)-\boldsymbol{v}_2(s))_{\boldsymbol{V}'\times\boldsymbol{V}}ds$$

where $m=\min(m_{\mathcal{A}^1},m_{\mathcal{A}^2})$. Then, using $2ab\leq\frac{a^2}{\delta}+\delta b^2$ we obtain

$$\int_0^t\|\boldsymbol{v}_1(s)-\boldsymbol{v}_2(s))\|_{\boldsymbol{V}}^2\,ds\leq C\int_0^t\|\eta_1(s)-\eta_2(s)\|_{\boldsymbol{V}'}^2ds. \tag{5.31}$$

On the other hand, from the Cauchy problem (5.17)–(5.18) we can write

$$\zeta_i(t)=\zeta_0-\int_0^t H_{ad}\big(\zeta_i(s),R_\nu(u_{i\nu}^1(s)+u_{i\nu}^2(s)),\boldsymbol{R}_\tau(\boldsymbol{u}_{i\tau}^1(s)-\boldsymbol{u}_{i\tau}^2(s))\big)ds$$

and then

$$\big\|\zeta_1(t)-\zeta_2(t)\big\|_{L^2(\Gamma_3)}\leq C\int_0^t\big\|\zeta_1(s)-\zeta_2(s)\big\|_{L^2(\Gamma_3)}ds$$
$$+C\int_0^t\big\|R_\nu(u_{1\nu}^1(s)+u_{1\nu}^2(s))-R_\nu(u_{2\nu}^1(s)+u_{2\nu}^2(s))\big\|_{L^2(\Gamma_3)}ds$$
$$+C\int_0^t\big\|\boldsymbol{R}_\tau(\boldsymbol{u}_{1\tau}^1(s)-\boldsymbol{u}_{1\tau}^2(s))-\boldsymbol{R}_\tau(\boldsymbol{u}_{2\tau}^1(s)-\boldsymbol{u}_{2\tau}^2(s))\big\|_{L^2(\Gamma_3)}ds.$$

Using the definition of R_ν and $\boldsymbol{R}_\tau$ and writing $\zeta_1=\zeta_1-\zeta_2+\zeta_2$, we get

$$\big\|\zeta_1(t)-\zeta_2(t)\big\|_{L^2(\Gamma_3)}\leq C\int_0^t\Big(\|\zeta_1(s)-\zeta_2(s)\|_{L^2(\Gamma_3)}+\big\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\big\|_{L^2(\Gamma_3)^d}\Big)ds.$$

Next, we apply Gronwall's inequality and use (2.1) to conclude that

$$\|\zeta_1(t)-\zeta_2(t)\|_{L^2(\Gamma_3)}\leq C\int_0^t\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|_V ds. \tag{5.32}$$

The definition (5.21) yields

$$\|\boldsymbol{k}_1(t)-\boldsymbol{k}_2(t)\|_Y^2\leq C\Big(\int_0^t\|\mu_1(s)-\mu_2(s)\|_Y^2ds\Big). \tag{5.33}$$

On the other hand, from (5.16) we deduce that

$$(\dot{\varsigma}_1 - \dot{\varsigma}_2, \varsigma_1 - \varsigma_2)_{L_0} + a(\varsigma_1 - \varsigma_2, \varsigma_1 - \varsigma_2) \leq \big(\theta_1 - \theta_2, \varsigma_1 - \varsigma_2\big)_{L_0}).$$

Integrating the previous inequality with respect to time, using the initial conditions $\varsigma_1(0) = \varsigma_2(0) = \varsigma_0$ and inequality $a(\varsigma_1 - \varsigma_2, \varsigma_1 - \varsigma_2) \geq 0$, to find

$$\frac{1}{2}\|\varsigma_1(t) - \varsigma_2(t)\|^2_{L_0} \leq \int_0^t \big(\theta_1(s) - \theta_2(s), \varsigma_1(s) - \varsigma_2(s)\big)_{L_0} ds,$$

which implies that

$$\|\varsigma_1(t) - \varsigma_2(t)\|^2_{L_0} \leq \int_0^t \|\theta_1(s) - \theta_2(s)\|^2_{L_0} ds + \int_0^t \|\varsigma_1(s) - \varsigma_2(s)\|^2_{L_0} ds.$$

This inequality, combined with Gronwall's inequality, leads to

$$\|\varsigma_1(t) - \varsigma_2(t)\|^2_{L_0} \leq C \int_0^t \|\theta_1(s) - \theta_2(s)\|^2_{L_0} ds \quad \text{a.e. } t \in (0, T). \tag{5.34}$$

From (5.19) we deduce that

$$(\dot{\tau}_1 - \dot{\tau}_2, \tau_1 - \tau_2)_{L_0} + a_0(\tau_1 - \tau_2, \tau_1 - \tau_2) = \big(\pi_1 - \pi_2, \tau_1 - \tau_2\big)_{L_0}.$$

We integrate this equality with respect to time, using the initial conditions $\tau_1(0) = \tau_2(0) = \tau_0$ and inequality $a_0(\tau_1 - \tau_2, \tau_1 - \tau_2) \geq 0$, to find

$$\frac{1}{2}\|\tau_1(t) - \tau_2(t)\|^2_{L_0} \leq \int_0^t \|\pi_1(s) - \pi_2(s)\|_{L_0}.\|\tau_1(s) - \tau_2(s)\|_{L_0} ds$$

which implies that

$$\|\tau_1(t) - \tau_2(t)\|^2_{L_0} \leq \int_0^t \|\pi_1(s) - \pi_2(s)\|^2_{L_0} ds + \int_0^t \|\tau_1(s) - \tau_2(s)\|^2_{L_0} ds.$$

This inequality combined with Gronwall's inequality leads to

$$\|\tau_1(t) - \tau_2(t)\|^2_{L_0} \leq C \int_0^t \|\pi_1(s) - \pi_2(s)\|^2_{L_0} ds \quad \text{a.e. } t \in (0, T). \tag{5.35}$$

Since $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ have the same initial value we get

$$\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|^2_{\boldsymbol{V}} \leq \int_0^t \|\boldsymbol{v}_1(s) - \boldsymbol{v}_2(s))\|^2_{\boldsymbol{V}} ds. \tag{5.36}$$

We substitute (5.31)–(5.36) in (5.30) to obtain

$$\big\|\Pi(\eta_1, \mu_1, \theta_1, \pi_1)(t) - \Pi(\eta_2, \mu_2, \theta_2, \pi_2)(t)\big\|^2_{\boldsymbol{V}' \times \boldsymbol{Y} \times L_0 \times L_0} \leq$$
$$C \int_0^t \big\|(\eta_1, \mu_1, \theta_1, \pi_1)(s) - (\eta_2, \mu_2, \theta_2, \pi_2)(s)\big\|^2_{\boldsymbol{V}' \times \boldsymbol{Y} \times L_0 \times L_0} ds.$$

Reiterating this inequality n times we obtain

$$\left\|\Pi^n(\eta_1,\mu_1,\theta_1,\pi_1)-\Pi^n(\eta_2,\mu_2,\theta_2,\pi_2)\right\|^2_{L^2(0,T;\boldsymbol{V}'\times\boldsymbol{Y}\times L_0\times L_0)}\leq \frac{C^nT^n}{n!}\left\|(\eta_1,\mu_1,\theta_1,\pi_1)-(\eta_2,\mu_2,\theta_2,\pi_2)\right\|^2_{L^2(0,T;\boldsymbol{V}'\times\boldsymbol{Y}\times L_0\times L_0)}.$$

Thus, for n sufficiently large, Π^n is a contraction on the Banach space $L^2(0,T;\boldsymbol{V}'\times\boldsymbol{Y}\times L_0\times L_0)$, and so Π has a unique fixed point. □

Now, we have all the ingredients to prove Theorem 2.

Proof. Existence. Let $(\eta^*,\mu^*,\theta^*,\pi^*)\in L^2(0,T;\boldsymbol{V}'\times\boldsymbol{Y}\times L_0\times L_0)$ be the fixed point of Π defined by (5.25)–(5.29) and denote

$$\boldsymbol{u}_*=\boldsymbol{u}_{\eta^*},\quad \xi_*=\xi_{\eta^*},\quad \varsigma_*=\varsigma_{\theta^*},\quad \zeta_*=\zeta_{\eta^*},\quad \boldsymbol{k}_*=\boldsymbol{k}_{\mu^*}\quad \tau_*=\tau_{\pi^*}, \tag{5.37}$$

$$\boldsymbol{\sigma}^\alpha_*=\mathcal{A}^\alpha\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\alpha_{\eta^*})+(\mathcal{E}^\alpha)^*\nabla\xi^\alpha_{\eta^*}+\boldsymbol{\sigma}^\alpha_{\eta^*\mu^*\theta^*\pi^*},\quad \alpha=1,2. \tag{5.38}$$

We prove $\{\boldsymbol{u}_*,\boldsymbol{\sigma}_*,\xi_*,\varsigma_*,\zeta_*,\boldsymbol{k}_*,\tau_*\}$ satisfies (4.9)–(4.16) and the regularites (4.17)–(4.24). Indeed, we write (5.1) for $\eta=\eta^*$ and use (5.37) to find

$$(\ddot{u}_*(t),v)_{\boldsymbol{V}'\times\boldsymbol{V}}+\sum_{\alpha=1}^{2}(\mathcal{A}^\alpha\varepsilon(\dot{\boldsymbol{u}}^\alpha_*(t)),\ \varepsilon(\boldsymbol{v}^\alpha))_{\mathcal{H}^\alpha}+(\eta^*(t),v)_{\boldsymbol{V}'\times\boldsymbol{V}} =(\mathbf{f}(t),\boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}}\quad\forall\boldsymbol{v}\in\boldsymbol{V}, a.e.\ t\in(0,T). \tag{5.39}$$

Equation $\Pi^1(\eta^*,\mu^*,\theta^*,\pi^*)=\eta^*$ combined with (5.26) and (5.38) show that

$$\begin{aligned}(\eta^*(t),v)_{\boldsymbol{V}'\times\boldsymbol{V}}&=\sum_{\alpha=1}^{2}\left(\mathcal{B}^\alpha\varepsilon(\boldsymbol{u}^\alpha_*(t)),\ \varepsilon(\boldsymbol{v}^\alpha)\right)_{\mathcal{H}^\alpha}+\sum_{\alpha=1}^{2}\left((\mathcal{E}^\alpha)^*\nabla\xi^\alpha_*,\ \varepsilon(\boldsymbol{v}^\alpha)\right)_{\mathcal{H}^\alpha}\\&+\sum_{\alpha=1}^{2}\left(\int_0^t\mathcal{G}^\alpha\big(\boldsymbol{\sigma}^\alpha_*-\mathcal{A}^\alpha\varepsilon(\dot{\boldsymbol{u}}^\alpha_*)-(\mathcal{E}^\alpha)^*\nabla\xi^\alpha_*,\varepsilon(\boldsymbol{u}^\alpha_*),\varsigma^\alpha_*,\boldsymbol{k}^\alpha_*,\tau^\alpha_*\big)(s)ds\ ,\ \varepsilon(\boldsymbol{v}^\alpha)\right)_{\mathcal{H}^\alpha}\\&+j_{ad}(\zeta_*(t),\boldsymbol{u}_*(t),\boldsymbol{v})+j_{\nu c}(\boldsymbol{u}_*(t),\boldsymbol{v}),\quad\forall\boldsymbol{v}\in\boldsymbol{V}.\end{aligned} \tag{5.40}$$

We substitute (5.40) in (5.39) and use (4.9) to see that (4.11) is satisfied. From $\Pi^2(\eta^*,\mu^*,\ \theta^*,\pi^*)=\mu^*$ and use (5.21) we see that (4.10) is satisfied. We write now (5.2) and (5.17) for $\boldsymbol{\eta}=\boldsymbol{\eta}^*$ and use (5.37) to find (4.14) and (4.15). The equalities $\Pi^3(\eta^*,\mu^*,\theta^*,\pi^*)=\theta^*$ and $\Pi^4(\eta^*,\mu^*,\theta^*,\pi^*)=\pi^*$, combined with (5.16), (5.19) show that (4.12)–(4.13) are satisfied. Next, (4.16) and the regularity (4.17), (4.19)–(4.23) follow from Lemmas 3, 4, 5, 6 and the relation (5.21). The regularity $\sigma_*\in L^2(0,T;\mathcal{H})$ follows from Lemma 7, assumptions H(1), H(8) and (5.38). Finally, (4.11) implies that

$$\rho^\alpha\ddot{\boldsymbol{u}}^\alpha_*=\mathrm{Div}\,\boldsymbol{\sigma}^\alpha_*+\boldsymbol{f}^\alpha_0\quad\text{a.e. } t\in[0,T],\ \alpha=1,2$$

and from H(12)(a), H(12)(b) and (4.17) we find that $(\text{Div}\,\boldsymbol{\sigma}_*^1, \text{Div}\,\boldsymbol{\sigma}_*^2) \in L^2(0,T;\boldsymbol{V}')$. We deduce that the regularity (4.18) holds.

Uniqueness. The uniqueness part of Theorem 2 is a consequence of the uniqueness of the fixed point of the operator Π defined by (5.25)-(5.29) and the unique solvability of the Problems $\text{PV}_\eta^{u\xi}$, PV_η^{ζ}, $\text{PV}_\theta^{\varsigma}$, PV_π^{τ} and $\text{PV}_{\eta\mu\theta\pi}^{\sigma}$. □

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Laid Maiza
Laboratory of Applied Mathematics, Department of Mathematics
University Kasdi Merbah,30000 Ouargla, Algeria.
e-mail: maiza.laid@univ-ouargla.dz

Tedjani Hadj Ammar
Departement of Mathematics, University of El-Oued,
39000 El-Oued, Algeria.
e-mail: $hadjammar\_tedjani@univ-eloued.dz$

Mohamed Laid Gossa
Department of Mathematics, University of Khenechla,
Khenechla, Algeria.
e-mail: gossa.med.laid@gmail.com

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