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THE SĂLĂGEAN-TYPE PROBABILITY DISTRIBUTION

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Abstract. The purpose of the present paper is to investigate the Sălăgean - type probability distribution of order α. In this paper, we obtain moments, factorial moments and moment generating function for this distribution. Finally, we obtain a entropy of this distribution. Our results improve and generalize the results of Porwal [5].

1 Introduction

The probability distribution is an interesting topic of study in statistics. However, it is a recent topic of study in Geometric Function Theory. A probability distribution is defined as a rule which distribute distinct values of the random variables with specified probabilities. If a discrete random variable X takes values x_i $(i = 1, 2, 3, \cdots)$, with probabilities p_i $(i = 1, 2, 3, \cdots)$. Then p_i is called the probability mass function if it satisfies the following conditions

1. $p_i \geq 0$
2. $\sum\limits_{i=1}^{\infty} p_i = 1.$

Recently, Porwal [5] introduced starlike and convex - type probability distribution and obtained results regarding moments, factorial moments, mean, variance and moment generating functions. In this paper we generalize and improve the results of [5].

Let $\mathcal{A}$ represent the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

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which are analytic in the open unit disk $\mathbb{U} = \{z \ : \ z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$ and satisfy the normalization condition $f(0) = f'(0) - 1 = 0$. Further, we represent by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions f of the form (1.1) which are also univalent in $\mathbb{U}$. In 1983, Sălăgean [7] introduced the Sălăgean derivative operator $\mathcal{D}^\kappa$ for functions f of the form (1.1) as

$$\begin{aligned} &\mathcal{D}^0 f(z) = f(z); \qquad \mathcal{D}\ f(z) = zf'(z); \qquad \cdots \qquad \text{and} \\ &\mathcal{D}^\kappa f(z) = \mathcal{D}^{\kappa-1}\left(\mathcal{D}f(z)\right) \qquad \kappa \in \mathbb{N} = \{1,\ 2,\ 3,\ \cdots\}. \end{aligned} \tag{1.2}$$

Also, we note that

$$\mathcal{D}^\kappa f(z) := z + \sum_{n=2}^{\infty} n^\kappa a_n z^n, \qquad \kappa \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0,\ 1,\ 2,\ 3,\ \cdots\}. \tag{1.3}$$

By using this operator Sălăgean [7] introduced a new subclass $\mathcal{S}(\kappa, \alpha)$ of functions of the form (1.1) satisfying the analytic criteria

$$\Re\left(\frac{\mathcal{D}^{\kappa+1} f(z)}{\mathcal{D}^\kappa f(z)}\right) > \alpha, \qquad (\kappa \in \mathbb{N}_0, \quad 0 \le \alpha < 1; \quad z \in \mathbb{U}). \tag{1.4}$$

Also, Owa et al. [4] and Kadioğlu [3] studied this class to discuss the further properties. It is easy to see that for $\kappa = 0$ we have the well known class of starlike functions denoted by $\mathcal{S}^*(\alpha)$ and defined by

$$\mathcal{S}^*(\alpha) := \left\{f \ : \ \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad 0 \le \alpha < 1; \quad z \in \mathbb{U}\right\}$$

and for $\kappa = 1$ it is well known class of convex functions denoted by $\mathcal{K}(\alpha)$ and defined by

$$\mathcal{K}(\alpha) := \left\{f \ : \ \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad 0 \le \alpha < 1; \quad z \in \mathbb{U}\right\}.$$

The classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ were introduced and studied by Robertson [6]. For the detailed study of these classes one may refer the works of Robertson [6], Silverman [8] and others [1, 3, 4, 7] for more details.

To introduce the Sălăgean-type probability distribution we require the following lemma:

Lemma 1. *[3] If $f(z)$ of the form (1.1) and satisfying the inequality*

$$\sum_{n=2}^{\infty} n^\kappa (n-\alpha)\,|a_n| \le 1-\alpha, \tag{1.5}$$

then $f \in \mathcal{S}(\kappa,\ \alpha)$.

Now, we introduce Sălăgean-type probability distribution associated with the function $f(z)$ of the form (1.1) with condition that $a_n \geq 0$ and satisfy the condition (1.5). This condition can be re-written in the following form

$$\sum_{n=2}^{\infty} \frac{n^{\kappa}(n-\alpha)}{1-\alpha} |a_n| = 1-\epsilon, \qquad 0 \leq \epsilon < 1. \tag{1.6}$$

Now we define the probability mass function of this distribution as

$$p(n) := \begin{cases} 0 & \text{if} \quad n=0 \\ \epsilon & \text{if} \quad n=1 \\ \dfrac{n^{\kappa}(n-\alpha)}{1-\alpha} |a_n| & \text{if} \quad n \geq 2. \end{cases} \tag{1.7}$$

Since $p(n) \geq 0$ and $\sum\limits_{n=0}^{\infty} p(n) = 1$. Hence $p(n)$ is a probability mass function.

By specializing the parameters in the above distribution, we have

1. For $\epsilon = k = 0$, then (1.7) reduces to starlike distribution of order α [5].
2. For $\epsilon = 0$, $k = 1$, then (1.7) reduces to convex distribution of order α [5].
3. For $\epsilon = k = \alpha = 0$, then (1.7) reduces to starlike distribution [5].
4. For $\epsilon = \alpha = 0$, $k = 1$, then (1.7) reduces to convex distribution of order α [5].

We consider the following definitions from the book of Fisz [2] to discuss our main results.

Definition 2. *If X is a discrete random variable which can take the values x_1, x_2, x_3, $\cdots$ with respective probabilities p_1, p_2, p_3, $\cdots$ then expectation of X, denoted by $E(X)$, is defined as*

$$E(X) = \sum_{n=1}^{\infty} p_n x_n. \tag{1.8}$$

Definition 3. *The r^{th} moment of a discrete probability distribution about $X = 0$ is defined by*

$$\mu_r^{'} = E(X^r).$$

Definition 4. *The mean of the distribution is given by*

$$Mean = \mu_1^{'}. \tag{1.9}$$

Definition 5. *The variance of the distribution is given by*

$$Variance = \mu_2^{'} - \left(\mu_1^{'}\right)^2. \tag{1.10}$$

Definition 6. *The r^{th} factorial moment of the discrete probability distribution is defined as*

$$\mu'_{(r)} = \sum_{n=0}^{\infty} n(n-1)\cdots(n-r+1)p(n) = \sum_{n=0}^{\infty} n^{(r)}p(n).$$

Definition 7. *The moment generating function (m.g.f.) of a random variable X is denoted by $M_X(t)$ and defined by*

$$M_X(t) = E(e^{tX}) = \sum_{n=1}^{\infty} p_n e^{tn}. \tag{1.11}$$

In the present paper we obtain the results of moments, factorial moments and moment generating function we improve and generalize the results of Porwal [5]. We also give a nice application of this distribution in information theory.

2 Main Results

In our first theorem, we obtain the r^{th} moment of the Sălăgean-type probability distribution about the origin.

Theorem 8. *The r^{th} moment of the Sălăgean-type probability distribution is defined by the relation*

$$\mu'_r = \frac{\mathcal{D}^{r+\kappa+1}f(1) - \alpha\mathcal{D}^{r+\kappa}f(1)}{1-\alpha} - 1 + \epsilon. \tag{2.1}$$

Proof. By Definition 3 of the r^{th} moment about origin we have

$$\begin{aligned}
\mu'_r &= \sum_{n=0}^{\infty} p_n n^r \\
&= \epsilon + \sum_{n=2}^{\infty} n^r \frac{n^\kappa(n-\alpha)}{1-\alpha} a_n \\
&= \epsilon + \sum_{n=2}^{\infty} \frac{n^{\kappa+r+1}a_n - \alpha n^{\kappa+r}a_n}{1-\alpha} \\
&= \epsilon + \frac{\mathcal{D}^{r+\kappa+1}f(1) - \alpha\mathcal{D}^{r+\kappa+1}f(1) - 1 + \alpha}{1-\alpha} \\
&= \epsilon + \frac{\mathcal{D}^{r+\kappa+1}f(1) - \alpha\mathcal{D}^{r+\kappa+1}f(1)}{1-\alpha} - 1.
\end{aligned}$$

$\square$

Example 9. *Let the function*

$$f(z) = z + \frac{(1-\epsilon)(1-\alpha)}{n^{\kappa}(n-\alpha)} z^n,$$

where $0 \leq \epsilon < 1$*. It is easy to see that* $f \in \mathcal{S}(\kappa, \alpha)$*.*

The r^{th} moment of the Sălăgean-type probability distribution associated with above function is $\mu'_r = \epsilon + (1-\epsilon)n^r$.

Remark 10. *For special values of parameters involved in the above result, we obtain the following cases studied by the first author of this work [5].*

1. *If we put* $r = 1, 2, 3, 4$, $\epsilon = 0$ *and* $k = 0$*, then we obtain the first four moments of starlike distribution of order* α*.*
2. *If we put* $r = 1, 2, 3, 4$, $\epsilon = 0$ *and* $k = 1$*, then we obtain the first four moments of convex distribution of order* α*.*
3. *If we put* $r = 1, 2, 3, 4$, $\epsilon = 0$, $k = 0$ *and* $\alpha = 0$ *then we obtain the first four moments of starlike distribution.*
4. *If we put* $r = 1, 2, 3, 4$, $\epsilon = 0$, $k = 1$ *and* $\alpha = 0$ *then we obtain the first four moments of convex distribution.*

Theorem 11. *The* r^{th} *factorial moment of the Sălăgean-type probability distribution is given by the relation*

$$\mu'_{(r)} = \begin{cases} \mu'_1 & \text{if} \quad r = 1 \\ \dfrac{d^r}{dz^r}\left(\dfrac{(\mathcal{D}^{\kappa+1} - \alpha\mathcal{D}^{\kappa})f(z)}{1-\alpha}\right)_{z=1} & \text{if} \quad r \geq 2. \end{cases} \tag{2.2}$$

Proof. By using the Definition 6, we have

$$\mu'_1 = \mu_1$$

and

$$\begin{aligned} \mu'_{(r)} &= \sum_{n=0}^{\infty} p_n n^{(r)} \\ &= \epsilon + \sum_{n=2}^{\infty} n^{(r)} \frac{n^{\kappa}(n-\alpha)}{1-\alpha} a_n \\ &= \frac{d^r}{dz^r}\left(\frac{(\mathcal{D}^{\kappa+1} - \alpha\mathcal{D}^{\kappa})f(z)}{1-\alpha}\right)_{z=1}. \end{aligned}$$

□

Example 12. *The* r^{th} *factorial moment of the Sălăgean-type probability distribution of order* α *associated with the following function*

$$f(z) = z + \frac{(1-\epsilon)(1-\alpha)}{n^{\kappa}(n-\alpha)} z^n, \qquad 0 \le \epsilon < 1$$

is $\mu_1' = \epsilon + (1-\epsilon)n$ *and*

$$\mu_r' = \begin{cases} (1-\epsilon)n(n-1)\cdots(n-r+1) & \text{if} \quad r \le n \\ 0 & \text{if} \quad r > n. \end{cases}$$

Theorem 13. *The moment generating function of the Sălăgean-type probability distribution of order* α *is given by the relation*

$$M_X(t) = \frac{\mathcal{D}^{\kappa+1} f(e^t) - \alpha \mathcal{D}^{\kappa} f(e^t)}{1-\alpha} - (1-\epsilon)e^t. \tag{2.3}$$

Proof. By using the Definition 7, we have

$$M_X(t) = E(e^{tX}) = \sum_{n=0}^{\infty} p(n) e^{tn}$$

and

$$\begin{aligned} M_X(t) &= \epsilon e^t + \sum_{n=2}^{\infty} e^{tn} \frac{n^{\kappa}(n-\alpha)}{1-\alpha} a_n \\ &= \epsilon e^t + \frac{\mathcal{D}^{\kappa+1} f(e^t) - \alpha \mathcal{D}^{\kappa} f(e^t)}{1-\alpha} - e^t \\ &= \frac{1}{1-\alpha} \left[\mathcal{D}^{\kappa+1} f(e^t) - \alpha \mathcal{D}^{\kappa} f(e^t)\right] - (1-\epsilon)e^t. \end{aligned}$$

□

Example 14. *The moment generating function for the Sălăgean-type probability distribution of order* α *associated with the following function*

$$f(z) = z + \frac{(1-\epsilon)(1-\alpha)}{n^{\kappa}(n-\alpha)} z^n, \qquad 0 \le \epsilon < 1$$

is

$$\begin{aligned} M_X(t) &= \frac{1}{1-\alpha} \left[e^t + (1-\epsilon)(1-\alpha)e^{nt} - \alpha e^t\right] - (1-\epsilon)e^t \\ &= (1-\epsilon)e^{nt} + e^t - (1-\epsilon)e^t \\ &= (1-\epsilon)e^{nt} + \epsilon e^t. \end{aligned}$$

3 Application in Information Theory

The information theory is a recent topic of study mathematics. The modern information theory can be classified into the following three major branches.

1. **Shannon Theory** : CE Shannon deals with mathematical models for communication problems. The concept of entropy given by Shannon in his mathematical model.

2. **Cybernetics** : Norbert Wiener, deals with the communication problems encountered in living being begins and social organization.

3. **Coding Theory** : A recently developed subject dealing with the theory of error correcting codes.

For detail study of this one may refer to excellent text book by Swarup et al. [9].

Definition 15. *Let there be a number of events E_i, $i \in \mathbb{N}$ with probabilities p_i, $i \in \mathbb{N}$ is defined to be mathematical expectation of information $h(p_i)$, concerning the occurrence of one of the events E_i. In other words, the entropy function is given by*

$$H(p_1,\, p_2,\, \cdots) = \sum_{i=1}^{\infty} p_i h(p_i) = -\sum_{i=1}^{\infty} p_i \log p_i, \qquad p_i \geq 0, \qquad \sum_{i=1}^{\infty} p_i = 1.$$

Theorem 16. *The entropy of the Sălăgean-type probability distribution of order α is given by the relation*

$$H = -\epsilon \log \epsilon - \frac{1}{1-\alpha}\left\{\mathcal{D}^{\kappa+1} f(1) - \alpha \mathcal{D}^{\kappa} f(1) - 1 + \alpha\right\} \log \left\{\frac{\mathcal{D}^{\kappa+1} f(1) - \alpha \mathcal{D}^{\kappa} f(1) - 1 + \alpha}{1-\alpha}\right\}.$$

Proof. The entropy of the Sălăgean-type probability distribution of order α is defined by the relation

$$\begin{aligned} H &= -\epsilon \log \epsilon - \sum_{n=2}^{\infty} \frac{n^{\kappa}(n-\alpha)}{1-\alpha} a_n \log \left(\frac{n^{\kappa}(n-\alpha)}{1-\alpha} a_n\right) \\ &= -\epsilon \log \epsilon - \frac{1}{1-\alpha}\left\{\mathcal{D}^{\kappa+1} f(1) - \alpha \mathcal{D}^{\kappa} f(1) - 1 + \alpha\right\} \log \left\{\frac{\mathcal{D}^{\kappa+1} f(1) - \alpha \mathcal{D}^{\kappa} f(1) - 1 + \alpha}{1-\alpha}\right\}. \end{aligned}$$

□

If we let $\epsilon = 0$, $\alpha = 0$ and $k = 0$ in Theorem 16, we have the following corollary:

Corollary 17. *The entropy of starlike distribution is given by*

$$H = -(f'(1) - 1) \log \left(f'(1) - 1\right).$$

Example 18. *The entropy of starlike distribution associated with the function*

$$f(z) = z + \frac{z^2}{4} + \frac{z^3}{6}$$

is given by $H = \log 2 = 1$.

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