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NUMERICAL SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS USING DAUBECHIES FILTER WITH ACCURACY ORDER SIX

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Abstract. This article considers the representation of differentiation operators using Daubechies wavelets to solve PDEs numerically. Derivative approximation using this compactly supported wavelets convert the action into a matrix multiplication. The vanishing moments, dilation equation and quadrature formulas play a significant role in the scheme. The computed solution of the PDEs seems to behave better than the results reported in the literature, and we could progress the solution up to a large time-bound. We experimented with Daubechies wavelet filters with six vanishing moments on test problems and summarised the results.

1 Introduction

In the 1800s, an idea to analyse functions with sines and cosines emerged when Jean Baptiste Fourier tried solving the most important problem, "How does heat diffuse in a continuous medium?". Understanding a complicated structure will be easier if simple components can analyse. In 1808 and 1812, Fourier faced rejection for his manuscript titled "Every periodic function can be expressed as a weighted sum of sines and cosines" by the committee consisting of Joseph-Louis Lagrange Pierre-Simon Laplace, Gaspard Monge and Sylvestre Francois Lacroix. His peers felt that the work was based on intuition and particularly failed to explain the convergence of the series rigorously. After Fourier became the secretary of "The French Academy of Sciences", he published this work in 1822 (see [7]). The convergence of the series was a serious issue, even though this remarkable work was highly influential in science and technology. In 1826, Cauchy published proof of the convergence of the Fourier series. G. Lejeune Dirichlet (a 23 years old boy) pointed out a flaw in Cauchy's proof and gave sufficient conditions for the convergence (Dirichlet pointwise convergence

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theorem [32]). Camille Jordan settled the answer to Dirichlet's concerns in 1881 (see [27]). In continuation, many researchers put forward different sufficient conditions for the convergence of the Fourier series (for details, see [6, 16, 50, 56, 57, 58]).

For various practical applications, discrete Fourier transforms are computationally inefficient. So to enhance the computational speed, the Fast Fourier transform was introduced, which became popular among scientists and engineers. Also, due to its inability to handle non-stationary signals, Gabor proposed Short Time Fourier Transform (STFT) in 1946 (see [18, 19]). His idea was to use the Gaussian function as a time-localised window to analyse each signal segment. The literature has made several attempts with this algorithm by replacing the window function. If the signal contains high-frequency contents for a short period and low-frequency contents for a longer period, it won't be easy to analyse the entire signal with a single-window function. To resolve this issue, J. Morlet suggested to use multiple window functions to analyse different frequency components present in a signal. These window functions, called wavelets of the same shape, are dilated copies of the Gaussian function. To make this idea rigorous, he worked with A. Grossman and derived an exact inversion formula for the Morlet integral transform (see [21]). Y. Meyer recognised this inversion formula is similar to the formula from A. Calderon in the context of Harmonic analysis. In addition, he identified the redundant nature of wavelets, and he struck at this point when trying to generate an orthonormal system. Finally, he discovered an orthogonal wavelet basis with better localisation properties both in time and frequency. In this sense, Meyer successfully overcame the uncertainty principle reasonably well. Readers are advised to look at [39] for more details. But surprisingly, this work coincided with J. O. Stromberg, a Harmonic analyst who found the same basics five years earlier than Meyer. Even more amazing is that German mathematician Alfred Haar built an orthonormal wavelet basis (see [22]) in 1909 without mentioning the name "wavelet". Similar constructions of Haar-like bases were also discovered independently by Paul Levy, Paley and Littlewood. Concurrently, to discretise the time and scale parameters of the wavelet transform, Ingrid Daubechies a student of Grossman, laid the foundations of Modern Wavelet theory by introducing Daubechies wavelet and coined the idea of wavelet frames. This provided freedom in choosing a basis while retaining the same redundancy rate. Later on, in 1986, Multiresolution Analysis was introduced by Y. Meyer and S. Mallat (see [25]). This remarkable discovery helped us to generate an orthonormal basis for the space of square-integrable functions that consists of compactly supported wavelets (see [12, 10]). In the last 32 years, many more wavelet families have been introduced, such as Coiflet wavelets, Block spline semi-orthogonal wavelets, Battle-Lemarie wavelets, Shannon wavelets, Meyer wavelets and Biorthogonal wavelets of the Cohen family etc.

Generally speaking, decomposition in terms of wavelets is an exciting method for solving several mystifying problems in science and technology. This method is successfully applied in diverse fields such as data compression, wave propagation, signal processing, medical imaging technology, computer graphics, pattern recognition, the study of the distant universe, the detection of aircraft in the remote sky and submarines in the deep sea. In the last 20 years, several attempts have been reported, such as Curvelets, Complex wavelets, Composite wavelets, Shearlets, Grouplets, and Bandelets, to fulfil wavelet decompositions in higher dimensions. An interested reader can also refer the applications in time series analysis [47, 44, 45], signal processing [59, 36], image processing [24, 49, 38], statistics [2, 1, 60, 48] and some others in [31, 52, 63, 30, 29, 34]. Some outstanding review works that include mathematical aspects of wavelets is found in [9, 54, 55, 11, 10, 61, 62, 41].

2 Wavelet base for $L^2(\mathbb{R})$

It is well known that Fourier transform of the absolutely integrable functions $f \in L^1(\mathbb{R})$ is defined as

$$\hat{f}(\omega) = \int_{\mathbb{R}} f(x)e^{-i\omega x}dx \qquad \omega \in \mathbb{R}, \tag{2.1}$$

and $f \mapsto \hat{f}$ is a uniformly continuous mapping that decays to zero as $\omega \to \pm\infty$. The definition in Eq.(2.1) is not a valid one when we deal with the equivalence class of square-integrable functions $f \in L^2(\mathbb{R})$. Fourier transform for L^2- functions is defined as an isometric extension from $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ to $L^2(\mathbb{R})$ since $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is a dense subset of $L^2(\mathbb{R})$ (see [20]). For $\mathbf{f} \in L^2(\mathbb{R})$, there exist a sequence $\{f_n\}$ with $f_n, \widehat{f_n} \in L^1(\mathbb{R})$ for each n and $\|f_n - \mathbf{f}\| \to \mathbf{0}$ as $n \to \infty$ in the L^2- sense. Using Plancherel's relation $\|\widehat{f_n} - \widehat{\mathbf{f}}\| = \|f_n - \mathbf{f}\|$, we can extend the notion of Fourier transform to L^2- functions.

Consider a nested sequence $\{0\} \subset \cdots V_{-1} \subset V_0 \subset V_1 \subset \cdots \subset L^2(\mathbb{R})$ of subspaces and a function $\varphi \in L^2(\mathbb{R})$. Also, the integer translates of φ form an orthonormal set, closure of its span is termed as V_0. We can change the resolution of the space V_0 using the dilation criteria. That is, for $f \in V_0, f(2^j x) \in V_j$ for each $j \in \mathbb{Z}$. More rigorously, $V_j = \overline{\text{span}\{\varphi_{j,k}(x) := 2^{j/2}\varphi(2^j x - k) : k \in \mathbb{Z}\}}$. These closed subspaces V_j's also satisfies $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$ and $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$. We denote the orthogonal complement of V_j in V_{j+1} as W_j, hence we can write $V_{j+1} = V_j \bigoplus W_j$. The denseness property of the nested subspaces ensures the representation $L^2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}} W_j$ and since $\varphi \in V_0 \subset V_1$, we have

$$\varphi(x) = \sqrt{2}\sum_{k\in\mathbb{Z}} h(k)\varphi(2x - k), \tag{2.2}$$

for some $h := \{h(k)\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ is termed as the dilation equation. Also, the corresponding wavelet equation can be obtained by defining the scalar sequence $g := \{g(k)\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ as $g(k) = (-1)^k\overline{h(1-k)}$ and

$$\psi(x) = \sqrt{2}\sum_{k\in\mathbb{Z}} g(k)\varphi(2x - k). \tag{2.3}$$

For a fixed j, the set $\{\psi_{j,k} : k \in \mathbb{Z}\}$ forms an orthonormal basis of W_j (see [35, 40]). More specifically, $L^2(\mathbb{R})$ has decomposed into an infinite sequence of distinct closed subspaces W_j's. The set $\{\psi_{j,k} : j, k \in \mathbb{Z}\}$ form an orthonormal basis of $L^2(\mathbb{R})$. Thus a function $f \in L^2(\mathbb{R})$ can be represented as

$$f = \sum_{j,k\in\mathbb{Z}} c_{j,k}\psi_{j,k}, \quad \text{where} \quad c_{j,k} = \langle f, \psi_{j,k}\rangle \text{ called wavelet coefficients.}$$

The two scalar sequences h and g defined in Eq.(2.2) and Eq.(2.3) are said to be the filters associated with the scaling function φ and the wavelet ψ respectively. Instead of working with the functions φ and ψ explicitly, the multiresolution analysis described by φ and ψ can be constructed using an appropriate filter h. The following theorem explains this fact precisely.

Theorem 1. *[13] Let $\{h(k)\}_{k\in\mathbb{Z}}$ be a sequence such that*

1. $\sum_{k\in\mathbb{Z}} |h(k)|\ |k|^\epsilon < \infty$ *for some* $\epsilon > 0$,
2. $\sum_{k\in\mathbb{Z}} h(k-2m)h(k-2n) = \delta_{m,n}$,
3. $\sum_{k\in\mathbb{Z}} h(k) = \sqrt{2}$.

Moreover, suppose that $m_0(\omega) = \frac{1}{\sqrt{2}} \sum_{k\in\mathbb{Z}} h(k)\ e^{-ik\omega}$ *can be written as*

$$m_0(\omega) = \left[\frac{1}{2}\left(1 + e^{-i\omega}\right)\right]^N \left[\sum_{k\in\mathbb{Z}} f(k) e^{-ik\omega}\right],$$

where $\sum_{k\in\mathbb{Z}} |f(k)|\ |k|^\epsilon < \infty$ *for* $\epsilon > 0$, *and* $\sup_{\omega\in\mathbb{R}} \left|\sum_{k\in\mathbb{Z}} f(k)\ e^{-ik\omega}\right| < 2^{N-1}$. *Define* $\hat{\varphi}(\omega) = \prod_{k=1}^{\infty} m_0\left(\frac{\omega}{2^k}\right)$ *and for the reflected sequence* $g(k) = (-1)^k h(1-k)$, $\{\psi_{j,k} : j, k \in \mathbb{Z}\}$ *forms a wavelet basis corresponding to the MRA induced by* φ, *where* $\psi(x)$ *will be as in Eq.(2.3).*

The condition (3) of the above theorem ensures the that $\hat{\varphi}(0) = \int_{\mathbb{R}} \varphi(x)dx = 1$. Applying Fourier transform on both sides of Eq.(2.2), we will get

$$\hat{\varphi}(\omega) = m_0\left(\frac{\omega}{2}\right) \hat{\varphi}\left(\frac{\omega}{2}\right), \tag{2.4}$$

where the quantity $m_0(\omega)$ defined as in Theorem 1, would satisfy

$$|m_0(\omega)|^2 + |m_0(\omega + \pi)|^2 = 1.$$

Eventually,

$$\hat{\varphi}(\omega) = m_0\left(\frac{\omega}{2}\right) m_0\left(\frac{\omega}{2^2}\right) \cdots m_0\left(\frac{\omega}{2^n}\right) \hat{\varphi}\left(\frac{\omega}{2^n}\right).$$

By the continuity of φ at 0 (see [17]), as $n \to \infty$ $\hat{\varphi}(\omega)$ can be represented as the following infinite product,

$$\hat{\varphi}(\omega) = \prod_{k=1}^{\infty} m_0\left(\frac{\omega}{2^k}\right).$$

Moreover, from Eq.(2.4) $m_0(0) = 1$, this is essentially the condition for the convergence of the infinite product. In a similar way, the wavelet equation in Eq.(2.3) can be expressed as

$$\hat{\psi}(\omega) = m_1\left(\frac{\omega}{2}\right) \prod_{k=2}^{\infty} m_0\left(\frac{\omega}{2^k}\right),$$

where $m_1(\omega) = \frac{1}{\sqrt{2}} \sum_{k\in\mathbb{Z}} g(k)\, e^{-ik\omega}$. The orthonormality condition(2) of Theorem 1 will give $m_1(0) = 0$ and $\hat{\psi}(0) = 0$. This means that the wavelet has zero mean. The closed linear span of the translates of this wavelet ψ for a fixed dilation j will be W_j and this will have the decomposition property $L^2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}} W_j$. The above process is known as Multiresolution analysis (MRA) [33, 37]. If this wavelet is orthogonal to polynomials $1, x, x^2 \ldots, x^{M-1}$, then the above process can express any polynomial of degree up to $M-1$ efficiently. In other words, ψ is orthogonal to the $\mathbb{P}_{M-1}-$the space of all polynomials of degree less than or equal to $M-1$. i.e.,

$$\int_{\mathbb{R}} x^m \psi(x) dx = 0, \text{ for } m = 0, 1, 2, \ldots M-1 \qquad (M \text{ vanishing moments for } \psi).$$

If the scaling function and the corresponding wavelet have compact support, we will have a finite number of nonzero filter coefficients in Eqs.(2.2), (2.3). This will change the dilation and wavelet equations as

$$\begin{aligned} \varphi(x) &= \sqrt{2} \sum_{k=0}^{L-1} h(k)\varphi(2x-k), \\ \psi(x) &= \sqrt{2} \sum_{k=0}^{L-1} g(k)\varphi(2x-k), \end{aligned} \tag{2.5}$$

with $g(k) = (-1)^k h(L-k-1)$, where L is the number of nonzero filter coefficients. To acquire better time-frequency localization and computational efficiency, we expect the underlying wavelet must have compact support. To construct a filter h with $2M$ nonzero filter coefficients that are capable to produce the polynomial space $\mathbb{P}_{M-1}$, we need to find the solution to the following system of equations

1. Sum of entries of the scaling sequence is $\sqrt{2}$:

$$\sum_{k=0}^{L-1} h(k) = \sqrt{2}. \tag{2.6}$$

2. Orthogonality of the scaling sequence:

$$\sum_{k=0}^{L-1-2l} h(k)h(k+2l) = \delta_{0,l}. \tag{2.7}$$

3. The polynomial reproduction property:

$$\sum_{k=0}^{L-1} (-1)^k k^m h(k) = 0 \text{ for } m = 0, 1, 2, \dots M-1. \tag{2.8}$$

For example, when $M = 2$ the system will become

$$\begin{aligned}
&h(0) + h(1) + h(2) + h(3) &&= \sqrt{2} && \text{(from Eq.(2.6))}\\
&h^2(0) + h^2(1) + h^2(2) + h^2(3) &&= 1 && \\
&h(0)h(2) + h(1)h(3) &&= 0 && \text{(from Eq.(2.7))}\\
&h(0) - h(1) + h(2) - h(3) &&= 0 && \\
&0 \cdot h(0) - 1 \cdot h(1) + 2 \cdot h(2) - 3 \cdot h(3) &&= 0. && \text{(from Eq.(2.8))}
\end{aligned}$$

Solving this system will give the filter coefficients as

$$h(0) = \frac{1+\sqrt{3}}{4\sqrt{2}}, \quad h(1) = \frac{3+\sqrt{3}}{4\sqrt{2}}, \quad h(2) = \frac{3-\sqrt{3}}{4\sqrt{2}}, \quad h(3) = \frac{1-\sqrt{3}}{4\sqrt{2}}.$$

Similarly, when $M = 3$ we have

$$\begin{aligned}
&h(0) + h(1) + h(2) + h(3) + h(4) + h(5) &&= \sqrt{2} && \text{(from Eq.(2.6))}\\
&h^2(0) + h^2(1) + h^2(2) + h^2(3) + h^2(4) + h^2(5) &&= 1 && \\
&h(0)h(2) + h(1)h(3) + h(2)h(4) + h(3)h(5) &&= 0 && \text{(from Eq.(2.7))}\\
&h(0)h(4) + h(1)h(5) &&= 0 && \\
&h(0) - h(1) + h(2) - h(3) + h(4) - h(5) &&= 0 && \\
&0 \cdot h(0) - 1 \cdot h(1) + 2 \cdot h(2) - 3 \cdot h(3) + 4 \cdot h(4) - 5 \cdot h(5) &&= 0 && \text{(from Eq.(2.8))}\\
&0^2 \cdot h(0) - 1^2 \cdot h(1) + 2^2 \cdot h(2) - 3^2 \cdot h(3) + 4^2 \cdot h(4) - 5^2 \cdot h(5) &&= 0 &&
\end{aligned}$$

and the solution to this system is

$$\begin{aligned}
h(0) &= 0.33267055295008261, & h(1) &= 0.80689150931109257,\\
h(2) &= 0.45987750211849157, & h(3) &= -0.13501102001025458,\\
h(4) &= -0.08544127388202666, & h(5) &= 0.03522629188570953.
\end{aligned}$$

These filters are known as Daubechies filters with four non-zero entries and six non-zero entries respectively. Using the iterative method named as Cascade algorithm derived from (1), we construct the corresponding scaling function and the wavelet. If we begin with the characteristic function on the interval $[0, L-1]$ as the initial function $\varphi^{(0)}(x)$, then next iterations can be calculated using the following formula

$$\varphi^{(k+1)}(x) = \sqrt{2}\sum_{l=0}^{L-1} h(l)\varphi^{(k)}(2x-l).$$

Also the corresponding wavelet will be obtained by the recursive application of the following equation

$$\psi^{(k+1)}(x) = \sqrt{2}\sum_{l=0}^{L-1} g(l)\varphi^{(k)}(2x-l).$$

The progress of this algorithm for a few iterations is depicted in Figure 1, this will converge to the scaling function and the wavelet.

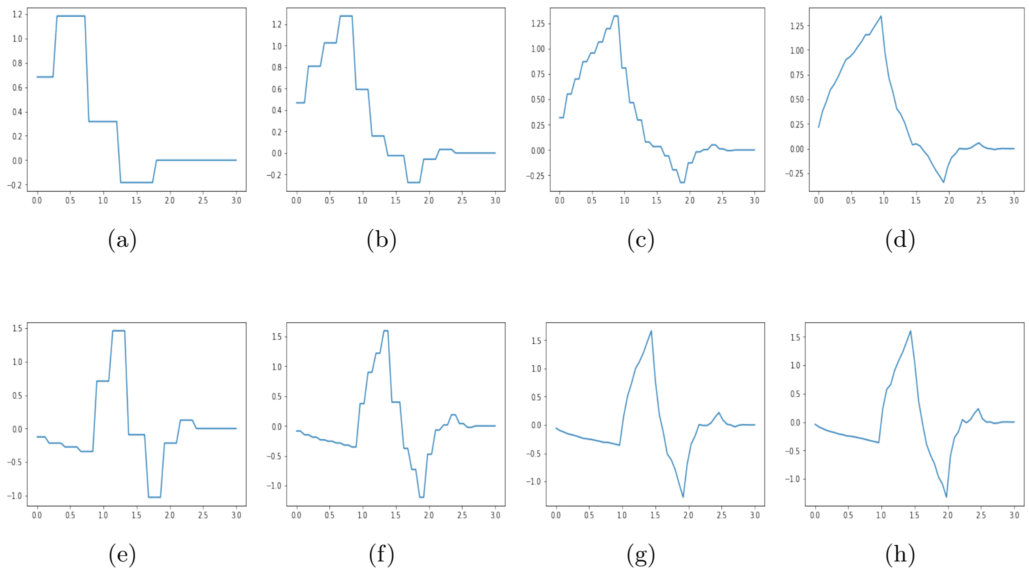


Figure 1: Iterates of the scaling filter $h(k)$ [(a) to (d)] and the wavelet filter $g(k)$ [(e) to (h)] with 2 vanishing moments.

3 Differentiation operator using compactly supported wavelets

The key interest is to approximate the derivative of a square-integrable function using Daubechies wavelets. To find the derivative of an $L^2(\mathbb{R})-$ function f, we consider its $J^{th}-$ level wavelet approximation. The derivative of this may not be in $L^2(\mathbb{R})$, so we again project the same to V_J. The projection operator on to the space V_J is defined as

$$P_{V_J} f(x) = \sum_{k\in\mathbb{Z}} s_{J,k}\ \varphi_{J,k}(x), \quad \text{where} \quad s_{J,k} = \langle f, \varphi_{J,k}\rangle. \tag{3.1}$$

The Differential operator $\mathcal{D}$ acting on either sides, we get

$$\mathcal{D}P_{V_J} f(x) = \sum_{k\in\mathbb{Z}} s_{J,k}\ \mathcal{D}\varphi_{J,k}(x).$$

Thus the notion of differentiation in this context may be considered as follows

$$P_{V_J}\mathcal{D}P_{V_J} f(x) = \sum_{l\in\mathbb{Z}}\langle \mathcal{D}P_{V_J} f,\ \varphi_{J,l}\rangle\ \varphi_{J,l}(x) = \sum_{l\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} s_{J,k}\langle \mathcal{D}\varphi_{J,k},\ \varphi_{J,l}\rangle\ \varphi_{J,l}(x). \tag{3.2}$$

Recall that $V_j = V_{j-1}\oplus W_{j-1}$ is one among the criteria of MRA, thus the projection of f onto V_J can be written as

$$P_{V_{J-1}\oplus W_{J-1}} = \sum_{k\in\mathbb{Z}} s_{J-1,k}\varphi_{J-1,k} + \sum_{k\in\mathbb{Z}} d_{J-1,k}\psi_{J-1,k}, \quad \text{where} \quad d_{J-1,k} = \langle f, \psi_{J-1,k}\rangle.$$

Hence,

$$\begin{aligned} P_{V_{J-1}\oplus W_{J-1}}\mathcal{D}P_{V_{j-1}\oplus W_{J-1}} = \quad & \sum_{l\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} s_{J-1,k}\langle \mathcal{D}\varphi_{J-1,k},\ \varphi_{J-1,l}\rangle\varphi_{J-1,l} \\ & + \sum_{l\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} s_{J-1,k}\langle \mathcal{D}\varphi_{J-1,k},\ \psi_{J-1,l}\rangle\psi_{J-1,l} \\ & + \sum_{l\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} d_{J-1,k}\langle \mathcal{D}\psi_{J-1,k},\ \varphi_{J-1,l}\rangle\varphi_{J-1,l} \\ & + \sum_{l\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} d_{J-1,k}\langle \mathcal{D}\psi_{J-1,k},\ \psi_{J-1,l}\rangle\psi_{J-1,l}. \end{aligned}$$

To simplify the expressions, we choose the following notations:

$$\begin{aligned} A_{J-1} &:= P_{W_{J-1}}\mathcal{D}\ P_{W_{J-1}}, \quad B_{J-1} := P_{W_{J-1}}\mathcal{D}\ P_{V_{J-1}}, \\ C_{J-1} &:= P_{V_{J-1}}\mathcal{D}\ P_{W_{J-1}}, \quad R_{J-1} := P_{V_{J-1}}\mathcal{D}\ P_{V_{J-1}}. \end{aligned}$$

The $(l,k)^{th}-$ entry of the matrix corresponds to the above operators are given below

$$\begin{aligned}
a_{l,k}^{J-1} &= \langle \mathcal{D}\psi_{J-1,k}, \psi_{J-1,l}\rangle = 2^{J-1}\int_{\mathbb{R}} \psi(2^{J-1}x-l)\,(\mathcal{D}\psi)\,(2^{J-1}x-k)2^{J-1}dx = 2^{J-1}a_{l-k},\\
b_{l,k}^{J-1} &= \langle \mathcal{D}\varphi_{J-1,k}, \psi_{J-1,l}\rangle = 2^{J-1}\int_{\mathbb{R}} \psi(2^{J-1}x-l)\,(\mathcal{D}\varphi)\,(2^{J-1}x-k)2^{J-1}dx = 2^{J-1}b_{l-k},\\
c_{l,k}^{J-1} &= \langle \mathcal{D}\psi_{J-1,k}, \varphi_{J-1,l}\rangle = 2^{J-1}\int_{\mathbb{R}} \varphi(2^{J-1}x-l)\,(\mathcal{D}\psi)\,(2^{J-1}x-k)2^{J-1}dx = 2^{J-1}c_{l-k},\\
r_{l,k}^{J-1} &= \langle \mathcal{D}\varphi_{J-1,k}, \varphi_{J-1,l}\rangle = 2^{J-1}\int_{\mathbb{R}} \varphi(2^{J-1}x-l)\,(\mathcal{D}\varphi)\,(2^{J-1}x-k)2^{J-1}dx = 2^{J-1}r_{l-k},
\end{aligned} \tag{3.3}$$

where $$a_k = \int_{\mathbb{R}} \psi(x-k)\ \ (\mathcal{D}\psi)\,(x)\ dx, \qquad b_k = \int_{\mathbb{R}} \psi(x-k)\ \ (\mathcal{D}\varphi)\,(x)\ dx,$$
$$c_k = \int_{\mathbb{R}} \varphi(x-k)\ \ (\mathcal{D}\psi)\,(x)\ dx, \qquad r_k = \int_{\mathbb{R}} \varphi(x-k)\ \ (\mathcal{D}\varphi)\,(x)\ dx.$$

The dilation equation seen in Eq.(2.5) will give us,

$$\begin{aligned}
a_k &= \int_{\mathbb{R}} \psi(x-k)\ \ (\mathcal{D}\psi)\,(x)\ dx\\
&= 2\sum_{l=0}^{L-1}\sum_{m=0}^{L-1} g(l)\ g(m)\int_{\mathbb{R}} \varphi(2x-2k-l)\ \varphi(2x-m)\,2\,dx\\
&= 2\sum_{l=0}^{L-1}\sum_{m=0}^{L-1} g(l)\ g(m)\ \ r_{2k+l-m}.
\end{aligned}$$

Similarly

$$b_k = 2\sum_{l=0}^{L-1}\sum_{m=0}^{L-1} g(l)\ h(m)\ \ r_{2k+l-m},$$
$$c_k = 2\sum_{l=0}^{L-1}\sum_{m=0}^{L-1} h(l)\ g(m)\ \ r_{2k+l-m},$$
$$r_k = 2\sum_{l=0}^{L-1}\sum_{m=0}^{L-1} h(l)\ h(m)\ \ r_{2k+l-m}.$$

The above set of equations indicates a significant role of the coefficients r_k in the determination of differential operator on the approximation space V_J and the wavelet space W_J. These actions can solely be determined by computing those coefficients r_k in the space V_0. The computation of the coefficients r_k can be characterized from the following results.

Lemma 2. *For the symbol of the scaling sequence* $m_0(\omega) = \dfrac{1}{\sqrt{2}} \sum_{k=0}^{L-1} h(k)\ e^{-ik\omega}$, *and the autocorrelation coefficients defined as*

$$\alpha_k := 2 \sum_{l=0}^{L-1-k} h(l)\ h(l+k) \tag{3.4}$$

has the following properties.

1. $\alpha_{2k} = 0$ *for* $k = 0, 1, 2, \ldots \frac{L}{2} - 1$.

2. $|m_0(\omega)|^2 = \dfrac{1}{2} + \dfrac{1}{2} \sum_{k=1}^{L/2} \alpha_{2k-1} \cos\left((2k-1)\omega\right).$

The orthonormality conditions given in Eq.(2.7), is equivalent to $\alpha_{2k} = 0$ for $k = 0, 1, 2, \ldots \frac{L}{2} - 1$. For the second assertion, consider

$$\begin{aligned}
|m_0(\omega)|^2 &= m_0(\omega) \cdot \overline{m_0(\omega)} \\
&= \frac{1}{2} \left(\sum_{k=0}^{L-1} h(k)\ e^{-ik\omega} \right) \left(\sum_{l=0}^{L-1} h(l)\ e^{il\omega} \right) \\
&= \frac{1}{2} \left(\sum_{k=0}^{L-1} h(k)\ [\cos(k\omega) - i \sin(k\omega)] \right) \left(\sum_{l=0}^{L-1} h(l)\ [\cos(l\omega) + i \sin(l\omega)] \right) \\
&= \frac{1}{2} \left[\sum_{k=0}^{L-1} h(k)^2 + 2 \sum_{k=1}^{L-1} \sum_{l=0}^{L-1-k} h(l)\ h(l+k)\ (\cos(k\omega) - i \sin(k\omega)) \right].
\end{aligned}$$

LHS of the above relation is real and thus by Eq.(2.7), and the definition of α_k we have

$$|m_0(\omega)|^2 = \frac{1}{2} + \frac{1}{2} \sum_{k=1}^{L/2} \alpha_{2k-1} \cos\left((2k-1)\omega\right).$$

Theorem 3. *[8] The coefficients* r_k *satisfy the following linear system of algebraic equations*

$$r_k = 2 \left[r_{2k} + \frac{1}{2} \sum_{l=1}^{L/2} \alpha_{2l-1} \left(r_{2k-2l+1} + r_{2k+2l-1} \right) \right] \quad \textit{and} \quad \sum_{k \in \mathbb{Z}} k\ r_k = -1,$$

provided the integral

$$r_k = \int_{\mathbb{R}} \varphi(x-k)\ (\mathcal{D}\varphi)(x)\ dx \tag{3.5}$$

or equivalently the integral in Fourier domain

$$r_k = \int_{\mathbb{R}} (-i\omega)|\hat{\varphi}(\omega)|^2 e^{-ik\omega} d\omega. \tag{3.6}$$

exists. Moreover, if the corresponding wavelet ψ *has* $M \geq 2$ *vanishing moments, then the above system has a unique solution with a finite number of nonzero* r_k*'s, i.e.,* $r_k \neq 0$ *for* $-L+2 \leq k \leq L-2$ *such that* $r_k = -r_{-k}$*.*

Proof.

$$\text{Claim I}: \quad r_k = 2\left[r_{2k} + \frac{1}{2}\sum_{l=1}^{L/2} \alpha_{2l-1}\left(r_{2k-2l+1} + r_{2k+2l-1}\right)\right] \tag{3.7}$$

Using Eq.(2.5) we have

$$\begin{aligned}
r_k &= \int_{\mathbb{R}} \varphi(x-k)\ \left(\mathcal{D}\varphi\right)(x)\ dx \\
&= 2\sum_{l=0}^{L-1}\sum_{m=0}^{L-1} h(l)\ h(m) \int_{\mathbb{R}} \varphi(2x-2k-l)\ \left(\mathcal{D}\varphi\right)(2x-m)\, 2\, dx \\
&= 2\sum_{l=0}^{L-1}\sum_{m=0}^{L-1} h(l)\ h(m)\ r_{2k+l-m} \\
&= 2\sum_{l=0}^{L-1}\sum_{m=l}^{l-L+1} h(l)\ h(l-m)\ r_{2k+m},\ \text{(by change of variable)} \\
&= 2\ r_{2k} + \sum_{l=1}^{L-1} \alpha_l\ \left(r_{2k-l} + r_{2k+l}\right),\ \left(\text{by } \sum_{k=0}^{L-1} h(k)^2 = 1\right) \\
&= 2\left[r_{2k} + \frac{1}{2}\sum_{l=1}^{L/2} \alpha_{2l-1}\left(r_{2k-2l+1} + r_{2k+2l-1}\right)\right].
\end{aligned}$$

The existence of the solution to the above system follows from the existence of the integral (3.5) or (3.6). Since φ has compact support, it is possible to write $r_k \neq 0$ for $-L+2 \leq k \leq L-2$ and from the result

$$\sum_{k=-\infty}^{+\infty} k^m \varphi(x-k) = x^m + \sum_{k=1}^{m} (-1)^k \binom{m}{k} M_k^{\varphi}\ x^{m-k}, \tag{3.8}$$

where $M_k^{\varphi} = \int_{\mathbb{R}} x^k \varphi(x)\ dx$ for $k = 1, 2, \ldots m$, called moments of the scaling function φ (see [8]). Combine Eq.(3.8) and Eq.(3.5) for $m = 1$ to obtain $\sum_{k\in\mathbb{Z}} k\ r_k = -1$. Rewriting Eq.(3.6) will give $r_k = -r_{-k}$.

$$\text{Claim II}: \quad \sum_{k\in\mathbb{Z}} r_k = 0.$$

Multiplying both sides of Eq.(3.7) by $e^{ik\omega}$ and summing over all $k \in \mathbb{Z}$, we have

$$\hat{r}(\omega) = 2\left[\hat{r}_e\left(\frac{\omega}{2}\right) + \frac{1}{2}\,\hat{r}_o e\left(\frac{\omega}{2}\right)\sum_{l=1}^{L/2}\alpha_{2l-1}\,\left(e^{-i(2l-1)\omega/2} + e^{i(2l-1)\omega/2}\right)\right]$$

where

$$\hat{r}(\omega) = \sum_{k\in\mathbb{Z}} r_k\, e^{ik\omega}, \quad \hat{r}_e(\omega) = \sum_{k\in\mathbb{Z}} r_{2k}\, e^{i2k\omega}, \quad \hat{r}_o(\omega) = \sum_{k\in\mathbb{Z}} r_{2k+1}\, e^{i(2k+1)\omega}.$$

Notice that,

$$2\hat{r}_e = \hat{r}(\omega) + \hat{r}(\omega+\pi), \quad 2\hat{r}_o = \hat{r}(\omega) - \hat{r}(\omega+\pi).$$

Thus,

$$\begin{aligned}\hat{r}(\omega) &= 2\hat{r}_e\left(\frac{\omega}{2}\right) + \; 2\hat{r}_o\left(\frac{\omega}{2}\right)\sum_{l=1}^{L/2}\alpha_{2l-1}\,\cos\left((2l-1)\omega\right)\\ &= \hat{r}\left(\frac{\omega}{2}\right) + \hat{r}\left(\frac{\omega}{2}+\pi\right) + \left(\hat{r}\left(\frac{\omega}{2}\right) - \hat{r}\left(\frac{\omega}{2}+\pi\right)\right)\left(|2m_0(\omega)|^2 - 1\right)\\ &= 2\left[\hat{r}\left(\frac{\omega}{2}\right)\,|m_0(\omega)|^2 + \hat{r}\left(\frac{\omega}{2}+\pi\right)\,|m_0(\omega+\pi)|^2\right].\end{aligned}$$

Choose $\omega = 0$, and see $\hat{r}(0) = 0$. This establishes our claim.

Claim III: Uniqueness of r_k.

Uniqueness of these coefficients r_k is immediate from the uniqueness of the following representation

$$(\mathcal{D}f)(x) = \sum_{k\in\mathbb{Z}}\langle \mathcal{D}f,\ \varphi_{0,k}\rangle\varphi_{0,k}(x) + \sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}}\langle \mathcal{D}f,\ \psi_{j,k}\rangle\psi_{j,k}(x),$$

which is written from the fact $V_j \bigoplus W_j = V_{j+1}$. Let $\Gamma_J := P_{V_J}\mathcal{D}P_{V_J}$ be the representation of the differentiation operator on the approximation space V_J. As $J \to \infty$, Γ_J and $\mathcal{D}$ act in the same fashion for smooth functions $f \in L^2(\mathbb{R})$. To see this we use Eq.(3.2) and Eq.(3.3)

$$\begin{aligned}(\Gamma_J f)(x) &= \sum_{l\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} 2^J\, s_{J,k}\, r_{l-k}\, \varphi_{J,l}(x)\\ &= 2^J \sum_{l\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} r_m\, s_{J,l-m}\, \varphi_{J,l}(x),\end{aligned}$$

where

$$
\begin{aligned}
s_{J,l-m} &= 2^{J/2}\int_{\mathbb{R}} f(x)\ \varphi(2^J - l + m)\ dx = 2^{J/2}\int_{\mathbb{R}} f(x-2^{-J}m)\ \varphi(2^J - l)\ dx \\
&= \int_{\mathbb{R}} f(x-2^{-J}m)\ \varphi_{J,l}\ dx, \qquad \left(\text{expand } f(x-2^{-J}m) \text{ in Taylor series}\right) \\
&= \int_{\mathbb{R}} \left(f(x) - 2^{-J}m\ \left(\mathcal{D}f\right)(x) + \frac{(2^{-J}m)^2}{2}\ \left(\mathcal{D}^2 f\right)(\tilde{x}) f(x) \right)\ \varphi_{j,l}\ dx \\
&= \int_{\mathbb{R}} f(x)\varphi_{J,l}\ dx - 2^{-J}m\int_{\mathbb{R}} \left(\mathcal{D}f\right)(x)\varphi_{J,l}\ dx + \frac{2^{-2J}}{2}m^2\int_{\mathbb{R}} \left(\mathcal{D}^2 f\right)(\tilde{x})\ \varphi_{J,l}\ dx.
\end{aligned}
$$

Thus we have,

$$
\begin{aligned}
(\Gamma_J f)(x) &= 2^J\sum_{l\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} r_m \left(\int_{\mathbb{R}} f(x)\ \varphi_{J,l}\ dx\right)\varphi_{J,l} \\
&\qquad - \sum_{l\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} m r_m \left(\int_{\mathbb{R}} \left(\mathcal{D}f\right)(x)\ \varphi_{J,l}\ dx\right)\varphi_{J,l} \\
&\qquad + 2^{-J}\sum_{l\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} \frac{1}{2}m^2 r_m \left(\int_{\mathbb{R}} \left(\mathcal{D}^2 f\right)(\tilde{x})\ \varphi_{J,l}\ dx\right)\varphi_{J,l} \\
&= \sum_{l\in\mathbb{Z}} \left(\int_{\mathbb{R}} \left(\mathcal{D}f\right)(x)\ \varphi_{J,l}\ dx\right)\varphi_{J,l} \\
&\qquad + 2^{-(J+1)}\sum_{l\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} m^2 r_m \left(\int_{\mathbb{R}} \left(\mathcal{D}^2 f\right)(\tilde{x})\ \varphi_{J,l}\ dx\right)\varphi_{J,l} \\
&= \sum_{k\in\mathbb{Z}} \langle \mathcal{D}f,\ \varphi_{0,k}\rangle \varphi_{0,k}(x) + \sum_{j=0}^{J-1}\sum_{k\in\mathbb{Z}} \langle \mathcal{D}f,\ \psi j, k\rangle \psi j, k(x) \\
&\qquad + 2^{-(J+1)}\sum_{l\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} m^2 r_m \left(\int_{\mathbb{R}} \left(\mathcal{D}^2 f\right)(\tilde{x})\ \varphi_{J,l}\ dx\right)\varphi_{J,l}.
\end{aligned}
$$

As $J\to\infty$,

$$
\begin{aligned}
\lim_{J\to\infty} (\Gamma_J f)(x) &= \sum_{k\in\mathbb{Z}} \langle \mathcal{D}f,\ \varphi_{0,k}\rangle \varphi_{0,k}(x) + \sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}} \langle \mathcal{D}f,\ \psi j, k\rangle \psi j, k(x) \\
&= \mathcal{D}f(x).
\end{aligned}
$$

Hence it is possible to write $\Gamma_\infty = \mathcal{D}$. Thus the uniqueness of the solution has been established. □

Repeat the procedure recursively to represent higher-order derivatives $\mathcal{D}^n$. This wavelet representation solely depends on the expansion in the approximation space

V_0, i.e., depends on the coefficients $r_k^{(n)}$. Analogously, we have the representation of higher-order derivatives as

$$P_{V_J}\mathcal{D}^n P_{V_J} f(x) = \sum_{l\in\mathbb{Z}} \langle \mathcal{D}^n P_{V_J} f,\ \varphi_{J,l}\rangle\ \varphi_{J,l}(x) = \sum_{l\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} s_{J,k}\langle \mathcal{D}^n\varphi_{J,k},\ \varphi_{J,l}\rangle\ \varphi_{J,l}(x),$$

where $\langle \mathcal{D}^n\varphi_{J,k},\ \varphi_{J,l}\rangle = \left(2^J\right)^n r_k^{(n)}$ with

$$r_k^{(n)} = \int_{\mathbb{R}} \varphi(x-k)\ \frac{d^n}{dx^n}\ \varphi(x)\ dx. \tag{3.9}$$

Equivalently,

$$\int_{\mathbb{R}} (-i\omega)^n |\hat{\varphi}(\omega)|^2 e^{-ik\omega}\ d\omega. \tag{3.10}$$

These coefficients can be characterized as follows:

Theorem 4. *[8] If the integral (3.9) or (3.10) exists, then the coefficients $r_k^{(n)}$ satisfy the following linear system of algebraic equations*

$$r_k^{(n)} = 2^n\left[r_{2k}^{(n)} + \frac{1}{2}\sum_{l=1}^{L/2}\alpha_{2l-1}\left(r_{2k-2l+1}^{(n)} + r_{2k+2l-1}^{(n)}\right)\right]$$
$$\text{and}\quad \sum_{k\in\mathbb{Z}} k\ r_k^{(n)} = (-1)^n\ n!,$$

where α_k defined as in Eq.(3.4). If ψ has $M \geq (n+1)/2$ vanishing moments, then the above system has a unique solution with a finite number of coefficients r_k are nonzero, i.e., $r_k^{(n)} \neq 0$ for $-L+2 \leq k \leq L-2$ such that for n even,

$$r_k^{(n)} = r_k^{(n)},$$
$$\sum_{k\in\mathbb{Z}} k^{2m} r_k^{(n)} = 0, \quad \text{for } m = 1, 2, \ldots n/2-1$$
$$\sum_{k\in\mathbb{Z}} r_k^{(n)} = 0$$

and for n odd,

$$r_k^{(n)} = -r_k^{(n)}$$
$$\sum_{k\in\mathbb{Z}} k^{2m-1} r_k^{(n)} = 0, \quad \text{for } m = 1, 2, \ldots (n-1)/2.$$

The advantage of this result is, the approximation of higher-order derivatives can be achieved in a single step. The proof is similar to Theorem 3.

4 Matrix form of the differentiation operator

In this section, we explain the matrix equivalent of the differentiation operator with the help of MRA in the limelight of the discussions in [8, 23]. The matrix representation of the differentiation operator with Daubechies wavelets has been established in [26]. Under the assumption of periodic boundary conditions, the representation is viewed in a matrix form. Let f be a compactly supported function on $[a, b]$ having periodic boundary behaviour. We uniformly discretize the interval into N subintervals with the node points $a = x_0 < x_1 < \cdots < x_N = b$. The function f is then treated as a vector of values $f(x_k)$ for $k = 0, 1, 2, \dots N$. This is chosen to be the initial level of resolution. To restrict the translation parameter k in the expansion Eq.(3.1), we consider f as a periodic function in the real line. Then on computation, the coefficients $s_{j,k}$ also will have periodic behaviour. Hence the projection of the compactly supported function f onto the initial space V_0 can be written in finite sum as

$$P_{V_0} f(x) = \sum_{k=0}^{N} s_{0,k}\ \varphi_{0,k}(x).$$

Thus by similar arguments as in the previous section, the derivative approximation will be of the form

$$P_{V_0} \frac{d}{dx} P_{V_0} f(x) = \sum_{l=0}^{N} \sum_{k=0}^{N} s_{0,k} \left(\frac{r_{l-k}}{h}\right)\ \varphi_{0,l}(x),$$

where $\frac{1}{h}$ denotes the number of subdivisions made within a unit interval. Even though we are in the initial level V_0, it is not the actual initial stage. We take refinement and set it as the initial stage for the easiness in expression. In order to indicate this refined stage, we should use this factor in the expansion. Corresponding to Daubechies wavelet, a differentiation matrix $\widetilde{\mathcal{D}}$ on the approximation space V_0 can be derived as in the form $\widetilde{\mathcal{D}} = C^{-1}\mathcal{D}C$. The action of this differentiation matrix $\widetilde{\mathcal{D}}$ on the vector f of function values will give the vector f' of function values of $f'(x)$. Here the matrix C is transforming the vector f to the estimate scaling function coefficients $s_{0,k}$ of $f(x)$ at the finest resolution level, D will take these coefficients to the scaling function coefficients $s'_{0,k}$ of $f'(x)$ and C^{-1} will carry these coefficients to its corresponding vector f' of function values.

Computation of the matrices C and $\mathcal{D}$:

The circulant matrix C constructed using the weight coefficients c_m for $m = 0, 1, \dots, M-1$ is obtained from the integral quadrature representation

$$s_{0,k} = \int_{\mathbb{R}} f(x)\ \varphi_{0,k}(x)\ \approxeq \sum_{m=0}^{M-1} c_m\ f(x_{m+k}). \tag{4.1}$$

Using the M vanishing moments of ψ, we have this representation as exact of order M. Thus, it is possible to write

$$\int_{\mathbb{R}} p(x)\ \varphi_{0,k}(x)\ dx = \sum_{m=0}^{M-1} c_m\ p(m+k) \quad \text{for all} \quad p(x) \in \mathbb{P}_{M-1}$$

$$\text{i.e.,} \quad \int_{\mathbb{R}} p(x+k)\ \varphi(x)\ dx = \sum_{m=0}^{M-1} c_m\ p(m+k) \quad \text{for all} \quad p(x) \in \mathbb{P}_{M-1}.$$

In short, we have

$$\int_{\mathbb{R}} x^k\ \varphi(x)\ dx = \sum_{m=0}^{M-1} m^k\ c_m \quad \text{for} \quad k = 0, 1, \ldots M-1. \tag{4.2}$$

This algebraic system of linear equations Eq.(4.2) can be solved for c_m. Using the periodicity of f, $f(x_{m+k}) = f(x_{m+k \pmod N})$ and thus the summation in Eq.(4.1) is valid. Hence it is possible to write in the matrix form as

$$\vec{s}_0 = C\vec{f},$$

where,

$$\vec{s}_0 = \begin{bmatrix} s_{0,0} \\ s_{0,1} \\ \cdots \\ \\ \cdots \\ \\ \cdots \\ \\ s_{0,N} \end{bmatrix}, \; C = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{M-1} & 0 & \cdots & & 0 \\ 0 & c_0 & c_1 & & \cdots & c_{M-1} & 0 & \cdots & 0 \\ 0 & 0 & c_0 & \cdots & & \cdots & c_{M-1} & \cdots & 0 \\ \vdots & & & \ddots & & & & & \vdots \\ 0 & & & 0 & c_0 & & \cdots & & c_{M-1} \\ \vdots & & & & & \ddots & & & \vdots \\ c_3 & \cdots & c_{M-1} & 0 & \cdots & 0 & c_0 & c_1 & c_2 \\ c_2 & & \cdots & c_{M-1} & 0 & \cdots & 0 & c_0 & c_1 \\ c_1 & & & \cdots & c_{M-1} & 0 & \cdots & 0 & c_0 \end{bmatrix}, \; \vec{f} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \cdots \\ \\ \cdots \\ \\ \cdots \\ \\ f(x_N) \end{bmatrix}.$$

In the case of Daubechies wavelets, the moments $M_k = \int_{\mathbb{R}} x^k\ \varphi(x)\ dx$ in Eq.(4.2) can be iteratively calculated (see [26]) using the following relation

$$M_k = \frac{1}{2^{k+1}-2} \sum_{m=0}^{M-1} \binom{k}{m} \mu_{k-m}\ M_m, \quad \text{where} \quad \mu_m = \sum_{l \in \mathbb{Z}} l^m h(m).$$

The matrix $\mathcal{D}$ will map from the scaling function coefficients of f to the scaling function coefficients of f' at the level $j = 0$. Here the role of $\mathcal{D}$ is carried out by the matrix $R_0 = \left[\frac{r_{k,l}}{h}\right]$ for $k, l = 0, 1, 2, \ldots, N$. Here the computation of these coefficients $r_{k,l}$ is described in the previous section. Since $V_0 = V_{-1} \oplus W_{-1}$, we can represent the derivative on V_0 as $P_{V_{-1}\oplus W_{-1}}\mathcal{D}P_{V_{-1}\oplus W_{-1}}$. In this case, $\mathcal{D}$ will be the block matrix $\begin{bmatrix} R_{-1} & C_{-1} \\ B_{-1} & A_{-1} \end{bmatrix}$, where $A_{-1}, B_{-1}, C_{-1}, R_{-1}$ are all square matrices of order $\frac{N+1}{2}$ each. The definition of R_0 itself guarantee its circulant nature. The finite difference accuracy of the coefficients r_k for a general compactly supported wavelet is illustrated in [8] and for coefficients r_k derived for Daubechies wavelet of M vanishing moments, it is proved that these coefficients can differentiate polynomials up to order $2M$ in [26]. Now let us see the derived r_k corresponding to Daubechies wavelets with various vanishing moments M.

Table 1: List of α_i's and r_i's for different values of M

M	α_i		r_i	
2	$\alpha_1 = \frac{9}{8}$	$\alpha_3 = -\frac{1}{8}$	$r_1 = -\frac{2}{3}$	$r_2 = \frac{1}{12}$
3	$\alpha_1 = \frac{75}{64}$	$\alpha_3 = -\frac{25}{128}$	$r_1 = -\frac{272}{365}$	$r_2 = \frac{53}{365}$
	$\alpha_5 = \frac{3}{128}$		$r_3 = -\frac{16}{1095}$	$r_4 = -\frac{1}{2920}$
4	$\alpha_1 = \frac{1225}{1024}$	$\alpha_3 = -\frac{245}{1024}$	$r_1 = -\frac{39296}{49553}$	$r_2 = \frac{76113}{396424}$
	$\alpha_5 = \frac{49}{1024}$	$\alpha_7 = -\frac{5}{1024}$	$r_3 = -\frac{1664}{49553}$	$r_4 = \frac{2645}{1189272}$
			$r_5 = \frac{128}{743295}$	$r_6 = -\frac{1}{1189272}$
5	$\alpha_1 = \frac{19845}{16384}$	$\alpha_3 = -\frac{2205}{8192}$	$r_1 = -\frac{957310976}{1159104017}$	$r_2 = \frac{265226398}{1159104017}$
	$\alpha_5 = \frac{567}{8192}$	$\alpha_7 = -\frac{405}{32768}$	$r_3 = -\frac{735232}{13780629}$	$r_4 = \frac{17297069}{2318208034}$
	$\alpha_9 = \frac{35}{32768}$		$r_5 = -\frac{1386496}{5795520085}$	$r_6 = -\frac{563818}{10431936153}$
			$r_7 = -\frac{2048}{8113728119}$	$r_8 = -\frac{5}{18545664272}$
6	$\alpha_1 = \frac{160083}{131072}$	$\alpha_3 = -\frac{38115}{131072}$	$r_1 = -\frac{3986930636128256}{4689752620280145}$	$r_2 = \frac{4850197389074509}{18759010481120580}$
	$\alpha_5 = \frac{22869}{262144}$	$\alpha_7 = -\frac{5445}{262144}$	$r_3 = -\frac{1019185340268544}{14069257860840435}$	$r_4 = \frac{136429697045009}{9379505240560290}$
	$\alpha_9 = \frac{847}{262144}$	$\alpha_{11} = -\frac{63}{262144}$	$r_5 = -\frac{7449960660992}{4689752620280145}$	$r_6 = \frac{483632604097}{11255406288672348O}$
			$r_7 = \frac{78962327552}{6565653668392203}$	$r_8 = \frac{31567002859}{75036041924482320}$
			$r_9 = -\frac{2719744}{937950524056029}$	$r_{10} = \frac{1743}{2501201397482744}$

5 Discretization of PDEs

5.1 Burgers' equation

The dissipative form of one-dimensional Burgers' equation (see [4]) is

$$\begin{cases} u_t + \alpha u u_x = \nu\, u_{xx}, \quad x \in \Omega := (a,b),\ t > t_0, \\ \text{with the initial condition} \quad u(x,t_0) = f(x), \\ \text{and boundary conditions} \quad u(a,t) = 0 = u(b,t). \end{cases} \tag{5.1}$$

Partitioning the spatial domain into uniform subintervals with node points namely $a = x_0 < x_1 < \ldots < x_N = b$. Here the mesh is considered to be $h = \frac{b-a}{N}$. If

u doesn't possess periodic boundary conditions, it is possible to take a periodic extension of u depending on the behaviour of the initial condition. Here we treat u as a vector $[u(x_0,t)\ \cdots\ u(x_N,t)]$ of function values at the node points $x_0,\ldots,x_N$. We apply the proposed differentiation operator on the periodically extended vector and pick up the derivative values at x_j's for $j=0,\ldots,N$. Let us denote the approximated n^{th} derivative as the vector $\tilde{\mathcal{D}}^n u$ where $\tilde{\mathcal{D}}^n$ is the differentiation matrix corresponding to the Daubechies wavelet. Here we have the spatial derivatives u_x and u_{xx}. Discretizing the Eq.(5.1) in the spatial domain and replacing the spatial derivatives in terms of the proposed differentiation operator, we get a system of ordinary differential equations as

$$\frac{du_i}{dt} = -\alpha u_i \sum_{k=0}^{N} \tilde{\mathcal{D}}^1_{ik} u_k + \nu \sum_{k=0}^{N} \tilde{\mathcal{D}}^2_{ik} u_k, \quad i = 0,1,\ldots,N. \tag{5.2}$$

We start the computation from the initial time t_0. Using SSPRK schemes described in [53] the system Eq.(5.2) can be solved for the solution vector u at time $t_0 + \Delta t$. By iterating the procedure we can progress the solution on the time domain $[t_0, T)$, where T denotes the required time level. Here we implemented the idea in some test examples and the numerical solution plotted when time progresses. The computations are done with $\alpha = 1$ and spatial discretization $N = 80$.

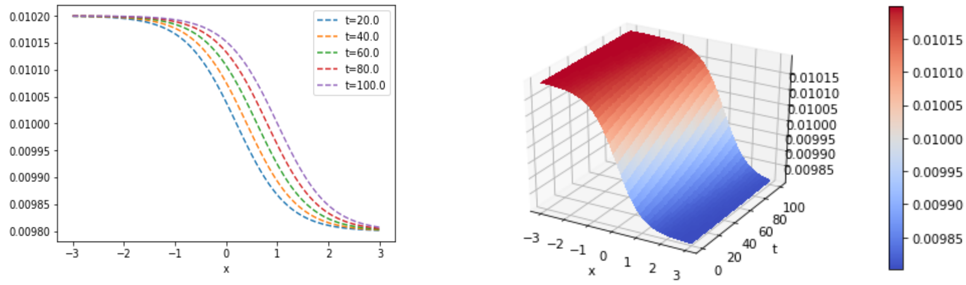
Example 5. Consider the one-dimensional Burgers' equation on the spatial domain $\Omega = (-3, 3)$. The exact solution is given by

$$u(x,t) = \frac{c}{\alpha} - \left[\frac{2\nu}{\alpha}\tanh(x - ct)\right], \quad t > 0.$$

Here we choose $c = 0.01$ for the computation purpose. The numerical solution is drawn as progress in time up to 100 s in Fig.2 for $\Delta t = 0.1$ and $\nu = 0.0001$. The numerical errors obtained from the proposed method have been compared with the works [28, 46, 15, 3, 4] in the literature for $\Delta t = 0.01$, $N = 80$ and $\alpha = 1$ (see Table 2). The numerical result shows that the present algorithm could produce more approximated solution to the problem as compared to many works in the literature. The L_2 error is also compared by taking Daubechies wavelets of different order with the works [28, 3, 4] and its clear that we could obtain better approximated solutions while increasing the smoothmess of the wavelet (see Table 3).

Example 6. Consider the equation Eq.(5.1) on the spatial domain $[0, 1]$ with the initial condition $f(x) = u(x, 0) = \frac{2\nu\pi\ \sin(\pi x)}{c+\cos(\pi x)}$ and zero boundary conditions. The exact solution found in [4] of the form

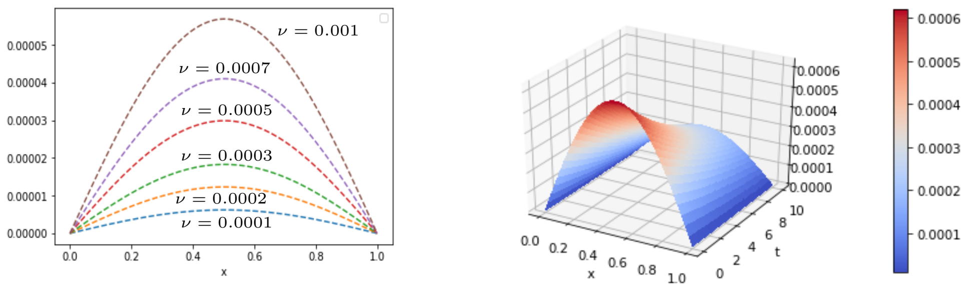
$$u(x,t) = \frac{2\nu\pi\, e^{-\pi^2\nu t}\ \sin(\pi x)}{c + e^{-\pi^2\nu t}\ \cos(\pi x)}, \quad c > 1. \tag{5.3}$$



(a) (b)

Figure 2: Example 5 - For $N = 80, \nu = 0.0001, c = 0.01, \alpha = 1, \Delta t = 0.1$ numerical solution drawn up to $t = 100$ s.

As time progresses, we observe that the computed solution behaves well as the exact solution even for small values of kinematic viscosity. The plot of the numerical solution up to 10 s is shown in Fig.3, here we take $c = 100$ with time step $\Delta t = 0.01$. The L_∞ errors are computed and compared for different spatial discretization and time in Table 4, where we choose $\Delta t = 0.001,\ \alpha = 1,\ c = 2$ and $\nu = 0.001$. The results are compared with the work[4] and it is clear from the comparison that the present method could better behave even for large time bounds. The comparison of L_2 error also tabulated in Table 5 by considering Daubechies wavelet of different order for a time bound $t = 1$ s.



(a) (b)

Figure 3: Example 6 - For $N = 80, \alpha = 1, c = 100, \Delta t = 0.01$ (a) numerical solution for different ν at 10 s (b) numerical solution up to $t = 10$ s for $\nu = 0.01$.

Table 2: Example 5: L_2 error comparison for $\Delta t = 0.01$, $N = 80$, and $\alpha = 1$.

Method	t	ν=0.01	ν=0.001	ν=0.0001
[28]	0.1	3.06×10^{-5}	3.10×10^{-7}	2.24×10^{-8}
[15]		3.06×10^{-5}	3.08×10^{-7}	3.40×10^{-9}
[3]		1.13×10^{-6}	3.30×10^{-8}	1.69×10^{-9}
[4]		1.10×10^{-9}	1.05×10^{-10}	1.05×10^{-11}
New		5.30×10^{-9}	7.13×10^{-11}	3.75×10^{-12}
[28]	0.2	6.11×10^{-5}	6.32×10^{-7}	5.22×10^{-8}
[15]		6.12×10^{-5}	6.37×10^{-7}	8.80×10^{-9}
[3]		1.31×10^{-6}	5.10×10^{-8}	1.94×10^{-9}
[4]		2.28×10^{-9}	2.11×10^{-10}	2.11×10^{-11}
New		9.20×10^{-9}	1.40×10^{-10}	7.49×10^{-12}
[28]	0.25	7.62×10^{-5}	7.99×10^{-7}	8.94×10^{-8}
[46]		7.64×10^{-5}	7.68×10^{-7}	7.69×10^{-9}
[15]		7.64×10^{-5}	7.98×10^{-7}	1.34×10^{-8}
[3]		1.63×10^{-6}	5.95×10^{-8}	2.25×10^{-9}
[4]		2.91×10^{-9}	2.64×10^{-10}	2.64×10^{-11}
New		1.08×10^{-8}	1.73×10^{-10}	9.36×10^{-12}

Table 3: Example 5 - L_2 error of the numerical solution for $\nu = 0.0001$ and $N = 80$.

t	DB_4	DB_6	DB_8	DB_{10}
0.1	4.05×10^{-12}	3.95×10^{-12}	3.89×10^{-12}	3.82×10^{-12}
0.2	8.09×10^{-12}	7.90×10^{-12}	7.77×10^{-12}	7.63×10^{-12}
0.25	1.01×10^{-11}	9.87×10^{-12}	9.71×10^{-12}	9.52×10^{-12}
	DB_{12}	[28]	[3]	[4]
0.1	3.75×10^{-12}	2.24×10^{-8}	1.69×10^{-9}	1.05×10^{-11}
0.2	7.49×10^{-12}	5.22×10^{-8}	1.94×10^{-9}	2.11×10^{-11}
0.25	9.36×10^{-12}	8.94×10^{-8}	2.25×10^{-9}	2.64×10^{-11}

Table 4: Example 6: The L_∞ errors comparison for $\alpha = 1$, $c = 2$, and $\nu = 0.001$.

	t	$N = 300$	$N = 500$	$N = 700$
	0.25	1.15×10^{-15}	1.15×10^{-15}	1.14×10^{-15}
	0.50	2.28×10^{-15}	2.28×10^{-15}	2.28×10^{-15}
New	0.75	3.41×10^{-15}	3.41×10^{-15}	3.41×10^{-15}
	1.0	4.52×10^{-15}	4.52×10^{-15}	4.52×10^{-15}
	2.0	8.86×10^{-15}	8.87×10^{-15}	8.87×10^{-15}
	0.25	4.43×10^{-8}	3.63×10^{-8}	3.32×10^{-8}
	0.5	2.17×10^{-7}	1.88×10^{-7}	1.76×10^{-7}
[4]	0.75	5.62×10^{-7}	5.00×10^{-7}	4.75×10^{-7}
	1.0	1.11×10^{-6}	1.00×10^{-6}	9.62×10^{-7}
	2.0	5.94×10^{-6}	5.53×10^{-6}	5.36×10^{-6}

Table 5: Example 6 - L_2 errors for $\nu = 0.01, c = 100$, and $\Delta t = 0.01$ at $t = 1$ s.

N	DB_4	DB_6	DB_8	DB_{10}
10	2.50×10^{-8}	7.85×10^{-10}	5.38×10^{-11}	5.21×10^{-12}
20	1.62×10^{-9}	1.30×10^{-11}	2.36×10^{-13}	7.34×10^{-15}
40	1.02×10^{-10}	2.11×10^{-13}	4.19×10^{-15}	3.86×10^{-15}
80	6.47×10^{-12}	6.61×10^{-15}	3.88×10^{-15}	3.88×10^{-15}
N	DB_{12}	[42]	[51]	
10	6.04×10^{-13}	3.28×10^{-7}	3.45×10^{-7}	
20	3.82×10^{-15}	8.19×10^{-8}	1.01×10^{-7}	
40	3.86×10^{-15}	2.04×10^{-8}	4.00×10^{-8}	
80	3.88×10^{-15}	5.11×10^{-9}	2.47×10^{-8}	

5.2 Telegraph equation

The one-dimensional Telegraph equation (see [5]) is of the form

$$u_{tt}(x,t) + 2\alpha u_t(x,t) + \beta^2 u(x,t) = u_{xx}(x,t) + f(x,t), \quad x \in \Omega := [a,b],\ t \in (t_0, T]\,,$$
$$\text{IC: } \begin{cases} u(x,t_0) = f_1(x), \\ u_t(x,t_0) = f_2(x), \end{cases} \qquad \text{BC: } \begin{cases} u(a,t) = g_1(x), \\ u(b,t) = g_2(x). \end{cases} \tag{5.4}$$

Spatial discretization is the same as in the case of Burgers' equation. The transformation $w = u_t$ and the derivative representation

$$u_{xx}(x,t)|_{x=x_i} = \sum_{k=0}^{N} \mathcal{D}^2_{ik} u_k, \quad i = 0, 1, \ldots, N,$$

will covert the equation Eq.(5.4) into a system first-order ODEs as follows:

$$\begin{aligned} \frac{d\,u_i}{dt} &= w_i, \\ \frac{d\,w_i}{dt} &= -2\alpha w_i - \beta^2 u_i + \sum_{k=0}^{N} \mathcal{D}^2_{ik} u_k + f(x_i,t), \end{aligned} \tag{5.5}$$

where we denote $u_k = u(x_k,t)$ and $w_k = w(x_k,t)$. This system Eq.(5.5) can be solved by SSPRK schemes for the next time step and the solution can be progressed up to a time-bound $T > t_0$. The method tested for some numerical examples and the result shows good agreement with the exact solutions. Here the computations are done with a spatial discretization $N = 80$ and time step $\Delta t = 0.001$.

Example 7. Consider the Eq.(5.4) on $\Omega = (0,1)$ for $\alpha = \frac{1}{2}$, $\beta = 1$ with the initial and boundary conditions $u(x,0) = 0 = u_t(x,0)$ and $u(0,t) = 0 = u(1,t)$ respectively.

The source term is $f(x,t) = (2t^2 + (x - x^2)(t^2 - 2t + 2))e^{-t}$. The exact solution to this problem is $u(x,t) = e^{-t}(x - x^2)t^2$. The numerical solution is plotted up to 5 s for a spatial discretization $N = 80$ with time step $\Delta t = 0.001$, shown in Fig.4. An error comparison table has been added in Table 6 for $N = 200$ and $\Delta t = 0.001$. The proposed method giving relatively small errors as compared with the works [43, 14] and which shows the efficiency of the method in predicting the numerical solution.

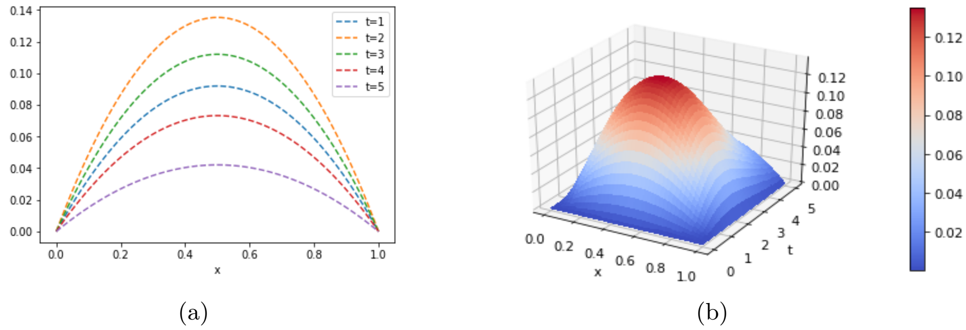
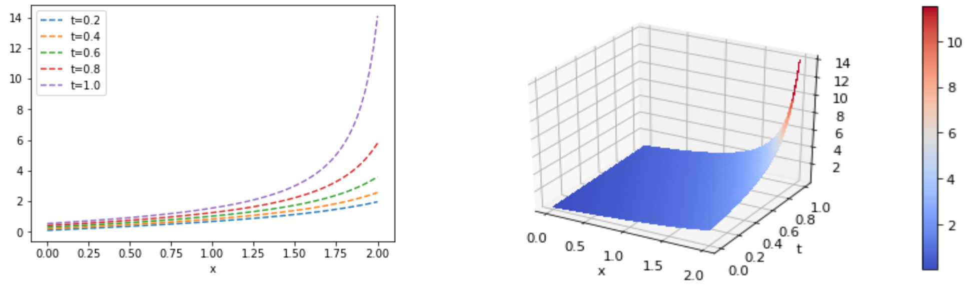


Figure 4: Example 7 - For $N = 80, \Delta t = 0.001$ numerical solution drawn upto $t = 5$s.

Table 6: Example 7: L_2 and L_∞ errors comparison for $N = 200$ and $\Delta t = 0.001$.

Error	t	New	[43]	[14]
L_2	1	1.2155×10^{-6}	4.5526×10^{-5}	1.4386×10^{-4}
	2	2.4025×10^{-6}	1.4307×10^{-5}	8.0879×10^{-5}
	3	1.7632×10^{-6}	6.4273×10^{-6}	1.2944×10^{-4}
	4	1.2472×10^{-6}	8.9203×10^{-6}	1.1845×10^{-4}
	5	6.6493×10^{-7}	3.0161×10^{-6}	7.5545×10^{-5}
L_∞	1	1.9235×10^{-6}	5.9153×10^{-5}	1.8479×10^{-5}
	2	2.9813×10^{-6}	1.7864×10^{-5}	1.0713×10^{-5}
	3	2.3633×10^{-6}	1.4309×10^{-5}	1.8161×10^{-5}
	4	1.5585×10^{-6}	1.3529×10^{-5}	1.6489×10^{-5}
	5	8.8842×10^{-7}	5.2032×10^{-6}	1.0455×10^{-5}

Example 8. Consider the Eq.(5.4) on the spatial domain $\Omega = (0,2)$ with $\alpha = 10$, $\beta = 5$. The initial and boundary conditions are $u(x,0) = \tan\left(\frac{x}{2}\right)$, $u_t(x,0) = \frac{1}{2}\left(1 + \tan^2\left(\frac{x}{2}\right)\right)$, and $u(0,t) = \tan\left(\frac{t}{2}\right)$, $u(2,t) = \tan\left(\frac{2+t}{2}\right)$, with the source term $f(x,t) = \alpha\left(1 + \tan^2\left(\frac{x+t}{2}\right)\right) + \beta^2 \tan\left(\frac{x+t}{2}\right)$. The exact solution to this problem is $u(x,t) = \tan\left(\frac{x+t}{2}\right)$. The numerical plots are given in Fig. 5 for a spatial discretization $N = 80$ and a time step $\Delta t = 0.001$ up to time $t = 1$ s.



(a) (b)

Figure 5: Example 8 - For $N = 80, \Delta t = 0.001$ numerical solution drawn up to $t = 1$ s.

6 Conclusion

We study the representation of the differentiation operator in this manuscript using compactly supported wavelets. The major idea of representation is the multiresolution analysis derived from a scaling function. The computation of the derivatives is performed at a particular level resolution. The elements in this approximation level are differentiable whenever the scaling function is smooth. Once we obtain the matrix equivalent of the differentiation operator, the function's derivative will be computed by multiplication with the differentiation matrix under the periodic boundary conditions. With this idea, we numerically solve the Burgers' and Telegraph equations in one dimension and results are depicted, and comparisons with known works in the same context are tabulated. Since the solutions do not possess periodic boundary behaviour, we use the periodic extension on the vector to approximate its derivatives. The numerical algorithm seems efficient in finding the solution to the partial differential equations. The method is tested for several examples, and the associated numerical solution is plotted as we progress in time.

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