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UNIT GROUPS OF GROUP ALGEBRAS ON CERTAIN QUASIDIHEDRAL GROUPS

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Abstract. Let F_q be any finite field of characteristic $p > 0$ having $q = p^n$ elements. In this paper, we have obtained the complete structure of unit groups of group algebras $F_q[QD_{2^k}]$, for $k = 4$ and 5, for any prime $p > 0$, where QD_{2^k} is quasidihedral group of order 2^k.

1 Introduction

Let FG be the group algebra of a group G over a field F. Let N be the normal subgroup of G. The natural homomorphism $g \mapsto gN$ such that $G \mapsto G/N$ can be extended to an algebra homomorphism from $FG \mapsto F[G/N]$ defined by

$$\sum_{g\in G} a_g g \to \sum_{g\in G} a_g gN$$

for $a_g \in F$. The kernal of this F-algebra homomorphism is $\omega(N)$, which is an ideal generated by $\{x-1,\, x \in N\}$ in FG. The augmentation ideal $\omega(FG)$ of the group algebra FG is defined as:

$$\omega(FG) = \left\{ \sum_{g\in G} a_g g \in FG \,|\, a_g \in F, \sum_{g\in G} a_g = 0 \right\}$$

Obviously, $\frac{FG}{\omega(N)} \cong F(\frac{G}{N})$. It is observed that $\omega(N) = \omega(FN)FG = FG\omega(FN)$. Now $\frac{FG}{\omega(G)} \cong F$ implies that $J(FG) \subseteq \omega(FG)$, where $J(FG)$ is Jacobson radical of FG. It is well known that for an ideal $I \subseteq J(FG)$, the natural homomorphism FG to FG/I induces an epimorphism from the unit group of FG, $U(FG)$ to $U(FG/I)$ with kernal $1 + I$ and

$$\frac{U(FG)}{1+I} \cong U\left(\frac{FG}{I}\right)$$

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We use V_1 as kernal of the epimorphism $U(FG)$ to $U(FG/I)$ and $V_1 = 1 + J(FG)$. For any group G with $g_1, g_2 \in G$ the commutator $(g_1, g_2) = g_1^{-1} g_2^{-1} g_1 g_2$. The lower central series of G is defined as

$$G = \gamma_1(G) \supseteq \gamma_2(G) \supseteq ... \supseteq \gamma_m(G) \supseteq ...$$

where $\gamma_{c+1}(G) = (\gamma_c(G), G)$ is the group generated by (g, h), $g \in \gamma_c(G)$, $h \in G$, for $c \geq 1$. A group G is said to be nilpotent of class c if $\gamma_{c+1}(G) = 1$ but $\gamma_c(G) \neq 1$.

Let F be a finite field of characteristic $p > 0$. An element $g \in G$ is called p-regular if $(p, o(g)) = 1$, where Char$F = p > 0$. Suppose m be the L.C.M. of the orders of p-regular elements of G and η be a primitive m-th root of unity. Now if T be the multiplicative group consisting of those integers t, taken modulo m such that $\eta \to \eta^t$ is an F-automorphism of $F(\eta)$ over F. Any two p-regular elements $g_1, g_2 \in G$ are said to be F-conjugate if $g_1^t = x^{-1} g_2 x$, for some $x \in G$ and $t \in T$. It gives an equivalence relation which partitions the p-regular elements of G into p-regular, F-conjugacy classes. In accordance of Witt-Berman theorem [6, Ch.17, Theorem 5.3], we have the number of non-isomorphic simple FG-modules is equal to the number of F-conjugacy classes of p-regular elements of G.

Problem based on the structure of unit group $U(FG)$ has generated considerable interest. A lot of papers has been appeared in this direction (see [1–3, 7, 9–13, 15]). Sharma and Srivastava [8, 17, 18] have obtained the structure of the unit group of FG for $G = S_3$, S_4 and A_4.

In this paper we have obtained the structure of unit groups of group algebras of quasidihedral groups QD_{16} and QD_{32} of order 16 and 32 respectively. The presentation of quasidihedral groups are given as

$$QD_{2^k} =< a, x \,|\, a^{2^{k-1}} = x^2 = 1, \, xax = a^{2^{k-2}-1} > .$$

The distinct conjugacy classes of QD_{16} are $\mathscr{C}_1 = \{1\}$, $\mathscr{C}_2 = \{a^4\}$, $\mathscr{C}_3 = \{a^2, a^6\}$, $\mathscr{C}_4 = \{a, a^3\}$, $\mathscr{C}_5 = \{a^5, a^7\}$, $\mathscr{C}_6 = \{x, a^2x, a^4x, a^6x\}$, $\mathscr{C}_7 = \{ax, a^3x, a^5x, a^7x\}$. Also $\widehat{\mathscr{C}_i}$, denotes the class sum of $\mathscr{C}_i$, where $1 \leq i \leq 7$.

The distinct conjugacy classes of QD_{32} are $\mathscr{C}'_1 = \{1\}$, $\mathscr{C}'_2 = \{a^8\}$, $\mathscr{C}'_3 = \{a, a^7\}$, $\mathscr{C}'_4 = \{a^{\pm 2}\}$, $\mathscr{C}'_5 = \{a^3, a^5\}$, $\mathscr{C}'_6 = \{a^{\pm 4}\}$, $\mathscr{C}'_7 = \{a^{\pm 6}\}$, $\mathscr{C}'_8 = \{a^{-1}, a^{-7}\}$, $\mathscr{C}'_9 = \{a^{-3}, a^{-5}\}$, $\mathscr{C}'_{10} = \{x, a^2x, a^4x, a^6x, a^8x, a^{10}x, a^{12}x, a^{14}x\}$, $\mathscr{C}'_{11} = \{ax, a^3x, a^5x, a^7x, a^9x, a^{11}x, a^{13}x, a^{15}x\}$. Also $\widehat{\mathscr{C}'_j}$, denotes the class sum of $\mathscr{C}'_j$, where $1 \leq j \leq 11$.

Throughout the paper, $M(n, F)$ denotes the algebra of all $n \times n$ matrices over F, $F^* = F \setminus \{0\}$, $GL(n, F)$ is the general linear group of degree n over F, $CharF$ is the characteristic of F and C_n is the cyclic group of order n.

2 Preliminaries

In this section, we give a complete characterization of unit group $U(F_q[QD_{2^k}])$, for $k = 4$ and 5 having char$F_q = p > 0$.

We shall use the following results frequently in our work.

Lemma 1. *[16] Let G be a group and R be a commutative ring. Then the set of all finite class sums forms an R-basis of $\zeta(RG)$, the center of RG.*

Lemma 2. *[16] Let FG be a semi-simple group algebra. If G' denotes the commutator subgroup of G, then*

$$FG = FG_{e_{G'}} \oplus \Delta(G, G')$$

where $FG_{e_{G'}} \cong F(G/G')$ is the sum of all commutative simple components of FG and $\Delta(G, G')$ is the sum of all the others.

Lemma 3. *[14, Theorem 7.2.7] Let H be a normal subgroup of G with $[G : H] = n < \infty$. Then $(J(FG))^n \subseteq J(FH)FG \subseteq J(FG)$. If in addition, $n \neq 0$ in F, then $J(FG) = J(FH)FG$.*

Lemma 4. *[5, Lemaa 1.17] Let G be a locally finite p-group and let F be a field of characteristic p. Then $J(FG) = \omega(FG)$.*

3 Unit Group of $F_q[QD_{16}]$

Theorem 5. *Let $U(F_q[QD_{16}])$ be the unit group of group algebra $F_q[QD_{16}]$ of quasidihedral group of order 16 over any finite field of positive characteristic p. Let $V = 1 + J(F_q[QD_{16}])$, where $J(F_q[QD_{16}])$ denotes the Jacobson radical of the group algebra $F_q[QD_{16}]$.*

1. *If $|F_q| = 2^n$, then*
   (a) $\frac{U(F_q[QD_{16}])}{V_1} \cong C_{q-1}$.

   (b) V is 2-group of order 2^{15n} and exponent 8.

   (c) Nilpotency class of V is 4.

   (d) V is centrally metabelian.

2. *If $|F_q| = p^n$ where $p > 2$, then $U(F_q[QD_{16}])$ is isomorphic to:*

(a) $C_{q-1}^4 \times GL(2, F_q)^3$, *if* $q \equiv 1$ *or* $3 \, mod \, 8$.

(b) $C_{q-1}^4 \times GL(2, F_q) \times GL(2, F_{q^2})$, *if* $q \equiv -1$ *or* -3 $mod \, 8$.

Proof. **1.(a)** Let Char$F_q = 2$ and $|F_q| = 2^n$. Since D_8 is normal subgroup of QD_{16} of index 2, which is not invertible. We have $\frac{QD_{16}}{D_8} \cong C_2$, $F_qC_2 \cong F_q[\frac{QD_{16}}{D_8}] \cong \frac{F_q[QD_{16}]}{\omega(D_8)}$, so that $dim_{F_q}(\omega(D_8)) = 14$. Further $\omega(D_8)^5 = 0$ and as QD_{16} is a 2-group, therefore by Lemma 4, $\omega(F_q[QD_{16}]) = J(F_q[QD_{16}])$. Hence

$$J(F_qC_2) \cong J\left(\frac{F_q[QD_{16}]}{\omega(D_8)}\right) \cong \frac{J(F_q[QD_{16}])}{\omega(D_8)}.$$

It is known that $dim_{F_q}(J(F_qC_2)) = 1$ and $J(F_qC_2)^2 = 0$, which implies that $dim_{F_q}(J(F_q[QD_{16}])) = 15$ and $J(\frac{F_q[QD_{16}]}{\omega(D_8)})^2 = 0$. Hence we have $dim_{F_q}(\frac{F_q[QD_{16}]}{J(F_q[QD_{16}])}) = 1$. Since $\frac{F_q[QD_{16}]}{J(F_q[QD_{16}])}$ is commutative, therefore

$$\frac{U(F_q[QD_{16}])}{V} \cong F_q^* \cong C_{q-1}.$$

Now the complete description for V is as follows:

(b) It can be easily seen that $y^8 = 1$, for all $y \in V$, hence exponent of V is 8.

(c) Since $V = 1 + J(F_q[QD_{16}])$ and $dim(J(F_q[QD_{16}])) = 15$ therefore $|V| = 2^{15n}$ and V is a 2-group. Now let $a, b \in J(F_q[QD_{16}])$. If we take $u_1 \equiv (1+a, 1+b)$, then $u_1 \equiv 1 + ba((a,b) - 1) \bmod (J^3)$, $u_2 = (u_1, 1+b) \equiv 1 + b^2a(a,b)((u_1,b) - 1) \bmod (J^5)$, $u_3 = (u_2, 1+b) \equiv 1 + b^3a(a,b)((u_1,b) - 1)((u_2,b) - 1) \bmod (J^7)$ and $u_4 \equiv 1$, therefore $\gamma_5(V_1) = 1$ (cf. see [4]). Hence V is nilpotent group of class 4.

(d) Now $\frac{U(F_q[QD_{16}])}{V}$ is an abelian group, thus $U(F_q[QD_{16}])' \subseteq V$. Also $U(F_q[QD_{16}])'' \subseteq V' \subseteq \zeta(F_q[QD_{16}])$, therefore $U(F_q[QD_{16}])$ is centrally metabelian.

2. Let Char$F_q = p (> 2)$. Now since p does not divides $|QD_{16}|$, therefore by Maschke's theorem $F_q[QD_{16}]$ is semi-simple over F_q and thus $J(F_q[QD_{16}]) = 0$. Hence $\frac{F_q[QD_{16}]}{J(F_q[QD_{16}])} \cong F_q[QD_{16}]$. Now using Wedderburn structure theorem, we have:

$$F_q[QD_{16}] \cong (\bigoplus_{i=1}^{c} M(n_i, K_i))$$

where K_i' s are finite dimensional division algebras over F_q and hence K_i' s are finite extensions of F_q.

Since $p > 2$, therefore $F_q(\frac{QD_{16}}{QD_{16}'}) \cong F_q(C_2 \times C_2) \cong F_q \oplus F_q \oplus F_q \oplus F_q$. Thus by Wedderburn structure theorem and by Lemma 2, we have

$$F_q[QD_{16}] \cong F_q \oplus F_q \oplus F_q \oplus F_q \oplus (\bigoplus_{i=1}^{r} M(n_i, K_i)).$$

Hence

$$\zeta(F_q[QD_{16}]) \cong F_q \oplus F_q \oplus F_q \oplus F_q \oplus \big(\bigoplus_{i=1}^{r} K_i\big)$$

and by Lemma 1, $dim_{F_q}(\zeta(F_q[QD_{16}])) = 7$, therefore $\sum_{i=1}^{r}[K_i : F_q] = 7 - 4 = 3$.

Now for any $s \in N$, $x^{q^s} = x$, $\forall x \in \zeta(F_q[QD_{16}])$ if and only if $\widehat{\mathscr{C}_i^{q^s}} = \widehat{\mathscr{C}}_i$, for all $1 \le i \le 7$. This holds good if and only if $8|q^s - 1$ or $8|q^s + 1$. Suppose $K_i^* =< y_i >$ for all i, $1 \le i \le r$, then $x^{q^s} = x$, $\forall x \in \zeta(F_q[QD_{16}])$ if and only if $y_i^{q^s} = 1$, which is equivalent to $[K_i : F_q]|s$, for all $1 \le i \le r$. Hence the least number l such that $8|q^l - 1$ or $8|q^l + 1$ can be given as

$$l = l.c.m.\{[K_i : F_q]|1 \le i \le r\}.$$

By calculation, we have the following possibilities for q

1. If $q \equiv 1 \, mod \, 8$, then $l = 1$,
2. If $q \equiv -1 \, mod \, 8$, then $l = 2$,
3. If $q \equiv 3 \, mod \, 8$, then $l = 1$,
4. If $q \equiv -3 \, mod \, 8$, then $l = 2$.

To find the number of simple components in the Wedderburn decomposition of $F_q[QD_{16}]$, we apply the Witt-Berman theorem. Here $m = 8$. Let $c(= r + 4)$ is the number of simple components. First we will find T and p-regular F_q conjugacy classes as described in introduction.

(a) $\mathbf{q \equiv 1 \, mod \, 8}$

$T \equiv \{1\} \, mod \, 8$. Thus $\mathscr{C}_i$, $1 \le i \le 7$ will be p regular F_q-conjugacy classes and hence $c = 7$.

(b) $\mathbf{q \equiv -1 \, mod \, 8}$

$T \equiv \{1, 7\} \, mod \, 8$. Thus $\{1\}$, $\{a^4\}$, $\{a, a^3, a^5, a^7\}$, $\{a^2, a^6\}$, $\{x, a^2x, a^4x, a^6x\}$, $\{ax, a^3x, a^5x, a^7x\}$ will be p regular F_q-conjugacy classes and hence $c = 6$.

(c) $\mathbf{q \equiv 3 \, mod \, 8}$

$T \equiv \{1, 3\} \, mod \, 8$. Thus $\{1\}$, $\{a^4\}$, $\{a, a^3\}$, $\{a^2, a^6\}$, $\{a^5, a^7\}$, $\{x, a^2x, a^4x, a^6x\}$, $\{ax, a^3x, a^5x, a^7x\}$ will be p regular F_q-conjugacy classes and hence $c = 7$.

(d)$\mathbf{q \equiv -3 \, mod \, 8}$

$T \equiv \{1, 5\} \, mod \, 8$. Thus $\{1\}$, $\{a^4\}$, $\{a, a^3, a^5, a^7\}$, $\{a^2, a^6\}$, $\{x, a^2x, a^4x, a^6x\}$, $\{ax, a^3x, a^5x, a^7x\}$ will be p regular F_q-conjugacy classes and hence $c = 6$.

Thus from the above cases, we have following possibilities for $S = ([K_i : F_q])_{i=1}^{r}$, depending on q:

1. $q \equiv 1 \, mod \, 8 \implies S = (1, 1, 1)$,
2. $q \equiv -1 \, mod \, 8 \implies S = (1, 2)$,

3. $q \equiv 3 \, mod \, 8 \implies S = (1, 1, 1)$,
4. $q \equiv -3 \, mod \, 8 \implies S = (1, 2)$.

Due to dimensions constraint $n_i = 2$, $\forall 1 \leq i \leq r$. Therefore

$$F_q[QD_{16}] \cong \begin{cases} F_q \oplus F_q \oplus F_q \oplus F_q \oplus M(2, F_q)^3, & \text{if}, q \equiv 1 \text{ or } 3 \, mod \, 8 \\ F_q \oplus F_q \oplus F_q \oplus F_q \oplus M(2, F_q) \oplus M(2, F_{q^2}), & \text{if}, q \equiv -1 \text{ or } -3 \, mod \, 8 \end{cases}$$

Hence the result follows. □

4 Unit Group of $F_q[QD_{32}]$

Theorem 6. *Let $U(F_q[QD_{32}])$ be the unit group of group algebra $F_q[QD_{32}]$ of quasidihedral group of order 32 over any finite field of positive characteristic p. Let $V_1 = 1 + J(F_q[QD_{32}])$, where $J(F_q[QD_{32}])$ denotes the Jacobson radical of the group algebra $F_q[QD_{32}]$.*

1. *If $|F_q| = 2^n$, then*
   (a) $\frac{U(F_q[QD_{32}])}{V_1} \cong C_{q-1}$.

   (b) V_1 is 2-group of order 2^{31n} and exponent 16.

2. *If $|F_q| = p^n$ where $p > 2$, then $U(F_q[QD_{32}])$ is isomorphic to:*

   (a) $C_{q-1}^4 \times GL(2, F_q)^7$, if $q \equiv 1$ or $7 \, mod \, 16$.

   (b) $C_{q-1}^4 \times GL(2, F_q)^3 \times GL(2, F_{q^2})^2$, if $q \equiv -1$ or -7 $mod \, 16$.

   (c) $C_{q-1}^4 \times GL(2, F_q) \times GL(2, F_{q^2}) \times GL(2, F_{q^4})$, if $q \equiv \pm 3$ or ± 5 $mod \, 16$.

Proof. **1(a)** Let Char$F_q = 2$, then $|F_q| = 2^n$. Since C_{16} is normal subgroup of QD_{32}, which is not invertible. Therefore $\frac{QD_{32}}{C_{16}} \cong C_2$, $F_q C_2 \cong F_q[\frac{QD_{32}}{C_{16}}] \cong \frac{F_q[QD_{32}]}{\omega(C_{16})}$ and then $dim_{F_q}(\omega(C_{16})) = 30$. Now as QD_{32} is a 2-group, therefore by Lemma 4, $\omega(F_q[QD_{32}]) = J(F_q[QD_{32}])$. Hence

$$J(F_q C_2) \cong J\left(\frac{F_q[QD_{32}]}{\omega(C_{16})}\right) \cong \frac{J(F_q[QD_{32}])}{\omega(C_{16})}.$$

It is known that $dim_{F_q}(J(F_qC_2)) = 1$ and $J(F_qC_2)^2 = 0$, which implies that $dim_{F_q}(J(F_q[QD_{32}])) = 31$ and $J(\frac{F_q[QD_{32}]}{\omega(C_{16})})^2 = 0$. Hence we have $dim_{F_q}(\frac{F_q[QD_{32}]}{J(F_q[QD_{32}])}) = 1$. Since $\frac{F_q[QD_{32}]}{J(F_q[QD_{32}])}$ is abelian, therefore

$$\frac{U(F_q[QD_{32}])}{V_1} \cong F_q^* \cong C_{q-1}.$$

The complete description for V_1 is as follows:
1(b) Since $dim_{F_q}(J(F_q[QD_{32}])) = 31$, therefore V_1 is a 2-group of order 2^{31n}. Also for every element $y \in V_1$, we have $y^{16} = 1$. Thus exponent of V_1 is 16. Now as $\frac{U(F_q[QD_{32}])}{V_1}$ is an abelian group, thus $U(F_q[QD_{32}])' \subseteq V_1$ and hence $U(F_q[QD_{32}])'' \subseteq V_1'$.

2. Let Char$F_q = p\,(> 2)$. Now as p does not divides $|QD_{32}|$, thus by Maschke's theorem $F_q[QD_{32}]$ is semi-simple over F_q and then $J(F_q[QD_{32}]) = 0$. Hence $\frac{F_q[QD_{32}]}{J(F_q[QD_{32}])} \cong F_q[QD_{32}]$. By using Wedderburn structure theorem, we have:

$$F_q[QD_{32}] \cong (\bigoplus_{i=1}^{c} M(n_i, K_i))$$

where K_i' s are finite dimensional division algebras over F_q and hence K_i' s are finite extensions of F_q.
Since $p > 2$, therefore $F_q(\frac{QD_{32}}{QD_{32}'}) \cong F_q(C_2 \times C_2) \cong F_q \oplus F_q \oplus F_q \oplus F_q$. Therefore from Wedderburn structure theorem and by Lemma 2, we have

$$F_q[QD_{32}] \cong F_q \oplus F_q \oplus F_q \oplus F_q \oplus (\bigoplus_{i=1}^{r} M(n_i, K_i)).$$

Hence

$$\zeta(F_q[QD_{32}]) \cong F_q \oplus F_q \oplus F_q \oplus F_q \oplus (\bigoplus_{i=1}^{r} K_i)$$

and by Lemma 1, $dim_{F_q}(\zeta(F_q[QD_{16}])) = 11$, therefore $\sum_{i=1}^{r}[K_i : F_q] = 11 - 4 = 7$.

Now via similar arguments as in Thorem 5, we have the least number l such that $16|q^l - 1$ or $16|q^l + 1$, which can be given as

$$l = l.c.m.\{[K_i : F_q] | 1 \leq i \leq r\}.$$

By calculation, we have the following possibilities for q

1. If $q \equiv 1$ or $7 \, mod\, 16$, then $l = 1$,
2. If $q \equiv -1$ or $-7 \, mod\, 16$, then $l = 2$,

3. If $q \equiv \pm 3$ or $\pm 5 \, mod \, 16$, then $l = 4$.

Now we will find out the number of simple components in the Wedderburn decomposition of $F_q[QD_{32}]$. We use the Witt-Berman theorem. Here $m = 16$. If $c\,(= r+4)$ be the number of simple components, then we will find T and p-regular F_q conjugacy classes as described in introduction.

(a) $\mathbf{q \equiv 1 \, mod \, 16}$

$T \equiv \{1\} \, mod \, 16$. Thus $\mathscr{C}'_j$, $1 \leq j \leq 11$ will be p regular F_q-conjugacy classes and hence $c = 11$.

(b) $\mathbf{q \equiv -1 \, mod \, 16}$

$T \equiv \{-1, 1\} \, mod \, 16$. Thus $\{1\}$, $\{a^{\pm 1}, a^{\pm 7}\}$, $\{a^{\pm 2}\}$, $\{a^{\pm 3}, a^{\pm 5}\}$, $\{a^{\pm 4}\}$, $\{a^{\pm 6}\}$ $\{a^8\}$, $\{x, a^2x, a^4x, a^6x, a^8x, a^{10}x, a^{12}x, a^{14}x\}$, $\{ax, a^3x, a^5x, a^7x, a^9x, a^{11}x, a^{13}x, a^{15}x\}$ will be p regular F_q-conjugacy classes and hence $c = 9$.

(c) $\mathbf{q \equiv 3 \, or \, -5 \, mod \, 16}$

$T \equiv \{1, 3, 9, 11\} \, mod \, 16$. Thus $\{1\}$, $\{a^{\pm 1}, a^{\pm 3}, a^{\pm 5}, a^{\pm 7}\}$, $\{a^{\pm 2}, a^{\pm 6}\}$, $\{a^{\pm 4}\}$, $\{a^8\}$, $\{x, a^2x, a^4x, a^6x, a^8x, a^{10}x, a^{12}x, a^{14}x\}$, $\{ax, a^3x, a^5x, a^7x, a^9x, a^{11}x, a^{13}x, a^{15}x\}$ will be p regular F_q-conjugacy classes and hence $c = 7$.

(d) $\mathbf{q \equiv -3 \, or \, 5 \, mod \, 16}$

$T \equiv \{1, 5, 9, 13\} \, mod \, 16$. Thus $\{1\}$, $\{a^{\pm 1}, a^{\pm 3}, a^{\pm 5}, a^{\pm 7}\}$, $\{a^{\pm 2}, a^{\pm 6}\}$, $\{a^{\pm 4}\}$, $\{a^8\}$, $\{x, a^2x, a^4x, a^6x, a^8x, a^{10}x, a^{12}x, a^{14}x\}$, $\{ax, a^3x, a^5x, a^7x, a^9x, a^{11}x, a^{13}x, a^{15}x\}$ will be p regular F_q-conjugacy classes and hence $c = 7$.

(e) $\mathbf{q \equiv 7 \, mod \, 16}$

$T \equiv \{1, 7\} \, mod \, 16$. Thus $\mathscr{C}'_j$, $1 \leq j \leq 11$ will be p regular F_q-conjugacy classes and hence $c = 11$.

(f) $\mathbf{q \equiv -7 \, mod \, 16}$

$T \equiv \{1, 9\} \, mod \, 16$. Thus $\{1\}$, $\{a^{\pm 1}, a^{\pm 7}\}$, $\{a^{\pm 2}\}$, $\{a^{\pm 4}\}$, $\{a^{\pm 3}, a^{\pm 5}\}$, $\{a^{\pm 6}\}$, $\{a^8\}$ $\{x, a^2x, a^4x, a^6x, a^8x, a^{10}x, a^{12}x, a^{14}x\}$, $\{ax, a^3x, a^5x, a^7x, a^9x, a^{11}x, a^{13}x, a^{15}x\}$ will be p regular F_q-conjugacy classes and hence $c = 9$.

Thus from the above cases, we have following possibilities for $S = ([K_i : F_q])_{i=1}^r$, depending on q:

1. $q \equiv 1$ or $7 \, mod \, 16 \implies S = (1, 1, 1, 1, 1, 1, 1)$,
2. $q \equiv -1$ or $-7 \, mod \, 16 \implies S = (1, 1, 1, 2, 2)$,
3. $q \equiv \pm 3$ or $\pm 5 \, mod \, 16 \implies S = (1, 2, 4)$.

Due to dimensions constraint $n_i = 2$, $\forall 1 \leq i \leq r$. Therefore

$$F_q[QD_{32}] \cong \begin{cases} F_q^4 \oplus M(2, F_q)^7, & \text{if}, q \equiv \ 1 \, \text{or} \, 7 \, mod \, 16 \\ F_q^4 \oplus M(2, F_q)^3 \oplus M(2, F_{q^2})^2, & \text{if}, q \equiv \ -1 \, \text{or} \, -7 \, mod \, 16 \\ F_q^4 \oplus M(2, F_q) \oplus M(2, F_{q^2}) \oplus M(2, F_{q^4}), & \text{if}, q \equiv \ \pm 3 \, \text{or} \, \pm 5 \, mod \, 16 \end{cases}$$

Hence the result follows.

□

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