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APPLICATION OF THE LITTLEWOOD-PALEY METHOD TO CALDERON-ZYGMUND OPERATORS

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Abstract. In this article, we establish the conditions for the pseudo-differential operator T under which this operator can be represented in convolution form with the singular kernel that satisfies $\left|\partial_x^\alpha \partial_z^\beta k(x,z)\right| \le A_{\beta\alpha}(L)\,|z|^{-l-m-|\beta|-L}$ for all $z \neq 0$, and all multi-indices α, β and $L \ge 0$ such that $l+m+|\beta|+L>0$. Also, applying the Littlewood-Paley method, we show the inverse: if a is a symbol such that $\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \le A_{\beta\alpha}(1-|\xi|)^{(|\beta|-|\alpha|)\delta}$ for some $0 \le \delta < 1$, then $T(f)(x) = \left\langle a(x,\,\cdot)\,\hat{f}(\cdot)\exp(2\pi i x\cdot)\right\rangle$ defines a bounded pseudo-differential operator $L^2(R^l) \mapsto L^2(R^l)$.

We establish the necessary and sufficient conditions on the kernel K under which there exists a bounded operator $T : L^2(R^l) \to L^2(R^l)$. Finally, we establish the necessary and sufficient conditions in terms of the operator $T : L^p(R^l) \to L^p(R^l)$ under which a nonnegative Borel measure μ is absolutely continuous $d\mu(x) = \omega(x)\,dx$ $\omega \in A_p$.

1 Introduction and discussion of the subject

The main object of research in harmonic analysis is linear operators that satisfy the Calderon-Zygmund conditions. The Calderon-Zygmund operator T is a linear operator defined by the singular integral with the kernel $K(x,y)$ in the form

$$\begin{aligned} T(f)(x) &= \textstyle\int_{R^l} K(x,y)\,f(y)\;d\mu(y) = \\ &= \langle K(x,\cdot)\,f(\cdot)\rangle_{\mu(\cdot)} \end{aligned} \tag{1.1}$$

for $f \in S(R^l)$, where μ is a Borel measure on the Borel σ-algebra of R^l.

The integral kernel $K(\cdot,\,\cdot) \in L^1_{loc}\left(R^l \times R^l \setminus \{(x,x)\ :\ x \in R^l\}\right)$ is singular near $x=y$ and satisfies the following conditions:

growth condition one

$$|K(x,y)| \le A\,|x-y|^{-l} \tag{1.2}$$

growth condition two for all $|x-\breve{x}| \le \frac{1}{2}|x-y|$ we have

$$|K(x,y) - K(\breve{x},y)| \le A\,|x-\breve{x}|^{\gamma}\,|x-y|^{-l-\gamma} \tag{1.3}$$

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and growth condition three for $|y-\breve{y}| \leq \frac{1}{2}|x-y|$ we have

$$|K(x,y)-K(x,\breve{y})| \leq A|y-\breve{y}|^{\gamma}|x-y|^{-l-\gamma} \tag{1.4}$$

for some $0<\gamma\leq 1$ and positive constant A.

The first works which considered properties of operators (1.1) that satisfy (1.2)-(1.3) conditions were published in the 1950s by A.P. Calderon and A. Zygmund [4], the main result establishes the boundness of operators given by singular integrals (1.1) in L^p space of real variables. Some new information can be found in [1–19], so W. Li and Q. Xue studied the multilinear case of the Calderon-Zygmund operator and established its boundedness in the space with weights [14]; in [7] the generalized weighted Morrey spaces and the generalized weighted weak Morrey spaces are considered, and authors studied the Hardy–Littlewood maximal operator and its application to the Calderon-Zygmund operator.

Let the operator T be expressed by (1.1) then the operator T is defined by three elements: the nonnegative Borel measure μ with correspondent measurable space, the singular kernel $K(\cdot,\ \cdot)\in L^1_{loc}\left(R^l\times R^l\backslash\left\{(x,x)\ :\ x\in R^l\right\}\right)$ that satisfies certain conditions, and the functional class on which the operator T is defined. The specter of problems pertaining to T can concern each of the three elements or their combination. So, assume that the operator T is well defined and bounded on the functional space $L^q\left(R^l\right),\quad p>1$ with the norm $\|T\|_{L^q}=A$, and the kernel K satisfies the estimation

$$\int_{\overline{B(y,C\theta)}} |K(x,\ y)-K(x,\ \tilde{y})|\,d\mu(x)\leq A$$

for all $\tilde{y}\in B(y,C\theta)$, then the operator T uniquely extends to the operator in all $L^p\left(R^l\right),\qquad 1<p<q$ and remains bounded in the L^p-norm.

In this article, we consider a classical approach to analytic constructs of the problems of harmonic analysis of real-variable functions. The main object of our investigation is the wide class of operators that satisfy certain conditions that are usually called under the umbrella name “Calderon-Zygmund conditions”. There are many ways to define such operators: first, the operator is given by integral (1.1) with the singular kernel; second, by employing the Fourier transform, the Calderon-Zygmund operator T corresponds with the symbol $a(x,\ \xi)$ according to $T(f)(x)=\left\langle a(x,\ \cdot)\hat{f}(\cdot)\exp(2\pi ix\cdot)\right\rangle$ where $\hat{f}=F(f)$ is a Fourier transform of the function f; third, the operator can be defined as operator multiplication by $\hat{T}(f)(\xi)=m(\xi)\hat{f}(\xi)$ where $\hat{T}$ is the Fourier transform of T. We study the conditions under which the first and the second definitions determine the same object and establish its properties. Theorems 3 and 7 deal with singular integrals and measures in terms of weights, thus assuming we are interested in the L^pfunctional class, then for the operator $T(f)(x)=\langle K(x-\cdot)f(\cdot)\rangle_{\mu(\cdot)}$ with the kernel satisfying the theorem assumption, we establish the class of weights A_p given by (2.1), (2.2), which corresponds with measures for which the inequality

$$\int_{R^l} |T(f)(y)|^p \, d\mu(y) \le A \int_{R^l} |(f)(y)|^p \, d\mu(y)$$

for all measures $\mu(y) = \omega(y)\, dy$ when $\omega \in A_p$.

2 Maximal operator

Let measure μ be absolutely continuous respective Lebesgue measure with the density $\omega(x)$ so that $d\mu(x) = \omega(x)\, dx$, the class of weights A_p consists of locally integrable functions ω such that the following inequality

$$(mes\,(B))^{-p} \langle\omega\rangle_B \left(\left\langle \omega^{\frac{1}{1-p}} \right\rangle_B\right)^{p-1} \le A < \infty \tag{2.1}$$

or the same

$$(\langle\omega\rangle_B)^q \left(\left\langle \omega^{-\frac{q}{p}} \right\rangle_B\right)^p \le A^q \,(mes\,(B))^{pq} \tag{2.2}$$

holds for all balls B in R^l and $p+q = pq$, $\quad p > 1$. The smallest constant A is called the bound of ω and is denoted by $A_p(\omega)$.

Lemma 1. *Function ω belongs to A_p class if and only if there are some constants c such that the following inequality*

$$\langle\omega\rangle_B \left(\frac{1}{mes\,(B)} \langle|f|\rangle_B\right)^p \le c\; mes\,(B) \left\langle |f|^p \omega\right\rangle_B \tag{2.3}$$

holds for all balls B in R^l and all locally integrable functions f on R^l. The minimal value of constant c equals $A_p(\omega)$.

Proof. Let $d\mu(x) = \omega(x)\, dx$ then application of the Holder equation yields inequality

$$(\langle|f|\rangle_B)^p \le \langle|f|^p \omega\rangle_B \left(\langle\omega^{1-q}\rangle_B\right)^{p-1} \tag{2.4}$$

therefore we obtain the first statement of the lemma. To show the truth of the reverse statement one can take $f = (\omega+\varepsilon)^{1-q}$ in (2.3) and show that inequality (2.1) holds for $A \le c$ and for all $\varepsilon > 0$. Next, take the limit as $\varepsilon \to 0$.

Definition 2. *A maximal operator M on R^l is defined by*

$$M(f)(x) = \sup_{r>0} \frac{c_l}{r^l} \int_{|y|\le r} |f(x-y)|\, dy \tag{2.5}$$

for an arbitrary locally integrable function f.

The reverse Holder inequality yields the following statements.

Statement 1. Let $f \in L^p(d\mu(x))$ and let $d\mu(x) = \omega(x)\,dx$ where $\omega \in A_p$then the maximal operator has an estimation

$$\langle (M(f))^p \rangle_{d\mu=\omega dx} = \langle (M(f)(\cdot))^p \omega(\cdot) \rangle \le A \langle |f(\cdot)|^p \omega(\cdot) \rangle . \tag{2.6}$$

Statement 2. Let the kernel $K(x,\ y)$ satisfies conditions (1.2), (1.3), and (1.4) with $\gamma = 1$ then the inequality

$$\left\langle \left(\sup_{\varepsilon>0} \left| \int_{|\cdot - y|>\varepsilon} K(\cdot,\ y) f(y)\,dy \right| \right)^p \omega(\cdot) \right\rangle \le A(p,\omega) \langle (M(f)(\cdot))^p \omega(\cdot) \rangle \tag{2.7}$$

holds for all bounded functions $f \in L^2(dx)$ with compact support and $p \in (1,\ \infty)$.

From statement 2 follows theorem 3.

Theorem 3. ***Let the operator*** T ***be given by***

$$T(f)(x) = \langle K(x - \cdot) f(\cdot) \rangle \tag{2.8}$$

for all $f \in L^2(dx)$***. Let the kernel*** K ***satisfies conditions*** (1.2)***,*** (1.3)***, and*** (1.4) ***with*** $\gamma = 1$ ***then we have an estimation***

$$\langle |T(f)(\cdot)|^p \omega(\cdot) \rangle \le A \langle |(f)(\cdot)|^p \omega(\cdot) \rangle \tag{2.9}$$

for all functions $f \in L^2(R^l)$***, all weights*** $\omega \in A_p$ ***, and all*** $p \in (1,\ \infty)$***.***

The estimation (2.9) holds for all functions $f \in L^p(R^l)$ and for all measures $d\mu(x) = \omega(x)\,dx$ in the form

$$\int_{R^l} |T(f)(y)|^p \,d\mu(y) \le A \int_{R^l} |(f)(y)|^p \,d\mu(y),$$

usually, firstly, we prove the estimation (2.9) for functions $f \in C^\infty(R^l)$ with compact supports then we employ the density of $C_0^\infty(R^l)$ in $L^p(R^l)$ in the topology of the L^p- norm.

3 Pseudo-differential operators and Fourier transform

Let function $a(x,\xi)$ be a symbol S^m of order m which means that a is a function of $C^\infty(R^{2l})$ and satisfies the following condition

$$\left| \partial_x^\alpha \partial_\xi^\beta a(x,\xi) \right| \le A_{\beta\gamma} (1 + |\xi|)^{m-|\beta|} \tag{3.1}$$

for all multi-indices α, β.

The Fourier transform $\hat{f}$ of the function f is given by

$$\hat{f}(\xi) = \langle f(\cdot) \exp(-2\pi i \xi \cdot) \rangle, \tag{3.2}$$

mapping T defined by

$$T(f)(x) = \left\langle a(x, \ \cdot) \hat{f}(\cdot) \exp(2\pi i x \cdot) \right\rangle \tag{3.3}$$

is called the pseudo-differential operator of the function f.

The pseudo-differential operator T can be presented in terms of the kernel by

$$T(f)(x) = \langle K(x, \cdot) f(\cdot) \rangle$$

with the appropriate kernel K.

Statement 3. *Let $a \in S^0$ and let mapping T be given by*

$$T(f)(x) = \left\langle a(x, \ \cdot) \hat{f}(\cdot) \exp(2\pi i x \cdot) \right\rangle \tag{3.4}$$

for all functions $f \in S\left(R^l\right)$, then mapping T extends to a bounded operator $L^2\left(R^l\right) \mapsto L^2\left(R^l\right)$, namely, the equality

$$\|T(f)\|_{L^2} \leq A \|f\|_{L^2} \tag{3.5}$$

holds with the constant A and for all $f \in L^2\left(R^l\right)$.

Proof can be found in standard work on harmonic analysis, the proof employs the density of $S\left(R^l\right)$ in $L^2\left(R^l\right)$ functional space.

We will use the notation $k(x, x-y) = K(x, y)$ so that the pseudo-differential operator can be presented in the form

$$T(f)(x) = \langle k(x, \cdot) f(x - \cdot) \rangle,$$

the $k(x, y)$ is a distribution for each fixed x such that

$$a(x, \xi) = \langle k(x, \cdot) \exp(-2\pi i \xi \cdot) \rangle.$$

Let $a \in S^m$. We are going to employ the Littlewood-Paley method to establish the following estimation

$$\left| \partial_x^\alpha \partial_z^\beta k(x, z) \right| \leq A_{\beta\gamma} |z|^{-l-m-|\beta|-L} \tag{3.6}$$

that holds for all $z \neq 0$ for all multi-indices $\alpha, \quad \beta$ and all $L \geq 0$ such that $l + m + |\beta| + L > 0$.

4 Littlewood-Paley method

The Littlewood-Paley dyadic decomposition employs the representation of the function as the composition of the functions with localized frequencies.

Let us fix a function $\eta \in C_0^\infty\left(R^l\right)$ such that $\eta(\xi) = 1$, $|\xi| \leq 1$ and $\eta(\xi) = 0$, $|\xi| \geq 2$; and let us define a function $\varphi(\xi) = \eta(\xi) - \eta(2\xi)$. We define the partitions of unity by

$$1 = \eta(\xi) + \sum_{k=1,2,\ldots} \varphi\left(2^{-k}\xi\right) \quad \forall \xi, \tag{4.1}$$

$$1 = \sum_{k=-\infty,\,+\infty} \varphi\left(2^{-k}\xi\right) \quad \xi \neq 0. \tag{4.2}$$

The difference operator is given by

$$\Delta_k(f) = f * \left(\phi_{2^{-k}} - \phi_{2^{-k+1}}\right), \tag{4.3}$$

where we put $\phi_s(x) = s^{-l}\phi\left(\frac{x}{s}\right)$, $\langle\phi\rangle = 1$ and inverse Fourier transform $\hat{\phi} = \eta$.

Let function f satisfies the Lipschitz conditions then there exists a constant M such that inequality

$$\|\Delta_k(f)\|_{L^\infty} \leq M2^{-kL}, \tag{4.4}$$

holds for the Lipschitz constant L.

The operator T can be represented as

$$T = \sum_{n=0,1,\ldots} T_n, \tag{4.5}$$

where $T_n = T\Delta_n$ and $T\Delta_0 f = T(f * \phi)$.

Every operator T_n is associated with a symbol $a_n(x,\xi) = a(x,\xi)\varphi(2^{-n}\xi)$ and $a_0(x,\xi) = a(x,\xi)\eta(\xi)$ for T_0.

The difference operators satisfy are almost projections, namely, they satisfy the following condition

$$\Delta_n = \Delta_n(\Delta_{n-1} + \Delta_n + \Delta_{n+1}). \tag{4.6}$$

We obtain the operator identity

$$\mathrm{I} = \sum_{n=-\infty,\,+\infty} \Delta_n. \tag{4.7}$$

Now, we consider the series

$$T(f) = \sum_{n=0,1,\ldots} T_n(\Delta_{n-1} + \Delta_n + \Delta_{n+1}) f, \tag{4.8}$$

where $\|(\Delta_{n-1} + \Delta_n + \Delta_{n+1}) f\|_{L^\infty} \leq M2^{-kL}$, L is a Lipschitz coefficient.

Since

$$\left\|\partial_x^\alpha T_n\left(\Delta_{n-1}+\Delta_n+\Delta_{n+1}\right)f\right\|_{L^\infty}\leq M_\alpha 2^{n(|m|+|\alpha|-L)}, \tag{4.9}$$

we have

$$\left\|\Delta_j\sum T_n\left(\Delta_{n-1}+\Delta_n+\Delta_{n+1}\right)\right\|_{L^\infty}\leq M2^{j(m-L)}. \tag{4.10}$$

Next, we want to establish the following estimation

$$\left|\partial_x^\alpha\partial_z^\beta k\left(x,z\right)\right|\leq A_{\beta\gamma}\left(L\right)|z|^{-l-m-|\beta|-L}$$

for all $z\neq 0$ for all multi-indices $\beta,\quad\gamma$ and all $L\geq 0$ such that $l+m+|\beta|+L>0$.

The kernel $k\left(x,z\right)$ can be decomposed into the sum $\sum_{n=0,1,...}k_n\left(x,z\right)$ converging for each x.

Statement 4. ***If the symbol*** a ***belongs to the class*** S^m ***then we have***

$$\left|\partial_x^\alpha\partial_z^\beta k_n\left(x,z\right)\right|\leq A_{\beta\gamma}\left(M\right)|z|^{-M}2^{n(l+m+|\beta|-M)} \tag{4.11}$$

for all $\alpha,\quad\beta$ ***and*** $M\geq 0$***.***

The statement straightforward follows from the representation

$$\left(2\pi iz\right)^\tau\partial_x^\alpha\partial_z^\beta k_n\left(x,z\right)=\left\langle\partial_\cdot^\tau\left(2\pi iz\right)^\beta\partial_x^\alpha a_n\left(x,\cdot\right)\exp\left(2\pi iz\cdot\right)\right\rangle. \tag{4.12}$$

Assume $|z|\geq 1$ and choose $M>l+m+|\beta|-L$ then

$$\sum_{n=0,1,..}\left|\partial_x^\alpha\partial_z^\beta k_n\left(x,z\right)\right|\leq A_{\beta\alpha}\left(M\right)O\left(|z|^{-M}\right) \tag{4.13}$$

for all $|z|\geq 1$, which can be estimated from above by $O\left(|z|^{-(l+m+|\beta|-L)}\right)$ with arbitrary large L.

For all $0<|z|\leq 1$, we divide the sum into two parts and estimate

$$\sum_{n=0,1,..}\left|\partial_x^\alpha\partial_z^\beta k_n\left(x,z\right)\right|\leq A_{\beta\alpha}\left(M\right)|z|^{-M}\sum_{2^n\leq\frac{1}{|z|}}2^{n(l+m+|\beta|-M)}+$$

$$+A_{\beta\alpha}\left(M\right)|z|^{-M}\sum_{2^n>\frac{1}{|z|}}2^{n(l+m+|\beta|-M)},$$

we take $M=0$ in the first sum, and assume $M>l+m+|\beta|$ in the second sum, so that, for $0<|z|\leq 1$ we have

$$\sum_{n=0,1,..}\left|\partial_x^\alpha\partial_z^\beta k_n\left(x,z\right)\right|\leq O\left(|z|^{-(l+m+|\beta|-L)}\right)$$

for all L.

So, we have obtained the following theorem.

Theorem 4. ***Assume the pseudo-differential operator T defined by***

$$T(f)(x) = \left\langle a(x,\ \cdot)\,\hat{f}(\cdot)\exp(2\pi i x\cdot)\right\rangle \tag{4.14}$$

for all symbols $a \in S^m$ has integral representation with singular kernel $k(x,z)$ in the form

$$T(f)(x) = \langle k(x,\cdot)\,f(x-\cdot)\rangle\,, \tag{4.15}$$

then the integral kernel satisfies the estimation

$$\left|\partial_x^\alpha \partial_z^\beta k(x,z)\right| \le A_{\beta\alpha}(L)\,|z|^{-l-m-|\beta|-L} \tag{4.16}$$

for all $z \neq 0$, and for all multi-indices $\alpha,\quad \beta$ and all $L \ge 0$ such that $l+m+|\beta|+L>0$.

From the Schwartz theorem, we obtain that for each symbol $a \in S\left(R^{2n}\right)$ there exists a kernel $K(x,\ y) = k(x,\ x-y)\,,\quad K \in S\left(R^{2n}\right)$ such that

$$T(f)(x) = \left\langle a(x,\ \cdot)\,\hat{f}(\cdot)\exp(2\pi i x\cdot)\right\rangle = \langle K(x,\ \cdot)\,f(\cdot)\rangle\,, \tag{4.17}$$

the reverse is also true, each kernel $K \in S\left(R^{2n}\right)$ corresponds with the symbol $a \in S\left(R^{2n}\right)$ so that $T_K = T_a$.

5 Calderon-Zygmund operator in the L^2-space

Let kernel $K(x,\ y)$ be defined for all $x \neq y$ and let K satisfies the estimations given by (1.2), (1.3), and (1.4) then $K(x,\ y)$ satisfies the differential inequality

$$\left|\partial_x^\alpha \partial_y^\beta K(x,y)\right| \le A_{\beta\alpha}\,|x-y|^{-l-|\alpha|-|\beta|} \tag{5.1}$$

for all multi-indices $\alpha,\quad \beta$. The operators corresponded to the kernel K under the condition (5.1) are not bounded in L^2-space.

Theorem 5. ***Assume a symbol a satisfies the inequality***

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \le A_{\beta\alpha}\,(1-|\xi|)^{(|\beta|-|\alpha|)\delta} \tag{5.2}$$

for $0 \le \delta < 1$. Then operator

$$T(f)(x) = \left\langle a(x,\ \cdot)\,\hat{f}(\cdot)\exp(2\pi i x\cdot)\right\rangle \tag{5.3}$$

defined for all $f \in S$ extends to a bounded operator $L^2\left(R^l\right) \mapsto L^2\left(R^l\right)$.

Proof. Let g be defined on R^{2l} a smooth function with compact support and such that $g(0,0)=1$. For all $0\le\varepsilon<1$, we define the symbol $a_\varepsilon(x,\xi)$ by

$$a_\varepsilon(x,\xi)=a(x,\xi)\,g(\varepsilon x,\varepsilon\xi) \tag{5.4}$$

then for all $f\in S$, we have that the set $\{T_{a_\varepsilon}\}$ of operators corresponding to kernels a_ε converges to the operator T_a as $\varepsilon\to 0$ in S-topology. Therefore, we can presume that the symbol a has a compact support.

Applying the Littlewood-Paley method, we decompose T into series

$$Tf=\sum_{n=0,1,\ldots}T_nf=T(f*\phi)+\sum_{n=1,\ldots}T\Delta_nf \tag{5.5}$$

where each operator T_n is associated with a symbol $a_n(x,\xi)=a(x,\xi)\,\varphi(2^{-n}\xi)$ is supported on $2^{n-1}\le|\xi|\le 2^{n+1}$ and T_0 associated with $a_0(x,\xi)=a(x,\xi)\,\eta(\xi)$ supported on $|\xi|\le 2$.

The sum (5.5) can be broken into two sums for even and for odd indices, the ξ-support of each sum is disjoint.

Since, for odd indices n and k, the intersection of supports of Δ_n and Δ_k is empty, we have $T_nT_k^*=T\Delta_n\Delta_k^*T=0$ for $n\ne k$.

If the operator T^*T is bounded in L^2 then the operator T is bounded in L^2. So, we can write

$$\begin{gathered}(T_k^*T_n)(f)(x)=\\ =\int_{R^l}\int_{R^l}\int_{R^l}\int_{R^l}\overline{a}_k(z,\nu)\,a_n(z,\xi)\exp\left(2\pi i\left(\xi(z-y)-\nu(z-x)\right)\right)dzd\nu d\xi dy.\end{gathered} \tag{5.6}$$

Now, we remark that

$$(I-\Delta_z)^N\exp\left(2\pi iz(\nu-\xi)\right)=\left(1+4\pi^2|\nu-\xi|^2\right)^N\exp\left(2\pi iz(\nu-\xi)\right), \tag{5.7}$$

using equality of (5.7) type, we integrate by parts first with the respect to z-variable next to ν and finally with the respect to ξ-variable. Applying estimation (5.2) and boundedness of the supports, we obtain

$$\begin{aligned}&\left|\int_{R^l}\int_{R^l}\int_{R^l}\overline{a}_k(z,\nu)\,a_n(z,\xi)\exp\left(2\pi i\left(\xi(z-y)-\nu(z-x)\right)\right)dzd\nu d\xi\right|\le\\ &\le 2^{2\max(k,n)((\delta-1)N+l)}\int_{R^l}(1+|x-z|)^{-2N}(1+|z-y|)^{-2N}dz\end{aligned}$$

for $n\ne k$. Let us denote

$$\begin{aligned}&\int_{R^l}\int_{R^l}(1+|x-z|)^{-2N}(1+|z-y|)^{-2N}dzdy=\\ &=\left(\int_{R^l}(1+|z|)^{-2N}dz\right)^2=A\end{aligned}$$

so, we obtain the estimation

$$\|T_k^*T_n\|\le A\cdot 2^{2\max(k,n)((\delta-1)N+l)}$$

therefore $\|T_k^* T_n\| \le A \cdot 2^{-\varepsilon k} 2^{-\varepsilon n}$ for all $\varepsilon > 0$ and let N be larger than $\frac{l}{1-\delta}$ so that we have $\varepsilon = (1-\delta) N - l$.

The next step is to show that all $\|T_n\|$ can be estimated by A from above. Using the Littlewood-Paley method, we have $a_n(x,\xi) = a(x,\xi)\varphi(2^{-n}\xi)$ and we denote $\breve{a}_n(x,\xi) = a_n\left(2^{-n\delta}x, 2^{n\delta}\xi\right)$. The symbol $\breve{a}_n$ satisfies the inequality

$$\left|\partial_x^\beta \partial_\xi^\alpha \breve{a}_n(x,\xi)\right| \le A_{\beta\alpha} \tag{5.8}$$

for all indices n .

A straightforward calculation shows $T_n = \Theta_n \breve{T}_n \Theta_n^{-1}$ where Θ_n is a mapping defined by $\Theta_n(f)(x) = f\left(2^{n\delta}x\right)$. Since $\|\Theta_n(f)\|_{L^2} = 2^{\frac{nl\delta}{2}} \|f\|_{L^2}$ and $\left\|\Theta_n^{-1}(f)\right\|_{L^2} = 2^{-\frac{nl\delta}{2}} \|f\|_{L^2}$, we have $\|T_k\| \le A$. Repeated the same process for even indices we obtain $\left\|\sum_{n=0,1,\ldots} T_n\right\|_{L^2} < \infty$.

Theorem 6. ***Assume kernel*** K ***satisfies conditions*** (1.2)-(1.4) ***then in order to exist a bounded operator*** $T : L^2\left(R^l\right) \to L^2\left(R^l\right)$ ***it is necessary and sufficient that there exists a constant*** $A > 0$ ***such that***

$$\int_{|x-\tilde{x}|<N} \left(\int_{\varepsilon<|x-y|<N} K(x,\ y)\, dy\right)^2 dx \le AN^l \tag{5.9}$$

holds for all $\varepsilon > 0,\ N$ ***and elements*** $\tilde{x} \in R^l$***.***

Proof. Applying the Littlewood-Paley method one can prove the following statement: if the kernel K satisfies (1.2)-(1.4) conditions then for the existence of the extension $T : L^2\left(R^l\right) \to L^2\left(R^l\right)$ it is necessary and sufficient that operators T and T^* satisfy the conditions

$\left\|T(\theta)\left(\frac{\cdot-\tilde{x}}{r}\right)\right\|_{L^2} \le Ar^{\frac{l}{2}}$ and $\left\|T^*(\theta)\left(\frac{\cdot-\tilde{x}}{r}\right)\right\|_{L^2} \le Ar^{\frac{l}{2}}$

for all $\tilde{x},\ r$, and where θ is a normalized test function for a ball $B(\tilde{x},\ r)$. The constant A does not depend on $\tilde{x},\ r$ or normalized test function θ.

First, we consider the necessity. We define an operator with a truncated kernel

$$K_\varepsilon(x,\ y) = \begin{cases} K(x,\ y), & |x-y| > \varepsilon \\ 0, & |x-y| \le \varepsilon \end{cases} \tag{5.10}$$

by

$$T_\varepsilon(f)(x) = \langle K_\varepsilon(x,\ \cdot)\, f(\cdot)\rangle \tag{5.11}$$

defined for all $f \in L^2\left(R^l\right)$.

We denote the characteristic function of the ball $\{y : |y-\tilde{x}| < N\}$ by $\chi_{\tilde{x},N}$ so we have an estimation

$$\left|\int_{\varepsilon<|x-y|<N} \left(K(x,\ y) - K_\varepsilon(x,\ y)\,\chi_{\tilde{x},N}(y)\right) dy\right| \le$$
$$\le \int_{\frac{N}{2}<|x-y|<\frac{3N}{2}} |K(x,\ y)|\, dy.$$

Since every ball can be covered by balls of half its radius, it is enough to show that

$$\int_{|x-\tilde{x}|<\frac{N}{2}} \left(\int_{\varepsilon<|x-y|<N} K\left(x,\ y\right) dy \right)^2 dx \leq AN^l, \tag{5.12}$$

so that, for $|x-\tilde{x}|<\frac{N}{2}$ we have

$$\left| \int_{\varepsilon<|x-y|<N} \left(K\left(x,\ y\right) - K_\varepsilon\left(x,\ y\right) \chi_{\tilde{x},N}\left(y\right)\right) dy \right| \leq \tilde{A}$$

since T_ε is bounded in L^2, we obtain the necessity of the condition (5.9).

Now, we are going to prove sufficiency. We define

$$\begin{aligned} &T_\varepsilon\left(f\right)\left(x\right) = \left\langle K_\varepsilon\left(x,\ \cdot\right) f\left(\cdot\right)\right\rangle = \\ &= \left\langle K\left(x,\ \cdot\right) \phi\left(\tfrac{x-\cdot}{\varepsilon}\right) f\left(\cdot\right)\right\rangle \end{aligned}$$

where we denote $K_\varepsilon\left(x,\ y\right) = K\left(x,\ y\right)\phi\left(\frac{x-y}{\varepsilon}\right)$ and $\phi \in C^\infty\left(R^l\right)$ such that

$$K_\varepsilon\left(x,\ y\right) = \begin{cases} 1, & |x| \geq 1 \\ 0, & |x| \leq \frac{1}{2}. \end{cases}$$

Let $\theta_{\tilde{x},\ r}$ be a normalized test function for a ball $B\left(\tilde{x},\ r\right)$ then we have

$$\begin{aligned} &T_\varepsilon\left(\theta_{\tilde{x},\ r}\right)\left(x\right) = \left\langle K_\varepsilon\left(x,\ \cdot\right)\theta_{\tilde{x},\ r}\left(\cdot\right)\right\rangle = \\ &= \int_{R^l} K_\varepsilon\left(x,\ y\right)\left(\theta_{\tilde{x},\ r}\left(y\right) - \theta_{\tilde{x},\ r}\left(x\right)\right)\chi_{x,3r}\left(y\right) dy + \\ &+ \theta_{\tilde{x},\ r}\left(x\right)\int_{|x-y|<3r} K_\varepsilon\left(x,\ y\right) dy \end{aligned}$$

for all $|x-\tilde{x}| \leq 2r$. We estimate

$$\begin{aligned} &\left|\int_{R^l} K_\varepsilon\left(x,\ y\right)\left(\theta_{\tilde{x},\ r}\left(y\right) - \theta_{\tilde{x},\ r}\left(x\right)\right)\chi_{x,3r}\left(y\right) dy\right| \leq \\ &\leq A\int_{|x-y|\leq 3r} |y-x|^{1-l} r^{-1} dy \leq \tilde{A} \end{aligned}$$

and

$$\left| \int_{|x-y|<3r} K_\varepsilon\left(x,\ y\right) dy - \int_{\varepsilon<|x-y|<3r} K\left(x,\ y\right) dy \right| \leq Const$$

From (5.12), we have

$$\int_{|x-\tilde{x}|<2r} \left|T_\varepsilon\left(\theta_{\tilde{x},\ r}\right)\left(x\right)\right|^2 dx \leq Ar^l$$

and the inequalities $\left\|T_\varepsilon\left(\theta_{\tilde{x},\ r}\right)\right\|_{L^2} \leq Ar^{\frac{l}{2}}$ and $\left\|T_\varepsilon^*\left(\theta_{\tilde{x},\ r}\right)\right\|_{L^2} \leq Ar^{\frac{l}{2}}$ hold for all $\varepsilon > 0$ uniformly.

If we choose a countable set $\{\varepsilon_n\}$ such that $\lim\limits_{n\to\infty} \varepsilon_n = 0$ then we obtain

$$L^2 weakly \quad \lim_{n\to\infty} T_{\varepsilon_n} = T,$$

and since $K_{\varepsilon_n}\left(x,\ y\right) \to K\left(x,\ y\right)$ pointwise, we have proven the statement of the theorem.

Theorem 7. ***For all*** $f \in L^2\left(R^l\right)$***, we define an operator*** T ***by***

$$T\left(f\right)\left(x\right) = \left\langle K\left(x - \cdot\right) f\left(\cdot\right)\right\rangle \tag{5.13}$$

with the kernel K***, which satisfies conditions*** (1.2)***,*** (1.3)***, and*** (1.4) ***where*** $\gamma = 1$ ***. Then a nonnegative Borel measure*** μ ***is absolutely continuous*** $d\mu\left(x\right) = \omega\left(x\right) dx$ ***and weight*** ω ***belongs to class*** A_p ***if and only if the following inequality***

$$\int_{R^l} \left|T\left(f\right)\left(y\right)\right|^p d\mu\left(y\right) \le A \int_{R^l} \left|\left(f\right)\left(y\right)\right|^p d\mu\left(y\right) \tag{5.14}$$

holds for all functions $f \in L^p\left(R^l\right)$***.***

Proof. The necessity is the consequence of theorem 3.

The set $C_0^\infty\left(R^l\right)$ is dense in the $L^p\left(R^l\right)$ in topology of the L^p- norm. So, we assume f is a nonnegative function of $C_0^\infty\left(R^l\right)$ with the support in the ball $B\left(\tilde{x},\, r\right)$ with the radius $r > 0$.

We denote the balls $B_1 = B\left(\tilde{x} + rz,\, r\right)$ and $B_2 = B\left(\tilde{x} - rz,\, r\right)$ where $x = rz$, $x = ru$, $|u| \le 2$. Applying conditions on K, we obtain

$$2\left|K\left(r\left(z + u\right)\right) - K\left(r\left(z\right)\right)\right| \le \left|K\left(r\left(z\right)\right)\right|,$$

we take $x = \tilde{x} + r\left(z + \widehat{x}\right)$ and $y = \tilde{x} + r\,\widehat{y}$ where we assume $\left|\widehat{x}\right| \le 1$ and $\left|\widehat{y}\right| \le 1$. So, $x - y = r\left(z + u\right)$, we note Lebesgue's measure of B by $mes\left(B\right)$, obtain that the estimation

$$2\left|K * f\right| mes\left(B\right) \ge \left|K\left(rz\right)\right| \int_B f\left(v\right) dv$$

holds for all $x \in B_1$.

Presuming the estimation

$$\int_{R^l} \left|T\left(f\right)\left(y\right)\right|^p d\mu\left(y\right) \le A \int_{R^l} \left(\left(f\right)\left(y\right)\right)^p d\mu\left(y\right),$$

we have

$$\mu\left(B_1\right)\left(\int_B f\left(v\right) dv\right)^p \le \left(mes\left(B\right)\right)^p A \int_B \left(\left(f\right)\left(y\right)\right)^p dv$$

and

$$\mu\left(B\right)\left(\int_{B_1} f\left(v\right) dv\right)^p \le \left(mes\left(B_1\right)\right)^p A \int_{B_1} \left(\left(f\right)\left(y\right)\right)^p dv.$$

The last two inequalities extend to all nonnegative functions with the support in B. Taking $f = \chi_{B_1}$, we have $\mu\left(B\right) \le A\mu\left(B_1\right)$ and therefore the estimation

$$\mu\left(B\right)\left(\int_B f\left(v\right) dv\right)^p \le \left(mes\left(B\right)\right)^p A \int_B \left(\left(f\right)\left(y\right)\right)^p dv \tag{5.15}$$

holds for all nonnegative functions $f \in L^p\left(R^l\right)$ and all balls B. Then a Borel measure μ is absolutely continuous relative to the Lebesgue measure dx so that $d\mu\left(x\right) = \omega\left(x\right) dx$ where density $\omega \in A_p$.

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