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ON WEAKLY BERWALD SPACE WITH A SPECIAL CUBIC (α, β)-METRIC

Brijesh Kumar Tripathi and Sadika Khan

Abstract. In 1997, S. Basco and M. Matsumoto introduced the concepts of Douglas spaces with a generalization of the notion of Berwald spaces. Weakly-Berwald space is one of the generalizations of Berwald spaces. In the present paper, we considered a cubic (α, β)-metric which is a special class of p-power Finsler metric and obtained the conditions under which the Finsler space with such special metric will be a Weakly-Berwald space.

1 Introduction

Let M be an n-dimensional C^∞ manifold. T_xM denotes the tangent space of M at x. The tangent bundle of M is the union of tangent spaces $TM = \cup_{x \epsilon M} T_xM$. We denote the elements of TM by (x, y), where $y \, \epsilon \, T_xM$. Let $TM_0 = TM - \{0\}$.

Definition 1. *A Finsler Metric on M is a function $L : TM \to [0, \infty)$ with the following properties:*

1. *L is C^∞ on TM_0*
2. *L is positively 1-homogeneous on the fibers of tangent bundle TM, and*
3. *the Hessian of L^2 with element $g_{ij} = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j}$ is regular on TM_0, i.e., det $(g_{ij}) \neq 0$*

The pair (M, L) is then called a Finsler space. L is called fundamental function and g_{ij} is called fundamental tensor.

Now, We define G_i as [9]

$$G_i = \left\{ y^h (\partial_h \dot{\partial}_i L^2) - \partial_i L^2 \right\} /4,$$

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and $G^i = g^{ij}G_j$ where the symbol ∂_i, $\dot{\partial}_i$ means $\frac{\partial}{\partial x^i}$, $\frac{\partial}{\partial y^i}$ and (g^{ij}) is the inverse matrix of (g_{ij}). The coefficients (G^i_{jk}, G^i_j) of the Berwald connection $B\Gamma$ are defined as $G^i_j = \dot{\partial}_j G^i$ and $G^i_{jk} = \dot{\partial}_k G^i_j$. A Berwald space is a Finsler space which satisfies the condition $G^h_{ijk} = 0$, that is to say, whose coefficients G^h_{ij} of the Berwald connection are functions of the position expressed by the local coordinates of (x^i) alone. Therefore the equation $y_h G^h_{ijk} = 0$ holds, so $2G^i = G^i_{jk}y^j y^k$ are homogeneous polynomials in (y^i) of degree two, so $D^{ij} = G^i y^j - G^j y^i$ are homogeneous polynomials in (y^i) of degree three.

Definition 2. *A Finsler space is called Berwald space if the Berwald connection* $B\Gamma = (G^i_{jk}, G^i_j, 0)$ *is linear.*

Definition 3. *A spray is called weakly affine spray if the* (hv)*-Ricci curvature tensor* $G_{ij(k)} = 0$ *where* $G_{ij(k)} = G^r_{rij(k)}$, *and the* (hv)*-curvature tensor is given by* $G^h_{ijk} = G^h_{ij(k)}$.

Definition 4. *A Finsler is said to be a Weakly-Berwald space if the* (hv)*-horizontal vertical Ricci curvature tensor of Berwald connection vanishes. i.e.,* $G_{ij(k)} = 0$.

Moreover, the functions G^i of a Finsler space with an (α,β)-metric is given by $2G^i = \gamma^i_{00} + 2B^i$, where γ^i_{jk} stands for the Christoffel symbols in the associated Riemannian space $R^n = (M^n, \alpha)$ with $F^n = (M^n, L(\alpha,\beta))$ and subscript 0 means a contraction by y^i. Then we have $G^i_j = \gamma^i_{0j} + B^i_j$ and $G^i_{jk} = \gamma^i_{jk} + B^i_{jk}$, where $\dot{\partial}_j B^i = B^i_j$ and $\dot{\partial}_k B^i_j = B^i_{jk}$. Hence a Finsler space with an (α,β)-metric is a weakly-Berwald space, iff $B^m_m = \frac{\partial B^m}{\partial y^m}$ is a one-form [9]. i.e., $B^m_m = \frac{\partial B^m}{\partial y^m}$ is a homogeneous polynomial in (y^i) of degree one. Further M. Matsumoto investigated that a Finsler space with an (α,β)-metric is a weakly-Berwald space, iff B^m are homogeneous polynomials in (y^i) of degree two [12].

S. Basco and B. Szilagi [3] in 2002 define the concept of Weakly-Berwald space as a generalization of Berwald spaces as well as investigated a necessary and sufficient condition for the existence of Weakly-Berwald space of Kropina type. Further S. Basco and R.Yoshikawa [4] found the conditions for Randers and Kropina spaces to be Weakly-Berwald spaces. Also, the conditions for generalized Kropina and Matsumoto spaces to be Weakly-Berwald spaces and Berwald spaces, too have been studied by R.Yoshikawa and K. Okubo [19]. Afterwards, the conditions for Finsler spaces to be Weakly Berwald space with different special kind of (α,β)-metrics have been studied by many authors [9, 17, 14, 15, 16, 6]

As a generalization of Rander's metric $L = \alpha + \beta$, M. Matsumoto [11] in 1972 introduced the concept of (α,β)-metrics in Finsler spaces where α is a regular Riemannian metric $\alpha = \sqrt{a_{ij}(x)y^i y^j}$, i.e., $det(a_{ij}) \neq 0$ and β is a one-form $\beta =$

$b_i(x)y^i$ and it has been studied by many authors [13, 10, 18, 7, 8, 17, 14]. Some important examples of (α,β)-metrics are namely Kropina metric $L=\frac{\alpha^2}{\beta}$, generalized Kropina metric $L=\frac{\alpha^{(n+1)}}{\beta^n}$, $(n\neq 0,-1)$ and Matsumoto metric $\frac{\alpha^2}{\alpha-\beta}$. The study of Finsler spaces with these metrics has greatly contributed to the growth of Finsler geometry and its applications to the theory of Relativity and allied areas [1]

Definition 5. *A Finsler metric $L=(\alpha,\beta)$ on a differential manifold M^n is called an (α,β)-metric, if L is a positively homogeneous function of degree one in α and β.*

A generalized form of an (α,β)-metric on an n-dimensional manifold M^n defined as

$$F=\alpha\left(1+\frac{\beta}{\alpha}\right)^p \tag{1.1}$$

is known as the class p-power (α,β)- metrics [18], where $p\neq 0$ is a real constant. If $p=1$ then equation (1.1) reduces to Rander's metric which has important and interesting curvature properties and firstly introduced by Ingarden in 1957. If $p=2$ then it becomes square metric and it is also known as Z. Shen's square metric. If $p=-1$, it reduces to Matsumoto metric which can be used in measurement of slope of a mountain and so on.

In the present paper we considered p = 3 in equation (1.1) and get special class of (α,β)- metric in the form of

$$L=\frac{(\alpha+\beta)^3}{\alpha^2} \tag{1.2}$$

and named as cubic (α,β)- metric in an n-dimensional manifold M^n and a n-dimensional Finsler space F^n equipped with this cubic (α,β)-metric is known as Finsler space with cubic (α,β)- metric. Further we investigate the conditions under which the Finsler space with this cubic (α,β)-metric will be Weakly-Berwald space.

2 Preliminaries

Let $F^n=(M^n,L)$ be an n-dimensional Finsler space equipped with an (α,β)-metric $L(\alpha,\beta)$. In this paper, the symbol (;) stands for h-covariant derivation with respect to the Riemannian connection in the associated Riemmanian space (M^n,α) and γ^i_{jk} denote the Christoffel symbols in the space (M^n,α).

We use the following notations [9].

$$r_{ij}=\tfrac{1}{2}\{b_{i;j}+b_{j;i}\},\quad r^i_j=a^{ih}r_{hj},\quad r_j=b_ir^i_j$$

$$s_{ij} = \frac{1}{2}\{b_{i;j} - b_{j;i}\}, \quad s^i_j = a^{ih}s_{hj}, \quad s_j = b_i s^i_j, \quad b^i = a^{ih}b_h, \quad b^2 = b^i b_i.$$

Now, we consider the function $G^i(x,y)$ of F^n with an (α,β)-metric. According to [9], they are written in the form

$$2G^m = \gamma^m_{00} + 2B^m,$$

where

$$B^m = \left(\frac{E^*}{\alpha}\right)y^m + \left(\frac{\alpha L_\beta}{L_\alpha}\right)s^m_0 - \left(\frac{\alpha L_{\alpha\alpha}}{L_\alpha}\right)C^*\left\{\left(\frac{y^m}{\alpha}\right) - \left(\frac{\alpha}{\beta}\right)b^m\right\} \tag{2.1}$$

where

$$E^* = \left(\frac{\beta L_\beta}{L}\right)C^*, \quad C^* = \frac{\alpha\beta(r_{00}L_\alpha - 2\alpha s_0 L_\beta)}{2(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})}, \quad \gamma^2 = b^2\alpha^2 - \beta^2 \tag{2.2}$$

Differentiating (2.1) by y^n and contracting m and n in the obtained equation, we have

$$\begin{aligned} B^m_m = &\left[\dot\partial_m\left(\frac{\beta L_\beta}{\alpha L}\right)y^m + \frac{n\beta L_\beta}{\alpha L} - \dot\partial_m\left(\frac{\alpha L_{\alpha\alpha}}{L_\alpha}\right)\left(\frac{\beta y^m - \alpha^2 b^m}{\alpha\beta}\right)\right]C^* \\ &- \frac{\alpha L_{\alpha\alpha}}{L_\alpha}\left[\dot\partial_m\left(\frac{1}{\alpha}\right)y^m + \frac{1}{\alpha}\delta^m_m - \dot\partial_m\left(\frac{\alpha}{\beta}\right)b^m\right]C^* \\ &+ \left(\frac{\beta L_\alpha L_\beta - \alpha L L_{\alpha\alpha}}{\alpha L L_{\alpha\alpha}}\right)\left(\dot\partial_m C^*\right)y^m \\ &+ \left(\frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha}\right)\left(\dot\partial_m C^*\right)b^m + \dot\partial_m\left(\frac{\alpha L_\beta}{L_\alpha}\right)s^m_0. \end{aligned} \tag{2.3}$$

Since $L = L(\alpha,\beta)$ is a positively homogeneous function of α and β of degree one, we have

$$L_\alpha\alpha + L_\beta\beta = L, \quad L_{\alpha\alpha}\alpha + L_{\alpha\beta}\beta = 0,$$
$$L_{\beta\alpha}\alpha + L_{\beta\beta}\beta = 0, \quad L_{\alpha\alpha\alpha}\alpha + L_{\alpha\alpha\beta}\beta = -L_{\alpha\alpha}.$$

From the above and the homogeneity of (y^i), we have the following terms:

$$\dot\partial_m\left(\frac{\beta L_\beta}{\alpha L}\right)y^m = -\frac{\beta L_\beta}{\alpha L}, \tag{2.4}$$

$$\dot\partial_m\left(\frac{\alpha L_{\alpha\alpha}}{L_\alpha}\right)\left(\frac{\beta y^m - \alpha^2 b^m}{\alpha\beta}\right) = \frac{\gamma^2}{(\beta L_\alpha)^2}\left\{L_\alpha L_{\alpha\alpha} + \alpha L_\alpha L_{\alpha\alpha\alpha} - \alpha(L_{\alpha\alpha})^2\right\}, \tag{2.5}$$

$$\left[\dot\partial_m\left(\frac{1}{\alpha}\right)y^m + \frac{1}{\alpha}\delta^m_m - \dot\partial_m\left(\frac{\alpha}{\beta}\right)b^m\right] = \frac{1}{\alpha\beta^2}\left\{\gamma^2 + (n-1)\beta^2\right\}, \tag{2.6}$$

$$(\dot{\partial}_m C^*) y^m = 2C^*, \tag{2.7}$$

$$(\dot{\partial}_m C^*) b^m = \frac{1}{2\alpha\beta\Omega^2}[\Omega\left\{\beta(\gamma^2+2\beta^2)W + 2\alpha^2\beta^2 L_\alpha r_0 - \alpha\beta\gamma^2 L_{\alpha\alpha} r_{00} - 2\alpha(\beta^3 L_\beta + \alpha^2\gamma^2 L_{\alpha\alpha})s_0\right\} \\ -\alpha^2\beta W\left\{2b^2\beta^2 L_\alpha - \gamma^4 L_{\alpha\alpha\alpha} - b^2\alpha\gamma^2 L_{\alpha\alpha}\right\}], \tag{2.8}$$

$$\dot{\partial}_m\left(\frac{\alpha L_\beta}{L_\alpha}\right) s_0^m = \frac{\alpha^2 L L_{\alpha\alpha} s_0}{(\beta L_\alpha)^2}, \tag{2.9}$$

where

$$\begin{aligned} &W = (r_{00} L_\alpha - 2\alpha s_0 L_\beta), \\ &\Omega = (\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha}), \ \ \textit{provided that} \ \ \Omega \neq 0. \\ &Y_i = a_{ir} y^r, \ \ s_{00} = 0, \ \ b^r s_r = 0, \ \ a^{ij} s_{ij} = 0. \end{aligned} \tag{2.10}$$

Substituting (2.4)-(2.9) into (2.3), we have

$$B^m_m = \frac{1}{2\alpha L(\beta L_\alpha)^2\Omega^2}\left\{2\Omega^2 AC^* + 2\alpha L\Omega^2 B s_0 + \alpha^2 L L_\alpha L_{\alpha\alpha}(Cr_{00} + Ds_0 + Er_0)\right\} \tag{2.11}$$

where

$$\begin{aligned} &A = (n+1)\beta^2 L_\alpha(\beta L_\alpha L_\beta - \alpha L L_{\alpha\alpha}) + \alpha\gamma^2 L\left\{\alpha(L_{\alpha\alpha})^2 - 2L_\alpha L_{\alpha\alpha} - \alpha L_\alpha L_{\alpha\alpha\alpha}\right\}, \\ &B = \alpha^2 L L_{\alpha\alpha}, \\ &C = \beta\gamma^2\left\{-\beta^2(L_\alpha)^2 + 2b^2\alpha^3 L_\alpha L_{\alpha\alpha} - \alpha^2\gamma^2(L_{\alpha\alpha})^2 + \alpha^2\gamma^2 L_\alpha L_{\alpha\alpha\alpha}\right\}, \\ &D = 2\alpha\left\{\beta^3(\gamma^2-\beta^2)L_\alpha L_\beta - \alpha^2\beta^2\gamma^2 L_\alpha L_{\alpha\alpha} - 2\alpha\beta\gamma^2(\gamma^2+2\beta^2)L_\beta L_{\alpha\alpha} \right. \\ &\quad \left. -\alpha^3\gamma^4(L_{\alpha\alpha})^2 - \alpha^2\beta\gamma^4 L_\beta L_{\alpha\alpha\alpha}\right\} \\ &E = 2\alpha^2\beta^2 L_\alpha\Omega \end{aligned} \tag{2.12}$$

Thus we have [9]

Theorem 6. *The necessary and sufficient condition for a Finsler space F^n with an (α, β)-metric to be Weakly-Berwald space is that $G_m = \gamma^m_{0m} + B^m_m$ and B^m_m is a homogeneous polynomial in (y^m) of degree one, where B^m_m is given by (2.11), provided that $\Omega \neq 0$.*

Lemma 7. *[5] If α^2 contains β as a factor, then the dimension is equal to two and $b^2 = 0$. Throughout this paper, we assume that the dimension is more than two and $b^2 \neq 0$, that is, $\alpha^2 \not\equiv 0 (mod \beta)$.*

3 Finsler space with special cubic metric $L = \frac{(\alpha+\beta)^3}{\alpha^2}$

In this section, we find the condition for a Finsler space F^n with a special cubic (α, β)-metric (1.2) to be a Weakly-Berwald space.

The partial derivative with respect to α and β of (1.2) are given by

$$\begin{aligned} &L_\alpha = \frac{(\alpha+\beta)^2(\alpha-2\beta)}{\alpha^3}, \quad L_\beta = \frac{3(\alpha+\beta)^2}{\alpha^2}, \quad L_{\alpha\beta} = L_{\beta\alpha} = -\frac{6\beta(\alpha+\beta)}{\alpha^3} \\ &L_{\alpha\alpha} = \frac{6\beta^2(\alpha+\beta)}{\alpha^4}, \quad L_{\beta\beta} = \frac{6(\alpha+\beta)}{\alpha^2} \quad \textit{and} \quad L_{\alpha\alpha\alpha} = -\frac{6\beta^2(3\alpha+4\beta)}{\alpha^5} \end{aligned} \tag{3.1}$$

where $L_\alpha = \frac{\partial L}{\partial \alpha}$, $L_\beta = \frac{\partial L}{\partial \beta}$, $L_{\alpha\beta} = \frac{\partial L_\alpha}{\partial \beta}$ $L_{\alpha\alpha} = \frac{\partial L\alpha}{\partial \alpha}$, $L_{\beta\beta} = \frac{\partial L_\beta}{\partial \beta}$, $L_{\alpha\alpha\alpha} = \frac{\partial L_{\alpha\alpha}}{\partial \alpha}$.

Since $L(\alpha, \beta) = \frac{(\alpha+\beta)^3}{\alpha^2}$ is a positively homogeneous function of α and β of degree one, we have

$$L_\alpha \alpha + L_\beta \beta = \frac{(\alpha+\beta)^2(\alpha-2\beta)}{\alpha^2} + \frac{3\beta(\alpha+\beta)^2}{\alpha^2} = \frac{(\alpha+\beta)^3}{\alpha^2} = L(\alpha, \beta),$$

$$L_{\alpha\alpha}\alpha + L_{\alpha\beta}\beta = \frac{6\beta^2(\alpha+\beta)}{\alpha^3} - \frac{6\beta^2(\alpha+\beta)}{\alpha^3} = 0,$$

$$L_{\beta\alpha}\alpha + L_{\beta\beta}\beta = -\frac{6\beta(\alpha+\beta)}{\alpha^2} + \frac{6\beta(\alpha+\beta)}{\alpha^2} = 0,$$

and

$$L_{\alpha\alpha\alpha}\alpha + L_{\alpha\alpha\beta}\beta = -\frac{6\beta^2(3\alpha+4\beta)}{\alpha^4} + \frac{6\beta(2\alpha+3\beta)}{\alpha^4} = -\frac{6\beta^2(\alpha+\beta)}{\alpha^4} = -L_{\alpha\alpha}(\alpha, \beta)$$

Substituting (3.1) into (2.1), (2.2), (2.10) and (2.12), we have

$$\begin{aligned} B^m =& \frac{3(\alpha+\beta)\left\{r_{00}(\alpha-2\beta) - 6\alpha^2 s_0\right\}}{\{\alpha^2(1+6b^2) - \alpha\beta - 8\beta^2\}} \left[\left(\frac{1}{2} - \frac{\beta}{(\alpha-2\beta)}\right) y^m + \left(\frac{\alpha^2}{(\alpha-2\beta)}\right) b^m\right] \\ &+ \left(\frac{3\alpha^2}{(\alpha-2\beta)}\right) s_0^m. \end{aligned} \tag{3.2}$$

$$
\begin{aligned}
A &= \frac{3\beta^2(\alpha+\beta)^5}{\alpha^8}\left[(n+1)(\alpha+\beta)(\alpha-2\beta)(\alpha-4\beta)\beta+2(\alpha^2b^2-\beta^2)(\alpha^2-3\beta^2)\right],\\
B &= \frac{6\beta^2(\alpha+\beta)^4}{\alpha^4},\\
C &= \frac{\beta^3\gamma^2(\alpha+\beta)^2}{\alpha^6}\left[-(\alpha+\beta)^2(\alpha-2\beta)^2+12b^2\alpha^2(\alpha-2\beta)(\alpha+\beta)-6\gamma^2\beta^2-6\gamma^2(\alpha-2\beta)(3\alpha+4\beta)\right],\\
D &= \frac{6\beta^3(\alpha+\beta)^2}{\alpha^4}\left[(\alpha^2b^2-2\beta^2)(\alpha-2\beta)(\alpha+\beta)^2-2\beta(\alpha^2b^2-\beta^2)(\alpha+\beta)(\alpha-2\beta)\right.\\
&\quad\left.-12(\alpha^2b^2-\beta^2)(\alpha^2b^2+\beta^2)(\alpha+\beta)-2(\alpha^2b^2-\beta^2)^2\beta+6(\alpha^2b^2-\beta^2)^2(3\alpha+4\beta)\right],\\
E &= \frac{2\beta^4(\alpha+\beta)^3(\alpha-2\beta)}{\alpha^4}\left[\alpha^2(1+6b^2)-\alpha\beta-8\beta^2\right],\\
\Omega &= \frac{\beta^2(\alpha+\beta)}{\alpha^3}\left[\alpha^2(1+6b^2)-\alpha\beta-8\beta^2\right],\\
C^* &= \frac{\alpha(\alpha+\beta)\left\{r_{00}(\alpha-2\beta)-6\alpha^2s_0\right\}}{2\beta\left\{\alpha^2(1+6b^2)-\alpha\beta-8\beta^2\right\}}.
\end{aligned}
\tag{3.3}
$$

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Substituting (3.3) into (2.11), we have

$$
\begin{aligned}
&\left\{\alpha^6\beta^2(-216b^4-96b^2-10)+\alpha^4\beta^4(288b^4+672b^2+130)+\alpha^2\beta^6(-768b^2-504)+512\beta^8\right.\\
&\left.+\alpha^7\beta(72b^4+24b^2+2)+\alpha^5\beta^3(-120b^2-18)+\alpha^3\beta^5(-96b^2+16)+128\alpha\beta^7\right\}B^m_m\\
&+\left\{\alpha^6\beta(-6b^2(3n+4)-3(n+1))+\alpha^4\beta^3(-216b^2n-18b^2-33n-27)\right.\\
&+\alpha^2\beta^5(288b^2n-46b^2+348n+150)+\beta^7(-384n-120)+\alpha^5\beta^2(126b^2n+78b^2+24(n+1))\\
&\left.+\alpha^3\beta^4(-72b^2n-96b^2-144n-96)+\alpha\beta^6(48n+72)\right\}r_{00}\\
&+\left\{\alpha^6\beta^2(-360b^4-540b^2n+36b^2-108n-60)+\alpha^4\beta^4(864b^2n+504b^2+828n+132)\right.\\
&+\alpha^2\beta^6(-1152n-336)+\alpha^7\beta(-360b^4+108b^2n+72b^2+18n+6)\\
&\left.+\alpha^5\beta^3(216b^2n+468b^2-18n+6)+\alpha^3\beta^5(-432n-278)\right\}s_0\\
&+\left\{\alpha^6\beta^2(216b^2+48)+\alpha^4\beta^4(-288b^2-336)+384\alpha^2\beta^6+\alpha^7\beta(-72b^2-12)\right.\\
&\left.+60\alpha^5\beta^3+48\alpha^3\beta^5\right\}r_0=0
\end{aligned}
\tag{3.4}
$$

Suppose that F^n is a Weakly-Berwald space, i.e., B^m_m is $hp(1)$. Since α is irrational in (y^i), the equation (3.4) is divided into two equations as follows

$$\beta A_1B^m_m+B_1r_{00}+\alpha^2\beta C_1s_0+\alpha^2\beta D_1r_0=0 \tag{3.5}$$

$$A_2B^m_m+\beta B_2r_{00}+\alpha^2C_2s_0+\alpha^2D_2r_0=0 \tag{3.6}$$

where

$$\begin{aligned}
A_1 &= \alpha^6(-216b^4 - 96b^2 - 10) + \alpha^4\beta^2(288b^4 + 672b^2 + 130) + \alpha^2\beta^4(-768b^2 - 504) + 512\beta^6\\
A_2 &= \alpha^6(72b^4 + 24b^2 + 2) + \alpha^4\beta^2(-120b^2 - 18) + \alpha^2\beta^4(-96b^2 + 16) + 128\beta^6\\
B_1 &= \alpha^6(-6b^2(3n+4) - 3(n+1)) + \alpha^4\beta^2(-216b^2n - 18b^2 - 33n - 27)\\
&\quad + \alpha^2\beta^4(288b^2n - 46b^2 + 348n + 150) + \beta^6(-384n - 120)\\
B_2 &= \alpha^4(126b^2n + 78b^2 + 24(n+1)) + \alpha^2\beta^2(-72b^2n - 96b^2 - 144n - 96) + \beta^4(48n + 72)\\
C_1 &= \alpha^4(-360b^4 - 540b^2n + 36b^2 - 108n - 60) + \alpha^2\beta^2(864b^2n + 504b^2 + 828n + 132)\\
&\quad + \beta^4(-1152n - 336)\\
C_2 &= \alpha^4(-360b^4 + 108b^2n + 72b^2 + 18n + 6) + \alpha^2\beta^2(216b^2n + 468b^2 - 18n + 6)\\
&\quad + \beta^4(-432n - 278)\\
D_1 &= \alpha^4(216b^2 + 48) + \alpha^2\beta^2(-288b^2 - 336) + 384\beta^4\\
D_2 &= \alpha^4(-72b^2 - 12) + 60\alpha^2\beta^2 + 48\beta^4.
\end{aligned}$$

Eliminating B^m_m from the equations (3.5) and (3.6), we have

$$Rr_{00} + \alpha^2\beta S s_0 + \alpha^2\beta T r_0 = 0 \tag{3.7}$$

where

$$R = A_2B_1 - \beta^2A_1B_2, \quad S = A_2C_1 - A_1C_2, \quad T = A_2D_1 - A_1D_2$$

Since only the term $\left\{(72b^4 + 24b^2 + 2)(-6b^2(3n+4) - 3(n+1))\right\}\alpha^{12}r_{00}$ of Rr_{00} in (3.7) does not contain β, we must have $hp(12)V_{12}$, such that

$$\alpha^{12}r_{00} = \beta V_{13} \tag{3.8}$$

Since we are concerned with $\alpha^2 \not\equiv 0(mod\beta)$ and $b^2 \neq 0$. (3.8) shows the existence of the function V^1 satisfying $V_{13} = V^1\alpha^{12}$, and hence $r_{00} = V^1\beta$. Then (3.7) reduces to

$$RV^1 + \alpha^2 S s_0 + \alpha^2 T r_0 = 0 \tag{3.9}$$

Only the term $\{128(-384n - 120) - 512(48n + 72)\}\,\beta^{12}V^1$ of the above equation (3.9) seemingly does not contain α^2, hence we must have $hp(11)V_{11}$ such that

$$\{128(-384n - 120) - 512(48n + 72)\}\,\beta^{12}V^1 = \alpha^2 V_{11}$$

Since $\alpha^2 \not\equiv 0(mod\beta)$, we have $V_{11} = 0$, i.e., $V^1 = 0$. Hence we obtain $r_{00} = 0$,

$r_{ij} = 0$, and $r_0 = 0$, $r_j = 0$. Substituting $V^1 = 0$, $r_0 = 0$ in (3.9), we get $Ss_0 = 0 \Rightarrow s_0 = 0$; $s_j = 0$ [Since $S \neq 0$].

Conversely, substituting $r_{00} = 0$, $s_0 = 0$ and $r_0 = 0$ into (3.4), we have $B^m_m = 0$, that is, the Finsler space with the cubic (α, β)-metric (1.2) is a Weakly-Berwald space.

On the other hand, we suppose that the Finsler space with the cubic (α, β)-metric (1.2) is a Berwald space then we have $r_{00} = 0$, $s_0 = 0$ and $r_0 = 0$. Because the space is a Weakly-Berwald space from the above discussion. Substituting above into (3.2), we have $B^m = 0$, i.e., the Finsler space with the cubic (α, β)-metric (1.2) is a Berwald space. Hence $s_{ij} = 0$ holds good.

Theorem 8. *A Finsler space with the cubic (α, β)-metric (1.2) is a Weakly-Berwald space if and only if $r_{ij} = 0$ and $s_j = 0$.. Also, a Finsler space with the cubic (α, β)-metric (1.2) is Berwald space if and only if $r_{ij} = 0$ and $s_{ij} = 0$.*

4 Conclusion

In the present paper, we have study a Finsler space where the (hv)-Ricci tensor $G_{ij(k)}$ vanishes but the (hv)-curvature tensor G^h_{ijk} does not always equals to zero investigated by S. Basco and B. Szilagi [3]. Further, we characterize a special cubic (α, β)-metric which is a special class of p-power Finsler metric in the form of $L = \frac{(\alpha+\beta)^3}{\alpha^2}$ and named it as cubic (α, β)-metric. The main aim of this paper is to study the conditions for the Finsler space F^n with special (α, β)-metric to be a Weakly-Berwald space and the conditions for the Finsler space with a special cubic (α, β)-metric (1.2) to be Berwald space is also found.

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Brijesh Kumar Tripathi
Department of Mathematics, L.D. College of Engineering Ahmedabad,
Gujarat, 380015, India.
E-mail: brijeshkumartripathi4@gmail.com

Sadika Khan
Research Scholar, Science_Mathematics Branch, Gujarat Technological University, Ahmedabad
Gujarat, 382424, India.
E-mail: sadikakhan013@gmail.com

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