DOUBLE CALCULUS

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Abstract. We present a streamlined, slightly modified version, in the two-variable situation, of a beautiful, but not so well known, theory by Bögel [1, 2], already from the 1930s, on an alternative higher dimensional calculus of real functions, a *double calculus*, which includes many two-variable extensions of classical results from single variable calculus, such as Rolle's theorem, Lagrange's mean value theorem, Cauchy's mean value theorem, Fermat's extremum theorem, the first derivative test, and the first and second fundamental theorems of calculus.

1 Introduction

The motivation for this article comes from the trivial observation that the difference operator Δ is connected to both the the derivative and the integral. To be more precise, if f is a single variable real function, then f has derivative

$$f'(a) = \lim_{b \to a} \frac{\Delta_a^b(f)}{b - a}$$

at a, if the limit exists exists, where $\Delta_a^b(f)$ denotes $f(b) - f(a)$, and if f is continuous, then, by the fundamental theorem of calculus,

$$\int_a^b f(x) \ dx = \Delta_a^b(F),$$

where F is a primitive function of f. A naive interpretation of this connection is that it should, in theory, be possible to obtain higher-dimensional analogues of the fundamental theorem of calculus by first defining a suitable difference operator given by the multiple integral, over suitable domains, and then, by reverse engineering, use this difference operator to define a derivative so that the fundamental theorem of calculus holds.

It may come as a surprise to some readers, and it certainly did so for the author of the present article, that Bögel [1, 2] already in the 1930s showed that it is indeed

possible, in any finite number of variables, to successfully carry out such a program. Since this theory should be of high interest to students and instructors of calculus in several variables, we have in this article produced an accessible and streamlined, slightly modified version, of this program in the case of two variables, that is, a theory of *double calculus*. Note that our notation, definitions, results and proofs, at times, somewhat differs from the approach by Bögel [1, 2]. In the presentation we therefore carefully point out whenever that happens.

The domains that Bögel considers are the natural two-dimensional analogues of intervals, namely *double intervals* $[a, b] = [a_1, b_1] \times [a_2, b_2]$ where $a = (a_1, a_2)$ and $b = (b_1, b_2)$ are points in $\mathbb{R}^2$ with $a_1 < b_1$ and $a_2 < b_2$. If $f : [a, b] \to \mathbb{R}$ is a continuous two-variable function, then, by iterated integration, it follows that $\iint_{[a,b]} f(x) \, dx_1 dx_2 = F(b_1, b_2) - F(b_1, a_2) - F(a_1, b_2) + F(a_1, a_2)$ where F is a two-variable function with the property that the iterated partial derivatives F_{12} and F_{21} exist and are equal to f on $[a, b]$. Therefore he defines the *double difference operator* by $\Delta_a^b(f) = f(b_1, b_2) - f(b_1, a_2) - f(a_1, b_2) + f(a_1, a_2)$ and he defines the *double derivative* of f at a by $f'(a) = \lim_{x_1 \to a_1; x_2 \to a_2} \Delta_a^b(f)/((b_1 - a_1)(b_2 - a_2))$ when it exists.

It turns out that the class of double differentiable functions thus obtained is much richer than it's one-dimensional counterpart. In fact, the class of double differentiable functions contains many examples of functions that are not partially differentiable and, in some cases, not even continuous. Indeed, if we pick *any* functions $g, h : \mathbb{R} \to \mathbb{R}$ and define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x) = g(x_1) + h(x_2)$ for $x \in \mathbb{R}^2$, then $\Delta_a^b(f) = 0$ for all $a, b \in \mathbb{R}^2$ and hence f is *double constant*, that is f is double differentiable with $f' = 0$. This has the unpleasant consequence that double differentiable functions may be *unbounded* on compact subsets of $\mathbb{R}^2$. Therefore, we can not expect to find a two-dimensional version of the Weierstrass extreme value theorem to hold within this framework.

Here is a detailed outline of this article.

In Section 2, we fix the notation concerning double intervals and double functions. We also show some elementary results that we need in subsequent sections.

In Section 3, we define the class of double continuous functions. We show that a function which is double continuous on an interval is automatically *globally* double continuous on that double interval. We also show a continuity result concerning the double difference map that we need in the following sections.

In Section 4, we define (signed) double limits and (signed) double derivatives. In analogue with the single variable situation, we show that double differentiable functions are double continuous. Then we show double versions of Rolle's theorem, the mean value theorem, Fermat's theorem and the first derivative test. At the end of this section, we introduce double primitive functions.

In Section 5, we define the double Newton integral. Using the double mean value theorem, we obtain a mean value theorem for double Newton integrals. After that, we connect the double Newton integral to the Riemann double integral in the first

and second double fundamental theorems of calculus. At the end of this section, we introduce improper double Newton integrals. We also discuss some examples of double integrals over non-rectangular regions.

In Section 6, we discuss Bögels extensions of the results in this article to higher dimensions, that is triple calculus, quadruple calculus and beyond. We also discuss the possibility for a higher-dimensional version of Schwarz's theorem and Darboux's theorem.

2 Double functions

In this section, we fix the notation concerning double intervals and double functions. We also show some elementary results that we need in subsequent sections (see Proposition 1 and Proposition 2).

Let $\mathbb{N}$ denote the set of positive integers. We let $\mathbb{R}$ denote the set of real numbers and we put $\mathbb{R}^2 := \mathbb{R} \times \mathbb{R}$. Let $\mathbb{R}_+$ and $\mathbb{R}_-$ denote the set of positive real numbers and the set of negative real numbers respectively; we put $\mathbb{R}^2_{++} := \mathbb{R}_+ \times \mathbb{R}_+$, $\mathbb{R}^2_{+-} := \mathbb{R}_+ \times \mathbb{R}_-$, $\mathbb{R}^2_{-+} := \mathbb{R}_- \times \mathbb{R}_+$ and $\mathbb{R}^2_{--} := \mathbb{R}_- \times \mathbb{R}_-$.

Suppose that $a \in \mathbb{R}^2$. We write $a = (a_1, a_2)$ for $a_1, a_2 \in \mathbb{R}$. More generally, if $A \subseteq \mathbb{R}^2$, then we put $A_1 = \{a_1 \mid a \in A\}$ and $A_2 = \{a_2 \mid a \in A\}$. Suppose that $b \in \mathbb{R}^2$. Then we write $a \sim b$ if $a_1 = b_1$ or $a_2 = b_2$; $a \nsim b$ if $a_1 \neq b_1$ and $a_2 \neq b_2$; $a < b$ if $a_1 < b_1$ and $a_2 < b_2$; $a \leq b$ if $a_1 \leq b_1$ and $a_2 \leq b_2$.

By a *double interval* we mean a subset I of $\mathbb{R}^2$ of the form $I_1 \times I_2$ where I_1 and I_2 are intervals in $\mathbb{R}$. If I_1 and I_2 are open (closed, compact), then we say that I is *double open (double closed, double compact)*. If $a, b \in \mathbb{R}^2$ and $a \leq b$, then we put $(a, b) = \{x \in \mathbb{R}^2 \mid a < x < b\}$ and $[a, b] = \{x \in \mathbb{R}^2 \mid a \leq x \leq b\}$.

By a *double δ-neighbourhood of a* we mean a set of the form $D(a, \delta) := (a - \delta, a + \delta)$ for some $\delta \in \mathbb{R}^2_{++}$. The *signed* double δ-neighbourhoods $D_{++}(a, \delta)$, $D_{-+}(a, \delta)$, $D_{--}(a, \delta)$ and $D_{+-}(a, \delta)$ are respectively defined as $[a_1, a_1 + \delta_1) \times [a_2, a_2 + \delta_2)$, $(a_1 - \delta_1, a_1] \times [a_2, a_2 + \delta_2)$, $(a_1 - \delta_1, a_1] \times (a_2 - \delta_2, a_2]$ and $[a_1, a_1 + \delta_1) \times (a_2 - \delta_2, a_2]$. By a *double punctured δ-neighbourhood of a* we mean a set of the form $P(a, \delta) := \{x \in D(a, \delta) \mid x \nsim a\}$ for some $\delta \in \mathbb{R}^2_{++}$. The *signed* double punctured δ-neighbourhoods $P_{++}(a, \delta)$, $P_{-+}(a, \delta)$, $P_{--}(a, \delta)$ and $P_{+-}(a, \delta)$ are defined as the intersection of $P(a, \delta)$ with, respectively, $D_{++}(a, \delta)$, $D_{-+}(a, \delta)$, $D_{--}(a, \delta)$ and $D_{+-}(a, \delta)$.

Let f denote a real-valued function with a domain $D(f)$ which is a subset of $\mathbb{R}^2$. In that case, we say that f is a *double function*. If $a, b \in \mathbb{R}^2$ are chosen so that $(b_1, b_2), (b_1, a_2), (a_1, b_2), (a_1, a_2) \in D(f)$, then we define the *double difference of f from a to b* as the real number $\Delta_a^b(f) := f(b_1, b_2) - f(b_1, a_2) - f(a_1, b_2) + f(a_1, a_2)$. If I is a double interval contained in $D(f)$, then we say that f is *double constant* on I if $\Delta_a^b(f) = 0$ for all $a, b \in I$.

Proposition 1. *Suppose that f is a double function which is defined on a double interval I. Then f is double constant on I if and only if there are functions $g : I_1 \to$*

$\mathbb{R}$ and $h : I_2 \to \mathbb{R}$ with $f(x) = g(x_1) + h(x_2)$ for $x \in I$.

Proof. Suppose f is double constant on I. Take $a \in I$. Define $g : I_1 \to \mathbb{R}$ by $g(s) = f(s, a_2)$ for $s \in I_1$ and define $h : I_2 \to \mathbb{R}$ by $h(t) = f(a_1, t) - f(a_1, a_2)$ for $t \in I_2$. If $x \in I$, then:

$$
\begin{aligned}
f(x_1, x_2) &= f(x) - 0 = f(x_1, x_2) - \Delta_a^x(f) \\
&= f(x_1, x_2) - f(x_1, x_2) - f(a_1, a_2) + f(x_1, a_2) + f(a_1, x_2) \\
&= g(x_1) + h(x_2).
\end{aligned}
$$

Now suppose that there are functions $g : I_1 \to \mathbb{R}$ and $h : I_2 \to \mathbb{R}$ with $f(x) = g(x_1) + h(x_2)$ for $x \in I$. If $a, b \in I$, then:

$$
\begin{aligned}
\Delta_a^b(f) &= f(b_1, b_2) + f(a_1, a_2) - f(b_1, a_2) - f(a_1, b_2) \\
&= g(b_1) + h(b_2) + g(a_1) + h(a_2) \\
&\quad - (g(b_1) + h(a_2) + g(a_1) + h(b_2)) = 0.
\end{aligned}
$$

Thus, f is double constant on I. $\square$

For future reference, we record some properties of the double difference.

Proposition 2. *Suppose that f is a double function which is defined on a double interval containing the points a, b and x. Then:*

(a) $\Delta_a^a(f) = 0$; $\quad \Delta_a^b(f) = \Delta_b^a(f)$; $\quad \Delta_a^b(f) = -\Delta_{(a_1,b_2)}^{(b_1,a_2)}(f)$;

(b) $\Delta_a^b(f) = \Delta_a^{(x_1,b_2)}(f) + \Delta_{(x_1,a_2)}^b(f)$; $\quad$ *(c)* $\Delta_a^b(f) = \Delta_a^{(b_1,x_2)}(f) + \Delta_{(a_1,x_2)}^b(f)$;

(d) $\Delta_a^b(f) = \Delta_a^x(f) + \Delta_x^b(f) + \Delta_{(a_1,x_2)}^{(x_1,b_2)}(f) + \Delta_{(x_1,a_2)}^{(b_1,x_2)}(f)$.

Proof. These properties follow immediately from the definition of Δ. $\square$

3   Double continuity

In this section, we define the class of double continuous functions and the class of globally double continuous functions. We show that a function which is double continuous on an interval is automatically *globally* double continuous on that double interval (see Proposition 7). We also show a continuity result (see Proposition 8), concerning the double difference map that we need in the next section.

Let $a \in \mathbb{R}^2$ and suppose that f is a double function. We say that f is *double continuous at a* if there is a double open interval I with $a \in I \subseteq D(f)$ such that $\forall x_1 \in I_1 \lim_{x_2 \to a_2} \Delta_a^x(f) = 0$ and $\forall x_2 \in I_2 \lim_{x_1 \to a_1} \Delta_a^x(f) = 0$.

Now we define *signed* double continuity at a point. We say that f is $+-$*double continuous at a* if there is $\delta \in \mathbb{R}^2_{++}$ with $a \in I := D_{+-}(a, \delta) \subseteq D(f)$ such that $\forall x_1 \in I_1 \lim_{x_2 \to a_2^-} \Delta_a^x(f) = 0$ and $\forall x_2 \in I_2 \lim_{x_1 \to a_1^+} \Delta_a^x(f) = 0$. Analogously, $++$double continuity, $-+$double continuity and $--$double continuity are defined.

Let I be a double interval such that $I \subseteq D(f)$. If f is double continuous at every $a \in I$, then we say that f is *double continuous on I*. In this definition it is understood that if a is a boundary point of I, then by double continuity at a we mean this in the sense of the signed double continuity defined above. Namely, suppose, for instance, that $I = (a_1, b_1] \times (a_2, b_2]$. By double continuity at the point b, we mean $--$double continuity, and by double continuity at a point (b_1, t), where $a_2 < t < b_2$, we mean both $-+$double continuity and $--$double continuity.

Proposition 3. *Suppose that f is a double function. Let I be a double interval.*

(a) *If f is continuous on I, then f is double continuous on I.*

(b) *If f is double constant on I, then f is double continuous on I.*

Proof. This is clear. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Example 4. *The class of double continuous functions on $\mathbb{R}^2$ contains many examples of functions which are not continuous. Indeed, suppose that g and h are functions $\mathbb{R} \to \mathbb{R}$ such that g is continuous but h is not. Put $f(x) = g(x_1) + h(x_2)$ for $x \in \mathbb{R}^2$. Then, clearly, f is not continuous. However, by Proposition 1 and Proposition 3(b), f is double continuous.*

Example 5. *Define the double function f by $f(x) = x_1^{x_2} \cdot x_2^{x_1}$, for $x > 0$, and $f(x) = 0$, if $x \le 0$. Note that f is not double constant since $\Delta_0^x(f) = f(x) \ne 0$ for all $x > 0$. Since $f(t,t) = t^t \cdot t^t = t^{2t} = e^{2t \ln(t)} \to e^0 = 1 \ne 0 = f(0,0)$, as $t \to 0^+$, it follows that f is not continuous at 0. However, f is double continuous at 0. In fact, if $x_1 > 0$, then $\Delta_0^x(f) = f(x) = x_1^{x_2} \cdot x_2^{x_1} \to 1 \cdot 0 = 0 = f(0,0)$, as $x_2 \to 0^+$; if $x_2 > 0$, then $\Delta_0^x(f) = f(x) = x_1^{x_2} \cdot x_2^{x_1} \to 0 \cdot 1 = 0 = f(0,0)$, as $x_1 \to 0^+$.*

Remark 6. *Our definition of double continuity is different from Bögel's [1] definition of this concept. Indeed, he defines f to be continuous at a if $\lim_{x \to a} \Delta_a^x(f) = 0$. Example 5 shows that there are double continuous functions in our sense that are not double continuous in the sense of Bögel. On the other hand, the function $f(x) = x_1^2 + x_2^2$, for $x \nsim 0$, and $f(x) = 0$, for $x \sim 0$, is continuous at 0 in the sense of Bögel but not in our sense. We have chosen to define double continuity as weak as possible but so that it still is implied by double differentiability (see Proposition 14).*

Suppose that f is a double function. Let I be a double interval such that $I \subseteq D(f)$. We say that f is *globally double continuous on I* if $\forall c, d \in I_1 \ \forall e \in I_2 \lim_{x_2 \to e; x_2 \in I_2} \Delta_{(d,e)}^{(c,x_2)}(f) = 0$, and $\forall c, d \in I_2 \ \forall e \in I_1 \lim_{x_1 \to e; x_1 \in I_1} \Delta_{(e,d)}^{(x_1,c)}(f) = 0$. To the knowledge of the author of the present article, Bögel [1, 2] does not define any concept resembling global double continuity.

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Proposition 7. *Let I be a double interval and suppose that f is a double function such that $I \subseteq D(f)$. Then f is double continuous on I if and only if f is globally double continuous on I.*

Proof. The "if" statement is clear. Now we show the "only if" statement. Suppose that f is double continuous on I. Take $a, b \in I$ with $a < b$. Take $e \in [a_2, b_2]$ and $c, d \in [a_1, b_1]$ with $c < d$. Put $J = [c, d]$. The case when $d < c$ is symmetrical and is therefore left to the reader. For all $z \in J$ choose $\epsilon_z > 0$ such that for all $y \in (z - \epsilon_z, z + \epsilon_z)$ the map $[a_2, b_2] \ni x_2 \mapsto f(y, x_2) - f(c, x_2)$ is continuous at $x_2 = e$. Since the open intervals $(z - \epsilon_z/2, z + \epsilon_z/2)$, for $z \in J$, is an open cover of the compact interval J we can choose a finite subcover $\{(z_i - \epsilon_{z_i}/2, z_i + \epsilon_{z_i}/2)\}_{i=0}^n$ of J. We may assume that $a_1 = z_0 < z_1 < \cdots < z_{n-1} < z_n = x_1$. We may also assume that for all $i, j \in \{0, \ldots, n\}$ if $(z_i - \epsilon_{z_i}/2, z_i + \epsilon_{z_i}/2) \subseteq (z_j - \epsilon_{z_j}/2, z_j + \epsilon_{z_j}/2)$, then $i = j$. It follows that $f(x_1, x_2) - f(a_1, x_2) = \sum_{i=1}^n f(z_i, x_2) - f(z_{i-1}, x_2) \to \sum_{i=1}^n f(z_i, a_2) - f(z_{i-1}, a_2) = f(x_1, a_2) - f(a_1, a_2)$, as $x_2 \to a_2$, since for all $i \in \{1, \ldots, n\}$, $z_{i-1} \in (z_i - \epsilon_{z_i}, z_i + \epsilon_{z_i})$ or $z_i \in (z_{i-1} - \epsilon_{z_{i-1}}, z_{i-1} + \epsilon_{z_{i-1}})$. The calculation involving the second limit is completely analogous to the above calculation and is left to the reader. $\square$

The next result is [1, Satz 12]. To prove this we use our notion of global double continuity, whereas Bögel loc. cit. resorts to an ad hoc argument (although in a more general context).

Proposition 8. *Let $a, b, h \in \mathbb{R}^2$ where $a < b$. Suppose that $f : [a, b] \to \mathbb{R}$ is a double continuous function. Then the map $x \mapsto \Delta_x^{x+h}$ is continuous at all $x \in (a, b)$ for which $x + h \in (a, b)$.*

Proof. It is enough to show that the map $x \mapsto \Delta_x^{x+h}$ is continuous separately in the variables x_1 and x_2. By Proposition 7 it follows that:

$$
\begin{aligned}
\Delta_x^{x+h}(f) - \Delta_{(a_1, x_2)}^{(a_1+h_1, x_2+h_2)} &= f(x_1 + h_1, x_2 + h_2) - f(x_1 + h_1, x_2) \\
&\quad - (f(a_1 + h_1, x_2 + h_2) - f(a_1 + h_1, x_2)) \\
&\quad - (f(x_1, x_2 + h_2) - f(x_1, x_2)) \\
&\quad + f(a_1, x_2 + h_2) - f(a_1, x_2) \ \to \ 0,
\end{aligned}
$$

as $x_1 \to a_1$, and:

$$
\begin{aligned}
\Delta_x^{x+h}(f) - \Delta_{(x_1, a_2)}^{(x_1+h_1, a_2+h_2)} &= f(x_1 + h_1, x_2 + h_2) - f(x_1 + h_1, x_2) \\
&\quad - (f(x_1 + h_1, a_2 + h_2) - f(x_1 + h_1, a_2)) \\
&\quad - (f(x_1, x_2 + h_2) - f(x_1, x_2)) \\
&\quad + f(x_1, a_2 + h_2) - f(x_1, a_2) \ \to \ 0,
\end{aligned}
$$

as $x_2 \to a_2$. $\square$

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Proposition 9. *Let $a, b \in \mathbb{R}^2$ where $a < b$. Suppose that f is a double function which is continuous on $[a, b]$. Let D denote the smallest closed interval in $\mathbb{R}$ containing $f(a_1, a_2)$, $f(b_1, b_2)$, $f(a_1, b_2)$ and $f(b_1, a_2)$. If d is an interior point in D, then there exists $c \in (a, b)$ with $f(c) = d$.*

Proof. From the assumptions it follows that there are $a', b' \in (a, b)$ with $f(a'_1, a'_2) < d < f(b'_1, b'_2)$. Let L denote the line segment from a' to b'. Since L is connected and f is continuous, $f(L)$ is an interval. Thus, there is $c \in L$ with $f(c) = d$. Since $c \in L \subsetneq [a, b]$ it follows that $c \in (a, b)$. $\square$

4   Double differentiability

In this section, we define (signed) double limits and (signed) double derivatives. In analogue with the single variable situation, we show that double differentiable functions are double continuous (see Proposition 14). Thereafter, we show a double Rolle's theorem (see Proposition 17), a double Lagrange's mean value theorem (see Proposition 18), a double Cauchy's mean value theorem (see Proposition 19), a double Fermat's theorem (see Proposition 25) and a double first derivative test (see Proposition 26). At the end of this section, we define double primitive functions.

Suppose that $a \in \mathbb{R}^2$, $L \in \mathbb{R}$ and that f is a double function. We say that f has *double limit L as x approaches a* if for every $\epsilon > 0$ there is $\delta \in \mathbb{R}^2_{++}$ with $P(a, \delta) \subseteq D(f)$ and $|f(x_1, x_2) - L| < \epsilon$ whenever $x \in P(a, \delta)$. In that case, we write $\lim_{x \rightsquigarrow a} f(x_1, x_2) = L$ or $f(x_1, x_2) \to L$ as $x \rightsquigarrow a$.

Now we define *signed* double limits. We say that f has $+-$*double limit L as x approaches a* if for every $\epsilon > 0$ there is $\delta \in \mathbb{R}^2_{++}$ with $P_{+-}(a, \delta) \subseteq D(f)$, and $|f(x_1, x_2) - L| < \epsilon$ whenever $x \in P_{+-}(a, \delta)$; then we write $\lim_{x \rightsquigarrow a^{+-}} f(x_1, x_2) = L$ or $f(x_1, x_2) \to L$ as $x \rightsquigarrow a^{+-}$. Analogously, $++$double limits, $-+$double limits and $--$double limits are defined.

More generally, one may analogously define (signed) double limits when a and L belong to $\overline{\mathbb{R}}^2 = \overline{\mathbb{R}} \times \overline{\mathbb{R}}$, where $\overline{\mathbb{R}}$ denotes the affinely extended real number system $\mathbb{R} \cup \{\infty, -\infty\}$. We leave the details of these definitions to the reader.

Suppose that $a, b \in \mathbb{R}^2$ and that $a \nsim b$. If f is defined at (a_1, a_2), (a_1, b_2), (b_1, a_2) and (b_1, b_2), then we define the *double mean slope of f from a to b* to be the quotient:

$$m_a^b(f) := \frac{\Delta_a^b(f)}{(b_1 - a_1)(b_2 - a_2)}.$$

We say that f is *double differentiable* at a if there is a double open interval I with $a \in I \subseteq D(f)$ and the double limit $\lim_{x \rightsquigarrow a} m_a^x(f)$ exists. In that case, we let $f'(a)$ denote this limit and we say that $f'(a)$ is the *double derivative* of f at a (cf. [10]). Using signed double limits, we can analogously define the $++$double derivative $f'_{++}(a)$, the $+-$double derivative $f'_{+-}(a)$, the $-+$double derivative $f'_{-+}(a)$ and the $--$double derivative $f'_{--}(a)$.

Let I be a double interval such that $I \subseteq D(f)$. If f is double differentiable at every $a \in I$, then we say that f is *double differentiable on I*. In this definition it is understood that when a is a boundary point of I, then by double differentiable at a we mean this in the sense of the signed double differentiability defined above. Namely, suppose, for instance, that $I = [a_1, b_1) \times (a_2, b_2]$. By differentiability at the point (a_1, b_2), we mean $+-$double differentiability, and by double differentiability at a point (a_1, t), where $a_2 < t < b_2$, we mean both $++$double differentiability and $+-$double differentiability.

The next result, more commonly known as Schwarz's theorem, Clairaut's theorem, or Young's theorem, is well-known (see e.g. [8, Theorem 9.40]). We have, nevertheless, chosen to include a proof of it since it involves, in a natural way, use of the double derivative. Note that for iterated partial derivatives, we will use the convention that f_{12} and f_{21} denote $(f_1)_2$ and $(f_2)_1$ respectively.

Proposition 10. *Suppose that f is a double function with the property that the mixed partial derivatives f_{12} and f_{21} exist and are continuous at $a \in \mathbb{R}^2$. Then f is double differentiable at a and $f'(a) = f_{12}(a) = f_{21}(a)$.*

Proof. The assumptions imply that f and the partial derivatives f_{12}, f_{21}, f_1 and f_2 are defined in some double open interval I containing a. Take $h \in \mathbb{R}^2$ with $h \nsim 0$. Define functions u and v by $u(x_1) = f(x_1, a_2 + h_2) - f(x_1, a_2)$, for $x_1 \in I_1$, and $v(x_2) = f(a_1 + h_1, x_2) - f(a_1, x_2)$, for $x_2 \in I_2$. Repeated use of the single variable mean value theorem yield $\theta_1, \theta_2 \in (0, 1)$ such that:

$$
\begin{aligned}
\Delta_a^{a+h}(f) &= u(a_1 + h_1) - u(a_1) = h_1 u'(a_1 + \theta_1 h_1) \\
&= h_1(f_1(a_1 + \theta_1 h_1, a_2 + h_2) - f_1(a_1 + \theta_1 h_1, a_2)) \\
&= h_1 h_2 f_{12}(a_1 + \theta_1 h_1, a_2 + \theta_2 h_2).
\end{aligned}
$$

Thus, $m_a^{a+h}(f) = f_{12}(a_1 + \theta_1 h_1, a_2 + \theta_2 h_2) \to f_{12}(a_1, a_2)$ as $h \rightsquigarrow 0$. Similarly, there exist $\theta_3, \theta_4 \in (0, 1)$ with:

$$
\begin{aligned}
\Delta_a^{a+h}(f) &= v(a_2 + h_2) - v(a_2) = h_3 v'(a_2 + \theta_3 h_2) \\
&= h_3(f_2(a_1 + h_1, a_2 + \theta_3 h_2) - f_2(a_1, a_2 + \theta_3 h_2)) \\
&= h_1 h_2 f_{21}(a_1 + \theta_4 h_1, a_2 + \theta_3 h_2).
\end{aligned}
$$

Hence, $m_a^{a+h}(f) = f_{21}(a_1 + \theta_4 h_1, a_2 + \theta_3 h_2) \to f_{21}(a_1, a_2)$ as $h \rightsquigarrow 0$. Thus, f is double differentiable at a and $f'(a) = f_{12}(a) = f_{21}(a)$. $\square$

Example 11. *By Proposition 10, all sufficiently smooth double functions are double differentiable. However, the class of double differentiable functions contains examples of functions which are not even partially differentiable. In fact, it is clear that all double constant functions are double differentiable with double derivative equal to*

zero everywhere. Therefore, the class of double differentiable functions even contains many examples of everywhere discontinuous functions (see Example 4).

Example 12. *It is easy to construct examples of double functions that are not double constant but double differentiable at a point where an iterated partial derivative fails to exist. Namely, suppose that g and h are single variable functions defined on $\mathbb{R}$. Furthermore, suppose that $g(x_1) \to 0$ as $x_1 \to 0$; $g(x_1) \neq 0$ for non-zero $x_1 \in \mathbb{R}$; $h(x_2)$ is bounded near the origin; in any open interval around the origin, there exists x_2 such that $h(x_2) \neq 0$; $\lim_{x_2 \to 0} h(x_2)$ does not exist. For instance, we can choose $g(x_1) = x_1$ and $h(x_2) = \sin(1/x_2)$, for $x_2 \neq 0$, and $h(0) = 0$. Now put $f(x) = x_1 g(x_1) \cdot x_2 h(x_2)$ for $x \in \mathbb{R}^2$. Then, in any double open interval containing the origin, there exists x such that $\Delta_0^x(f) = f(x) \neq 0$. Also, clearly, the double derivative $f'(0,0)$ exists and is equal to zero. However, for any fixed nonzero $x_1 \in \mathbb{R}$, $f_2(x_1, 0) = \lim_{x_2 \to 0} f(x)/x_2 = \lim_{x_2 \to 0} x_1 g(x_1) h(x_2)$ does not exist. Thus, the iterated partial derivative $f_{21}(0,0)$ does not exist.*

In analogy with the single variable situation, there is a "first order" approximation to Δ:

Proposition 13. *Suppose that f is a double function and that $a \in \mathbb{R}^2$. Then f is double differentiable at a if and only if there is $d \in \mathbb{R}$, a double neighbourhood $D(a, \delta)$, where f is defined, and a function $\rho : P(a, \delta) \to \mathbb{R}$ such that $\Delta_a^x(f) = (x_1 - a_1)(x_2 - a_2)d + (x_1 - a_1)(x_2 - a_2)\rho(x)$, for $x \in P(a, \delta)$, and $\lim_{x \rightsquigarrow a} \rho(x) = 0$. In that case, $f'(a) = d$.*

Proof. This follows immediately from the definition of Δ. $\square$

Proposition 14. *Suppose that f is a double function which is defined at $a \in \mathbb{R}^2$. If f is double differentiable at a, then f is double continuous at a.*

Proof. Suppose that f is double differentiable at a. Take δ and ρ satisfying the conditions in the formulation of Proposition 13. If $x \nsim a$, then:

$$
\begin{aligned}
f(x_1, x_2) - f(a_1, x_2) &= \Delta_a^x(f) + f(x_1, a_2) - f(a_1, a_2) \\
&= (x_1 - a_1)(x_2 - a_2)d + (x_1 - a_1)(x_2 - a_2)\rho(x) \\
&\quad + f(x_1, a_2) - f(a_1, a_2) \ \to \ f(x_1, a_2) - f(a_1, a_2),
\end{aligned}
$$

as $x_2 \to a_2$, and:

$$
\begin{aligned}
f(x_1, x_2) - f(x_1, a_2) &= \Delta_a^x(f) + f(a_1, x_2) - f(a_1, a_2) \\
&= (x_1 - a_1)(x_2 - a_2)d + (x_1 - a_1)(x_2 - a_2)\rho(x) \\
&\quad + f(a_1, x_2) - f(a_1, a_2) \ \to \ f(a_1, x_2) - f(a_1, a_2),
\end{aligned}
$$

as $x_1 \to a_1$. $\square$

****

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We now proceed to show double versions of Rolle's theorem and various versions of the mean value theorem. To this end, we need two propositions. We adapt, to the situation at hand, an approach which originally was invented by Cauchy (see [5, p. 169]) for the single variable situation and then corrected and clarified by Plante [7].

Proposition 15. *Suppose* $a, b \in \mathbb{R}^2$, $a < b$ *and* $x \in (a, b)$. *Let* $f : [a, b] \to \mathbb{R}$ *be a double function. Consider the following four real numbers:*

$$m_1 := m_a^x(f), \ m_2 := m_x^b(f), \ m_3 := \frac{\Delta_{(a_1,x_2)}^{(x_1,b_2)}(f)}{(x_1 - a_1)(b_2 - x_2)}, \ m_4 := \frac{\Delta_{(x_1,a_2)}^{(b_1,x_2)}(f)}{(b_1 - x_1)(x_2 - a_2)}.$$

Then either (i) all of them are equal to $m_a^b(f)$, *or (ii) at least one of them is greater than* m_a^b *and at least one of them is less than* m_a^b.

Proof. Suppose that (i) does not hold. Seeking a contradiction, suppose that $m_a^b(f) \leq m_i$ for all $i \in \{1, 2, 3, 4\}$ with strict inequality for at least one index. By Proposition 2(d), we get that:

$$
\begin{aligned}
\Delta_a^b(f) &= m_a^b(f)(b_1 - a_1)(b_2 - a_2) \\
&= m_a^b(f)(x_1 - a_1)(x_2 - a_2) + m_a^b(f)(b_1 - x_1)(b_2 - x_2) \\
&+ m_a^b(f)(x_1 - a_1)(b_2 - x_2) + m_a^b(f)(b_1 - x_1)(x_2 - a_2) \\
&< m_1(x_1 - a_1)(x_2 - a_2) + m_2(b_1 - x_1)(b_2 - x_2) \\
&+ m_3(x_1 - a_1)(b_2 - x_2) + m_4(b_1 - x_1)(x_2 - a_2) \\
&= \Delta_a^x(f) + \Delta_x^b(f) + \Delta_{(a_1,x_2)}^{(x_1,b_2)}(f) + \Delta_{(x_1,a_2)}^{(b_1,x_2)}(f) = \Delta_a^b(f)
\end{aligned}
$$

and hence the contradiction $\Delta_a^b(f) < \Delta_a^b(f)$. Similarly, if we assume that $m_a^b(f) \geq m_i$, for all $i \in \{1, 2, 3, 4\}$, with strict inequality for at least one index, then we get the contradiction $\Delta_a^b(f) > \Delta_a^b(f)$. Thus (ii) holds. $\qquad\square$

Proposition 16. *Suppose that* $a, b \in \mathbb{R}^2$ *satisfy* $a < b$. *Let* $f : (a, b) \to \mathbb{R}$ *be a double function which is double differentiable at* $c \in (a, b)$. *Let* $a(1) \leq a(2) \leq a(3) \leq \cdots$ *and* $b(1) \geq b(2) \geq b(3) \geq \cdots$ *be sequences in* (a, b) *with* $\lim_{n \to \infty} a(n) = \lim_{n \to \infty} b(n) = c$ *and satisfying one of the following properties:*

(i) $\forall n \in \mathbb{N} \ a(n) < c < b(n)$;

(ii) $\forall n \in \mathbb{N} \ a(n) < c$ *and* $\exists N \in \mathbb{N} \ \forall n \geq N \ c = b(n)$;

(iii) $\forall n \in \mathbb{N} \ c < b(n)$ *and* $\exists N \in \mathbb{N} \ \forall n \geq N \ a(n) = c$.

Then $\lim_{n \to \infty} m_{a(n)}^{b(n)}(f) = f'(c)$.

Proof. Suppose that (i) holds. For every $n \in \mathbb{N}$, put:

$$m_1(n) := \frac{\Delta^c_{a(n)}(f)}{(c_1 - a_1(n))(c_2 - a_2(n))} \quad m_2(n) := \frac{\Delta^{b(n)}_c(f)}{(b_1(n) - c_1)(b_2(n) - c_2)}$$

$$m_3(n) := \frac{\Delta^{(c_1, b_2(n))}_{(a_1(n), c_2)}(f)}{(c_1 - a_1(n))(b_2(n) - c_2)} \quad m_4(n) := \frac{\Delta^{(b_1(n), c_2)}_{(c_1, a_2(n))}(f)}{(b_1(n) - c_1)(c_2 - a_2(n))}.$$

From the definition of the double derivative we get that $\lim_{n \to \infty} m_i(n) = f'(c)$ for all $i \in \{1, 2, 3, 4\}$. Thus, from Proposition 15, it follows that $\lim_{n \to \infty} m^{b(n)}_{a(n)}(f) = f'(c)$.

If (ii) (or (iii)) holds, then, from the definition of the double derivative, we get that $\lim_{n \to \infty} m^{b(n)}_{a(n)}(f) = \lim_{n \to \infty} m^c_{a(n)}(f) = f'(c)$ (or $\lim_{n \to \infty} m^{b(n)}_{a(n)}(f) = \lim_{n \to \infty} m^{b(n)}_c(f) = f'(c)$). $\qquad\square$

The next result is [1, Satz 13]. Noteworthily, Bögel proves this result by an argument which is not identical, but similar in spirit, to our proof. Thus, in particular, for single variable functions, Bögel's proof produces the classical Rolle's theorem without resorting to the Weierstrass extremum theorem (cf. [7]).

Proposition 17 (Double Rolle's theorem). *Let $a, b \in \mathbb{R}^2$ satisfy $a < b$. Suppose $f : [a, b] \to \mathbb{R}$ is a double continuous function which is double differentiable on (a, b). If $\Delta^b_a(f) = 0$, then there exists $c \in (a, b)$ with $f'(c) = 0$.*

Proof. We claim that there are $p, q \in (a, b)$ with $\Delta^q_p(f) = 0$, $p < q$ and $q - p \le (b-a)/2$. Let us assume for a moment that the claim holds. Then we can inductively define sequences $a(1) \le a(2) \le a(3) \le \cdots$ and $b(1) \ge b(2) \ge b(3) \ge \cdots$ in (a, b) satisfying $a(n) < b(n)$, $\Delta^{b(n)}_{a(n)} = 0$ and $b(n) - a(n) \le (b - a)/2^n$, for every $n \in \mathbb{N}$. Then $\{a(n)\}_{n=1}^\infty$ and $\{b(n)\}_{n=1}^\infty$ have a common limit $c \in (a, b)$ satisfying one of the properties (i)-(iii) in Proposition 16. Thus, from the same proposition, it follows that $f'(c) = 0$.

Now we show the claim. Seeking a contradiction, suppose that $\Delta^q_p(f) \ne 0$ for all $p, q \in (a, b)$ such that $p < q$ and $q - p \le (b - a)/2$. Put $h = (b - a)/2$.

Case 1: $\Delta^{a+h}_a(f) \ne 0$. Consider the map $g(x) = \Delta^{x+h}_x(f)$ for $x \in [a, a+h]$. From Proposition 2 it follows that $g(a) + g(a_1, (b_2 + a_2)/2) + g((b_1 + a_1)/2, a_2) + g((a+b)/2) = \Delta^b_a(f) = 0$. Therefore, since $g(a) = \Delta^{a+h}_a(f) \ne 0$, at least one the real numbers $g(a)$, $g(a_1, h_2/2)$, $g((b_1 + a_1)/2, a_2)$ and $g((a+b)/2)$ is positive and at least one is negative. By Proposition 8 and Proposition 9 it follows that there is $p \in (a, (a + b)/2)$ with $g(p) = 0$. Put $q = p + h = p + (b - a)/2$. Then $\Delta^q_p(f) = g(p) = 0$, $p, q \in (a, b)$, $p < q$ and $q - p \le (b - a)/2$. This is a contradiction.

Case 2: $\Delta^{a+h}_a(f) = 0$. Put $b' = a + h$ and $h' = h/2$. Then $\Delta^{b'}_a(f) = 0$ and, by the assumptions, we get that $\Delta^{b'}_{a+h'}(f) \ne 0$. Consider the map $g(x) = \Delta^{x+h'}_x(f)$ for $x \in [a, a + h']$. From Proposition 2 it follows that $g(a) + g(a_1, (b_2 + a_2)/4) + g((b_1 +$

$a_1)/4, a_2) + g((a+b)/4) = \Delta_a^{b'}(f) = 0$. Therefore, since $g(a) = \Delta_{a+h'}^{b'}(f) \neq 0$, at least one the real numbers $g(a)$, $g(a_1, (b_2+a_2)/4)$, $g((b_1+a_1)/4, a_2)$ and $g((a+b)/4)$ is positive and at least one is negative. By Proposition 8 and Proposition 9 it follows that there is $p' \in (a, (a+b)/4)$ with $g(p') = 0$. Put $q' = p' + h' = p' + (b-a)/4$. Then $\Delta_{p'}^{q'}(f) = g(p') = 0$, $p', q' \in (a, b)$, $p' < q'$ and $q' - p' \leq (b-a)/4$. This is a contradiction. $\qquad\square$

Proposition 18 (Double Lagrange's mean value theorem). *Let $a, b \in \mathbb{R}^2$ satisfy $a < b$. Suppose that $f : [a, b] \to \mathbb{R}$ is a double continuous function which is double differentiable on (a, b). Then there exists $c \in (a, b)$ with $f'(c) = m_a^b(f)$.*

Proof. Consider the map $g(x) = f(x) - m_a^b(f)(x_1 - a_1)(x_2 - a_2)$ for $x \in [a, b]$. Then it is clear that $\Delta_a^b(g) = \Delta_a^b(f) - \Delta_a^b(f) = 0$. Proposition 17 implies the existence of an element $c \in (a, b)$ with $g'(c) = 0$. From Proposition 10 it follows that $0 = g'(c) = f'(c) - m_a^b(f) \cdot 1$. Therefore $f'(c) = m_a^b(f)$. $\qquad\square$

Proposition 19 (Double Cauchy's mean value theorem [1, Satz 14]). *Let $a, b \in \mathbb{R}^2$ satisfy $a < b$. Suppose $f, g : [a, b] \to \mathbb{R}$ are double continuous functions which are double differentiable on (a, b). Then there is $c \in (a, b)$ with $f'(c)\Delta_a^b(g) = g'(c)\Delta_a^b(f)$.*

Proof. We consider two cases.

Case 1: $\Delta_a^b(g) = 0$. By Proposition 17 there is $c \in (a, b)$ with $g'(c) = 0$. For this c we get that $f'(c)\Delta_a^b(g) = f'(c) \cdot 0 = 0 \cdot \Delta_a^b(f) = g'(c)\Delta_a^b(f)$.

Case 2: $\Delta_a^b(g) \neq 0$. Consider the map $h(x) = f(x) - \Delta_a^b(f)g(x)/\Delta_a^b(g)$ for $x \in [a, b]$. Then, clearly, $\Delta_a^b(h) = 0$. By Proposition 17 there is $c \in (a, b)$ with $h'(c) = 0$ form which the claim follows. $\qquad\square$

Remark 20. *(a) If we specialize $g(x) = x_1 x_2$ in Proposition 19, then we get Proposition 18.*

(b) In [4] the concept of double derivative (there named bidimensional derivative) as well as Propositions 17-19 were rediscovered (independently from Bögel it seems) by Dobrescu and Siclovan.

Suppose that f is a double function defined on a double interval I. We say that f is *double increasing (double decreasing)* on I if $\Delta_a^b(f) > 0$ ($\Delta_a^b(f) < 0$) for all $a, b \in I$ with $a < b$.

Example 21. *Let $a, b, c \in \mathbb{R}^2$ and $D \in \mathbb{R}$. Define a double function f on $\mathbb{R}^2$ by $f(x) = D(x_1 - c_1)(x_2 - c_2)$ for $x \in \mathbb{R}^2$. Then:*

$$
\begin{aligned}
\Delta_a^b(f) &= D(b_1 - c_1)(b_2 - c_2) - D(b_1 - c_1)(a_2 - c_2) \\
&\quad - D(a_1 - c_1)(b_2 - c_2) + D(a_1 - c_1)(a_2 - c_2) \\
&= D(b_1 - a_1)(b_2 - a_2).
\end{aligned}
$$

Thus, f is double increasing on $\mathbb{R}^2 \Leftrightarrow D > 0$; f is double decreasing on $\mathbb{R}^2 \Leftrightarrow D < 0$; f is double constant on $\mathbb{R}^2 \Leftrightarrow D = 0$.

Proposition 22. *Suppose that $a, b \in \mathbb{R}^2$ satisfy $a < b$. Let $f : [a, b] \to \mathbb{R}$ be a double continuous function which is double differentiable on (a, b). Then:*

(a) f is double increasing on $[a, b] \Leftrightarrow f'(x) > 0$ for every $x \in (a, b)$;

(b) f is double decreasing on $[a, b] \Leftrightarrow f'(x) < 0$ for every $x \in (a, b)$;

(c) f is double constant on $[a, b] \Leftrightarrow f'(x) = 0$ for every $x \in (a, b)$.

Proof. This follows immediately from Proposition 18. □

Remark 23. *Consider the function f defined in Example 21. Then, by Proposition 10, it follows that f is double differentiable with $f'(x) = D$ for $x \in \mathbb{R}^2$. Thus, the conclusions in this example follow from Proposition 22.*

Suppose that f is a double function defined on a double open interval I. Let $a \in I$. We say that a is a *double maximum point (double minumum point)* for f on I if $\Delta_a^b(f) < 0$ $(\Delta_a^b(f) > 0)$ for all $b \in I$ with $b \asymp a$. We say that a is a *double extreme point* for f on I if a is a double minumum point or a double maximum point for f on I

Example 24. *Let $a, b, c \in \mathbb{R}^2$ and $D \in \mathbb{R}$. Define a double function f on $\mathbb{R}^2$ by $f(x) = D(x_1 - c_1)^2(x_2 - c_2)^2$ for $x \in \mathbb{R}^2$ where D is a non-zero real number. Then:*

$$
\begin{aligned}
\Delta_a^b(f) &= D(b_1 - c_1)^2(b_2 - c_2)^2 - D(b_1 - c_1)^2(a_2 - c_2)^2 \\
&\quad - D(a_1 - c_1)^2(b_2 - c_2)^2 + D(a_1 - c_1)^2(a_2 - c_2)^2 \\
&= D(b_1 - a_1)(b_2 - a_2)(b_1 + a_1 - 2c_2)(b_2 + a_2 - 2c_2).
\end{aligned}
$$

Suppose now that a is a double extreme point for f on $\mathbb{R}^2$. The above calculation shows that $\Delta_a^{2c-a} = 0$. Therefore $a = 2c - a$ and thus $a = c$. Hence $\Delta_a^b(f) = D(b_1 - a_1)^2(b_2 - a_2)^2$. Thus, a is a double maximum (minimum) point for f on $\mathbb{R}^2$ if and only if $a = c$ and $D < 0$ $(D > 0)$.

Suppose that f is a double function which is defined on a double open interval I and let $a \in I$. If f is double differentiable at a and $f'(a) = 0$, then f we say that a is a *double stationary point* for f. We say that a is a *double a critical point for f* if either f is not double differentiable at a or a is a stationary point for f.

Proposition 25 (Double Fermat's theorem). *Suppose that f is a double function which is defined on a double open interval I and let $a \in I$. If a is a double extreme point for f on I, then a is a double critical point for f.*

Proof. Suppose that f is double differentiable at a and that a is a double minimum point for f on I. Then $\Delta_a^b(f) > 0$ for all b with $b \asymp a$. Hence:

$$
f'(a) = \lim_{b \rightsquigarrow a^{++}} m_a^b(f) = \lim_{b \rightsquigarrow a^{++}} \frac{\Delta_a^b(f)}{(b_1 - a_1)(b_2 - a_2)} \geq 0
$$

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and:

$$f'(a) \quad = \quad \lim_{b \rightsquigarrow a^{+-}} m_a^b(f) \quad = \quad \lim_{b \rightsquigarrow a^{+-}} \frac{\Delta_a^b(f)}{(b_1 - a_1)(b_2 - a_2)} \quad \leq \quad 0.$$

Thus $f'(a) = 0$. The proof is analogous in the case when f is double differentiable at a and a is a double maximum point for f on I. $\square$

Proposition 26 (Double first derivative test). *Suppose that f is a double function which is double differentiable on a double open interval (a,b). Let $c \in (a,b)$ be a double stationary point for f.*

(a) *Suppose that $f'(x) < 0$, when $a < x < c$ or $c < x < b$, and $f'(x) > 0$ when $(a_1, c_2) < x < (c_1, b_2)$ or $(c_1, a_2) < x < (b_1, c_2)$. Then c is a double maximum point for f on (a,b).*

(b) *Suppose that $f'(x) > 0$, when $a < x < c$ or $c < x < b$, and $f'(x) < 0$ when $(a_1, c_2) < x < (c_1, b_2)$ or $(c_1, a_2) < x < (b_1, c_2)$. Then c is a double minimum point for f on (a,b).*

(c) *If f' has the same sign throughout the formulation of the statement in (a) (or in (b)), then c is neither a double maximum point nor a double minimum point for f on (a,b).*

Proof. (a) Take $x \in (a,b)$ with $x \nsim c$. By Proposition 22(b) it follows that f is double decreasing on the intervals (a,c) and (c,b). Thus $\Delta_x^c(f) < 0$ for all x in those intervals. By Proposition 22(a) it follows that f is double increasing on the intervals $((a_1, c_2), (c_1, b_2))$ and $((c_1, a_2), (b_1, c_2))$. By Lemma 2(c) it follows that $\Delta_x^c(f) > 0$ for all x in those intervals. The proofs of (b) and (c) are similar to the proof of (a). $\square$

Remark 27. *One can reach the conclusion in Example 24, using the double derivative. Namely, let a b, c, D and f be defined as in that example. By Proposition 10, it follows that f is double differentiable with $f'(x) = 4D(x_1 - c_1)(x_2 - c_2)$ for $x \in \mathbb{R}^2$.*

Suppose that a is a double maximum point for f on $\mathbb{R}^2$. By Proposition 25 we get that $f'(a) = 0$, that is $4D(a_1 - c_1)(a_1 - c_1) = 0$. Therefore $a_1 = c_1$ or $a_2 = c_2$. We consider the case when $a_1 = c_1$ (the case when $a_2 = c_2$ reaches the same conclusion) so that $f'(x) = 4D(x_1 - a_1)(x_2 - c_2)$. By Proposition 26 $f'(x) < 0$ for large enough x. Thus $D < 0$. Also, by the same proposition, if $x_1 \neq a_1$, then $f'(x)$ should change sign as x_2 goes from a value less than a_2 to a value larger than a_2. Hence $c_2 = a_2$ so that $f'(x) = 4D(x_1 - a_1)(x_2 - a_2)$. Using this and Proposition 26, it is easy to see that a now is a double maximum point for f on $\mathbb{R}^2$.

A similar analysis reveals that a is double minimum point for f on $\mathbb{R}^2$ if and only if $a = c$ and $D > 0$.

Remark 28. *To the best of our knowledge, Bögel neither treats double critical points nor a double Fermat's theorem. However, in [2, §6] Bögel studies monotone double functions in the context of functions of bounded variation.*

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Suppose that f is a double function defined on a double interval I. We say that a double function F defined on I is a *double primitive function of f on I* if F is double differentiable on I and $F'(x) = f(x)$ for $x \in I$.

Proposition 29. *Suppose that f is a double function defined on a double interval I. If F and G are double primitive functions of f on I, then there is a double constant function H, defined on I, such that $G = F + H$.*

Proof. Put $H = G - F$. Then $H' = G' - F' = f - f = 0$. Proposition 22(c) implies that H is a double constant function. Clearly $G = F + H$. $\qquad\square$

5 Double integrability

In this section, we define the double Newton integral. Using the double mean value theorem, we obtain a mean value theorem for double Newton integrals (see Proposition 34). After that, we connect the double Newton integral to the Riemann double integral in the first and second double fundamental theorems of calculus (see Proposition 36 and Proposition 37). At the end of this section, we introduce improper double Newton integrals. We also discuss examples of double integrals over non-rectangular regions. Most of the material in this section (except the discussion on improper integrals) can be extracted and specialized from [2]. However, since we only restrict ourselves to the double calculus, our presentation can be more streamlined.

Let f be a double function defined on a double interval I. Suppose that there exists a double primitive function F of f on I. Let $a, b \in I$. We say that the *double Newton integral of f from a to b* is the real number:

$$\int_a^b f := \Delta_a^b(F). \tag{5.1}$$

Proposition 30 (The double Newton integral is well defined)**.** *The value of (5.1) does not depend on the choice of the double primitive function.*

Proof. Let f, F, G be double functions defined on I where F and G are double primitive functions of f on I. By Proposition 29, $G = F + H$ for some double constant function H defined on I. Take $a, b \in I$. Then it follows that $\Delta_a^b(G) = \Delta_a^b(F) + \Delta_a^b(H) = \Delta_a^b(F) + 0 = \Delta_a^b(F)$. $\qquad\square$

Example 31. *Suppose that F and f are the double functions defined on $I := [0,2] \times [1,3]$ by $F(x) = x_1^2 x_2^3 / 2$ and $f(x) = 3x_1 x_2^2$ for $x \in I$. By Proposition 10, F is double differentiable on I with $F' = f$. Therefore:*

$$\int_{(0,1)}^{(2,3)} f \;=\; \Delta_{(0,1)}^{(2,3)}(F) \;=\; F(2,3) - F(2,1) - F(0,3) + F(0,1) \;=\; 52.$$

Proposition 32 (Properties of the double Newton integral). *Let f be a double function defined on a double interval I. Suppose that there exists a double primitive function F of f on I. If $a, b, c \in I$, then:*

(a) $\displaystyle\int_a^a f = 0; \quad \int_a^b f = \int_b^a f; \quad \int_a^b f = -\int_{(a_1,b_2)}^{(b_1,a_2)} f;$

(b) $\displaystyle\int_a^b f = \int_a^{(c_1,b_2)} f + \int_{(c_1,a_2)}^b f; \quad$ *(c)* $\displaystyle\int_a^b f = \int_a^{(b_1,c_2)} f + \int_{(a_1,c_2)}^b f;$

(d) $\displaystyle\int_a^b f = \int_a^c f + \int_c^b f + \int_{(a_1,c_2)}^{(c_1,b_2)} f + \int_{(c_1,a_2)}^{(b_1,c_2)} f.$

Proof. This follows immediately from Proposition 2. $\square$

Proposition 33 (The double Newton integral is a primitive function). *Let f be a double function defined on a double interval I. Suppose that there exists a double primitive function F of f on I. Take $a \in I$ and define the map $G : I \to \mathbb{R}$ by $G(x) \mapsto \int_a^x f$ for $x \in I$.*

(a) The identity $\Delta_b^x(G) = \Delta_b^x(F)$ holds for all $b, x \in I$.

(b) The function G is double differentiable on I and $G'(b) = f(b)$ for $b \in I$.

Proof. Take $a, b, x \in I$. First we show (a). By Proposition 32 we get that:

$$
\begin{aligned}
\Delta_b^x(G) &= \int_a^x f + \int_a^b f - \int_a^{(x_1,b_2)} f - \int_a^{(b_1,x_2)} f \\
&= \int_a^x f + \int_b^a f + \int_{(a_1,b_2)}^{(x_1,a_2)} f + \int_{(a_1,x_2)}^{(b_1,a_2)} f = \int_b^x f = \Delta_b^x(F).
\end{aligned}
$$

Next we show (b). By (a) we get that:

$$
\begin{aligned}
G'(b) &= \lim_{x \rightsquigarrow b} m_b^x(G) = \lim_{x \rightsquigarrow b} \frac{\Delta_b^x(G)}{(x_1 - b_1)(x_2 - b_2)} \\
&= \lim_{x \rightsquigarrow b} \frac{\Delta_b^x(F)}{(x_1 - b_1)(x_2 - b_2)} = \lim_{x \rightsquigarrow b} m_b^x(F) = F'(b) = f(b).
\end{aligned}
$$

Alternatively, the equality $G' = f$ follows from the fact that G equals F plus the double constant function $I \ni (x_1, x_2) \mapsto F(a_1, a_2) - F(x_1, a_2) - F(a_1, x_2)$ which by Proposition 1 has zero double derivative. $\square$

Proposition 34 (The mean value theorem for double Newton integrals). *Let f be a double function defined on a double interval $[a, b]$ for some $a, b \in \mathbb{R}^2$ with $a < b$. Suppose that there exists a double primitive function F of f on $[a, b]$. Then there exists $c \in (a, b)$ such that $\int_a^b f = f(c)(b_1 - a_1)(b_2 - a_2)$.*

Proof. This follows from Propositions 14, 18 and 33. □

We now recall some classical notions (cf. e.g. [3]). Suppose that f is a double function defined on a double interval $[a, b]$ for some $a, b \in \mathbb{R}^2$ with $a < b$. A *partition* P of $[a, b]$ is a choice of points $x_{1,0}, x_{1,1}, \ldots, x_{1,m} \in [a_1, b_1]$ and $x_{2,0}, x_{2,1}, \ldots, x_{2,n} \in [a_2, b_2]$ such that $a_1 = x_{1,0} < x_{1,1} < \cdots < x_{1,m-1} < x_{1,m} = b_1$ and $a_2 = x_{2,0} < x_{2,1} < \cdots < x_{2,n-1} < x_{2,n} = b_2$. Given P, define the mn rectangles $R_{ij} = [x_{1,i-1}, x_{1,i}] \times [x_{2,j-1}, x_{2,j}]$, for $1 \le i \le m$ and $1 \le j \le n$. The *norm* $|P|$ of P is the largest of the diagonals in these mn rectangles. Pick an arbitrary point $(x^*_{1,i,j}, x^*_{2,i,j})$ in each of the rectangles R_{ij}. For all i and j put $\Delta x_{1,i} = x_{1,i} - x_{1,i-1}$ and $\Delta x_{2,j} = x_{2,j} - x_{2,j-1}$. The corresponding *double Riemann sum* is defined as $R(f, P) := \sum_{i=1}^{m} \sum_{j=1}^{n} f(x^*_{1,i,j}, x^*_{2,i,j}) \Delta x_{1,i} \Delta x_{2,j}$. The double function f is said to be *Riemann integrable* over $[a, b]$ and have *double integral* $I = \iint_{[a,b]} f(x) \, dx_1 dx_2$, if for every $\epsilon \in \mathbb{R}_+$ there is $\delta \in \mathbb{R}_+$ such that $|R(f, P) - I| < \epsilon$ holds for every partition P of $[a, b]$ satisfying $|P| < \delta$ and for all choices of $(x^*_{1,i,j}, x^*_{2,i,j})$ in the subrectangles of P. If f is continuous on $[a, b]$, then f is Riemann integrable there (see e.g. [3, p. 293]).

Proposition 35 (The mean value theorem for double Riemann integrals). *Let $a, b \in \mathbb{R}^2$ and $a < b$. Suppose that $f : [a, b] \to \mathbb{R}$ is a double function which is continuous. Then there exists $c \in (a, b)$ such that $\iint_{[a,b]} f(x) \, dx_1 dx_2 = f(c)(b_1 - a_1)(b_2 - a_2)$.*

Proof. See e.g. [3, p. 292]. □

Proposition 36 (The first double fundamental theorem of calculus). *Let $a, b \in \mathbb{R}^2$ and $a < b$. Suppose that $f : [a, b] \to \mathbb{R}$ is a double function which is continuous. Define $G : [a, b] \to \mathbb{R}$ by $G(x) = \iint_{[a,x]} f(x) \, dx_1 dx_2$ for $x \in [a, b]$. Then G is double differentiable on $[a, b]$ with $G' = f$.*

Proof. Take $x \in [a, b]$. We consider four cases.

Case 1: $x < b$. Take $h \in \mathbb{R}^2_{++}$ such that $x + h < b$. By Proposition 35:

$$m_x^{x+h}(G) = \frac{\iint_{[x,x+h]} f(x) \, dx_1 dx_2}{h_1 h_2} = f(t)$$

for some $t \in (x, x + h)$. Letting $h \to 0^{++}$ yields that $G'_{++}(x) = f(x)$.

Case 2: $a < x$. Take $h \in \mathbb{R}^2_{--}$ such that $a < x + h$. By Proposition 35:

$$m_x^{x+h}(G) = \frac{\iint_{[x+h,x]} f(x) \, dx_1 dx_2}{(-h_1)(-h_2)} = f(t)$$

for some $t \in (x + h, x)$. Letting $h \to 0^{++}$ yields that $G'_{--}(x) = f(x)$.

Case 3: $(a_1, x_2) < (x_1, b_2)$. Take $h \in \mathbb{R}^2_{-+}$ such that $x_1 + h_1 > a_1$ and $x_2 + h_2 < b_2$. By Proposition 35:

$$m_x^{x+h}(G) = \frac{\iint_{[x_1+h_1,x_1] \times [x_2,x_2+h_2]} f(x) \, dx_1 dx_2}{(-h_1) h_2} = f(t)$$

for some $t \in (x_1+h_1, x_1) \times (x_2, x_2+h_2)$. Letting $h \to 0^{-+}$ yields that $G'_{-+}(x) = f(x)$.

Case 4: $(x_1, a_2) < (b_1, x_2)$. Take $h \in \mathbb{R}^2_{+-}$ such that $x_1+h_1 < b_1$ and $x_2+h_2 > a_2$. By Proposition 35:

$$m_x^{x+h}(G) = \frac{\iint_{[x_1, x_1+h_1] \times [x_2+h_2, x_2]} f(x)\ dx_1 dx_2}{h_1(-h_2)} = f(t)$$

for some $t \in (x_1, x_1+h_1) \times (x_2+h_2, x_2)$. Letting $h \to 0^{+-}$ yields that $G'_{+-}(x) = f(x)$.

Cases 1-4 show that G is double differentiable on $[a, b]$ with $G' = f$. $\square$

Proposition 37 (The second double fundamental theorem of calculus)**.** *Let f be a double function defined on a double interval $[a, b]$ for some $a, b \in \mathbb{R}^2$ with $a < b$. Suppose that there exists a double primitive function F of f on $[a, b]$. If f is Riemann integrable on $[a, b]$, then $\iint_{[a,b]} f(x)\ dx_1 dx_2 = \int_a^b f$.*

Proof. Take $\epsilon \in \mathbb{R}_+$ and put $I := \int_a^b f$. Since $[a, b]$ is a compact interval, f is uniformly continuous on $[a, b]$ (see e.g. [8, Theorem 4.19]). Therefore, there exists $\delta \in \mathbb{R}_+$ such that:

$$|f(c) - f(d)| < \frac{\epsilon}{(b_1 - a_1)(b_2 - a_2)}$$

whenever $c, d \in [a, b]$ and $\sqrt{(c_1 - d_1)^2 + (c_2 - d_2)^2} < \delta$. Consider a fixed double Riemann sum:

$$R := \sum_{i=1}^m \sum_{j=1}^n f(x^*_{1,i,j}, x^*_{2,i,j}) \Delta x_{1,i} \Delta x_{2,j}$$

defined by a partition P, with $|P| < \delta$, and a choice of points $(x^*_{1,i,j}, x^*_{2,i,j})$ in the corresponding rectangles. We wish to show that $|R - I| < \epsilon$. By Proposition 34 there exist $c_{i,j} \in (x_{1,i-1}, x_{1,i}) \times (x_{2,j-1}, x_{2,j})$ with:

$$\int_{(x_{1,i-1}, x_{2,j-1})}^{(x_{1,i}, x_{2,j})} f = f(c_{i,j}) \Delta x_{1,i} \Delta x_{2,i}$$

for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. By repeated application of Proposition 32(b)(c) we therefore get that:

$$I = \sum_{i=1}^m \sum_{j=1}^n \int_{(x_{1,i-1}, x_{2,j-1})}^{(x_{1,i}, x_{2,j})} f = \sum_{i=1}^m \sum_{j=1}^n f(c_{i,j}) \Delta x_{1,i} \Delta x_{2,i}$$

which in turn implies that:

$$
\begin{aligned}
|R - I| &= \left| \sum_{i=1}^{m} \sum_{j=1}^{m} (f(x_{1,i,j}^*, x_{2,i,j}^*) - f(c_{i,j})) \Delta x_{1,i} \Delta x_{2,i} \right| \\
&\leq \sum_{i=1}^{m} \sum_{j=1}^{m} \left| f(x_{1,i,j}^*, x_{2,i,j}^*) - f(c_{i,j}) \right| \Delta x_{1,i} \Delta x_{2,i} \\
&< \frac{\epsilon}{(b_1 - a_1)(b_2 - a_2)} \sum_{i=1}^{m} \sum_{j=1}^{m} \Delta x_{1,i} \Delta x_{2,i} \\
&= \frac{\epsilon}{(b_1 - a_1)(b_2 - a_2)} \cdot (b_1 - a_1)(b_2 - a_2) = \epsilon.
\end{aligned}
$$

We have now shown that $\iint_{[a,b]} f(x) \, dx_1 dx_2 = \int_a^b f$. $\square$

Let f be a double function defined on a double *open* interval (a, b) for some $a, b \in \overline{\mathbb{R}}^2$ with $a < b$. Suppose that there exists a double primitive function F of f on (a, b). If the double signed limit:

$$
\lim_{y \to b^{--}; \; x \to a^{++}} \Delta_x^y(F) \tag{5.2}
$$

exists, then we say that $\int_a^b f$ is a *convergent* improper double Newton integral with *value* equal to the limit (5.2). If the limit in (5.2) does not exist, then we say that the improper double Newton integral $\int_a^b f$ is *divergent*. Note that if all of the following four signed limits exist:

$$
A := \lim_{x \to b^{--}} F(x) \qquad B := \lim_{x \to a^{++}} F(x)
$$

$$
C := \lim_{x \to (b_1, a_2)^{-+}} F(x) \qquad D := \lim_{x \to (a_1, b_2)^{+-}} F(x)
$$

then $\int_a^b f$ is convergent with value equal to $A + B - C - D$.

Example 38. *(a) Suppose that F and f are the double functions defined on $I := (0, 1) \times (0, 1)$ by $F(x) = (x_1 + x_2) \ln(x_1 + x_2)$ and $f(x) = 1/(x_1 + x_2)$ for $x \in I$. By Proposition 10, F is double differentiable on I with $F' = f$. Since*

$$
\lim_{x \to (1,1)^{--}} F(x_1, x_2) = 2 \ln(2)
$$

and

$$
\lim_{x \to (0,0)^{++}} F(x_1, x_2) = \lim_{x \to (1,0)^{-+}} F(x_1, x_2) = \lim_{x \to (0,1)^{+-}} F(x_1, x_2) = 0
$$

the improper double Newton integral $\int_{(0,0)}^{(1,1)} 1/(x_1+x_2)$ is convergent with value $2\ln(2)$.

(b) Suppose that F and f are the double functions defined on $I := (0,1] \times (0,1]$ by $F(x) = x_1/(x_1 + x_2)$ and $f(x) = (x_1 - x_2)/(x_1 + x_2)^3$ for $x \in I$. By Proposition 10, F is double differentiable on I with $F' = f$. However, since $\lim_{s \to 0^+} \Delta_{(s,s)}^{(1,1)}(F) = 0$ and $\lim_{t \to 0^+} \Delta_{(t,2t)}^{(1,1)}(F) = -1/6$ the improper double Newton integral

$$\int_{(0,0)}^{(1,1)} (x_1 - x_2)/(x_1 + x_2)^3$$

is divergent.

Many standard calculus textbook problems concerning double integrals over non-rectangular regions are solved by iterated integration. If, however, the region in question can be mapped bijectively onto a double interval, then such integrals can instead be considered as improper double Newton integrals. In fact, by Proposition 37 and the result in [9] we get the following:

Proposition 39. Let $a, b \in \overline{\mathbb{R}}^2$ with $a < b$. Suppose that D is an open subset of $\mathbb{R}^2$ and that $h : (a, b) \to D$ is a bijection such that h and h^{-1} are continuous and have continuous partial derivatives. Let $J(x)$ denote the absolute value of the Jacobian determinant of h at $x \in (a, b)$. If $f : D \to \mathbb{R}$ is a function which is integrable on D and $g := (f \circ h) \cdot J$ has a double primitive function on (a, b), then the improper double Newton integral of g from a to b is convergent and $\iint_D f(x)\, dx_1 dx_2 = \int_a^b g$.

Example 40. (a) We wish to evaluate $\iint_D f(x)\, dx_1 dx_2$ where $f(x) = x_1 x_2$ for $x \in D$, and D is the interior of the triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$. Define $h : (0,1) \times (0,1) \to D$ by $h(u,v) = (u, uv)$. Then $J(u,v) = u$ and $\iint_D f(x)\, dx_1 dx_2 = \int_{(0,0)}^{(1,1)} u \cdot uv \cdot u = \int_{(0,0)}^{(1,1)} u^3 v = \Delta_{(0,0)}^{(1,1)}(u^4 v^2/8) = 1/8$.

(b) We wish to evaluate $\iint f(x)\, dx_1 dx_2$ where $f(x) = 1/(x_1 + x_2)^2$ for $x \in D$, and D is defined by $0 < x_1 < 1$ and $0 < x_2 < x_1^2$. Define $h : (0,1) \times (0,1) \to D$ by $h(u,v) = (u, u^2 v)$ for $u, v \in (0,1)$. Then $J(u,v) = u^2$ and $\iint_D f(x)\, dx_1 dx_2 = \int_{(0,0)}^{(1,1)} u^2/(u + vu^2)^2 = \int_{(0,0)}^{(1,1)} 1/(1+uv)^2 = \Delta_{(0,0)}^{(1,1)}(\ln(1+uv)) = \ln(2)$.

(c) We wish to evaluate $\iint f(x)\, dx_1 dx_2$ where $f(x) = e^{-x_1^2}$ for $x \in D$, and D is defined by $x_1 > 0$ and $-x_1 < x_2 < x_1$. Define $h : (0,\infty) \times (-1,1) \to D$ by $h(u,v) = (u, uv)$ for $u \in (0,\infty)$ and $v \in (-1,1)$. Then $J(u,v) = u$ and $\iint_D f(x)\, dx_1 dx_2 = \int_{(0,-1)}^{(\infty,1)} e^{-u^2} u = \lim_{s \to \infty} \Delta_{(0,-1)}^{(s,1)}(-e^{-u^2}v/2) = 1$.

(d) We wish to evaluate $\iint f(x)\, dx_1 dx_2$ where $f(x) = 1/(x_1 + x_2)$ for $x \in D$, and D is defined by $1 < x_1$ and $0 < x_2 < 1/x_1$. Define $h : (1,\infty) \times (0,1) \to D$ by $h(u,v) = (u, v/u)$ for $u \in (1,\infty)$ and $v \in (0,1)$. Then $J(u,v) = u^{-1}$ and

$$\iint_D f(x)\, dx_1 dx_2 = \int_{(1,0)}^{(\infty,1)} (u + v/u)^{-1} \cdot u^{-1} = \int_{(1,0)}^{(\infty,1)} (u^2 + v)^{-1}$$
$$= \quad \lim_{t \to \infty;\ s \to 0^+} \Delta_{(1,s)}^{(t,1)} \left(u\ln(u^2 + v) + 2\sqrt{v}\arctan(u/\sqrt{v}) \right) = \pi/2 - \ln(2).$$

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6   Triple calculus, quadruple calculus and beyond

It is easy to work out the corresponding difference operators in higher dimensions. Namely, if $a, b \in \mathbb{R}^n$ and f is an n-variable function, then:

$$\Delta_a^b(f) \;\;=\;\; \sum_{s \in \{0,1\}^n} (-1)^s f(s_1 a_1 + (1 - s_1)b_1, \ldots, s_n a_n + (1 - s_n)b_n)$$

where $(-1)^s := (-1)^{s_1 + s_2 + \cdots + s_n}$. So, for instance, if $n = 3$, then we get that:

$$
\begin{aligned}
\Delta_a^b(f) \;\;=\;\;& f(b_1, b_2, b_3) \;-\; f(a_1, b_2, b_3) \;-\; f(b_1, a_2, b_3) \;-\; f(b_1, b_2, a_3) \\
+\;\;& f(a_1, a_2, b_3) \;+\; f(a_1, b_2, a_3) \;+\; f(b_1, a_2, a_3) \;-\; f(a_1, a_2, a_3).
\end{aligned}
$$

In [1, 2] higher-dimensional analogues of all the results established in this article are shown to hold, that is there is also a triple calculus, a quadruple calculus and beyond, at our disposal.

It seems to the author of the present article that Bögel did not consider higher-dimensional versions of Schwarz's theorem. To be more precise, suppose, for instance, that f is a five-variable function, that f is double differentiable with respect to the first two variables, with double derivative denoted by f'_{12}, and that f is triple differentiable with respect to the last three variables, with triple derivative denoted by f'_{345}. If we also suppose that the iterated derivatives $(f'_{345})'_{12}$ and $(f'_{12})'_{345}$ exist and are double respectively triple continuous, does it then follow that f is quintuple differentiable with $f'_{12345} = (f'_{345})'_{12} = (f'_{12})'_{345}$? Since the proof of Proposition 10 only depends on the mean value theorem in each variable, it seems reasonable to believe that this proof is generalizable to higher dimensions if we use Bögel's higher-dimensional mean value theorem.

Another classical result from calculus that neither we nor Bögel has considered is Darboux's theorem. Recall that this result states that if a single variable function is differentiable on an open interval, then the derivative enjoys the intermediate value property on this interval. It is not clear to the author of the present article if there is a double (or higher-dimensional) analogue of this result. Note that the usual text book proof for Darboux's theorem uses the Weierstrass extreme value theorem (see e.g. [6]), which, as we have pointed out earlier, is not at our disposal for double functions. However, there are proofs of Darboux's theorem which are based only on the mean value theorem for differentiable functions and the intermediate value theorem for continuous functions (see loc. cit.). Therefore, it is plausible that there indeed is a double (and higher) version(s) of Darboux's theorem which is (are) reachable by the methods used in this article.

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