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FIXED FILTER BASES OF MULTIPLICATIVE CONTRACTIONS ON MULTIPLICATIVE METRIC SPACES

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Abstract. Fixed filter base $\mathbb{B}$ of a mapping f consists of multiplicative open sets such that for given $B \in \mathbb{B}$, there is an $A \in \mathbb{B}$ satisfying $f(A) \subseteq B$. Also, the constructions of fixed filter bases have been established and some results for fixed filter bases of different types of multiplicative contraction mappings are derived in multiplicative metric spaces. Moreover, the concept of a fixed point at infinity is obtained in multiplicative metric spaces.

1 Introduction

A point in a multiplicative metric space is the intersection of members of its local base of the neighbourhood system, when the point is considered as a singleton set. A local base of multiplicative open sets containing a specific point is a filter base. So, a fixed point of a multiplicative contraction may also be considered as a filer base of the neighbourhood system. The concept of a fixed filter base was introduced by C. G. Moorthy and S.I. Raj in [8]. Also, the concept of a fixed point at infinity in metric spaces was provided in [8].

Multiplicative metric space(or, MMS) was introduced by A.E. Bashirov et al in [3]. Topological properties of multiplicative metric spaces(or, MMSs) were derived and fixed point results of multiplicative contractions were proved by M. Ozavsar, and A. C. Cevikel in [9]. There are many articles appearing on fixed point theory in MMSs; see [1, 2, 4, 5, 6, 7].

In this article, we introduce the concept of a fixed filter base in MMSs. Also, we prove some results for fixed filter bases of different types of multiplicative contractions on MMS. Moreover, we modify the concept of a fixed point at infinity from metric spaces to MMSs.

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2 Multiplicative Metric Spaces

Definition 1. *[3] Let $H \neq \emptyset$ be a set. A multiplicative metric is a mapping $d : H \times H \to [1, \infty)$ satisfying the following axioms.*

(i) $d(k, l) = 1$ *if and only if* $k = l$ *in* H,

(ii) $d(k, l) = d(l, k)$, $\forall\ k, l \in H$,

(iii) $d(k, l) \leq d(k, p)d(p, l)$, $\forall\ k, l, p \in H$.

Then the pair (H, d) is called a MMS.

Definition 2. *[9] Let (H, d) be a MMS and $A \subseteq H$. Then $k \in A$ is called multiplicative interior point of A if there exists an $\epsilon > 1$ such that $B_\epsilon(k) \subseteq A$, where $B_\epsilon(k) = \{l : d(k, l) < \epsilon\}$.*

Definition 3. *[9] Let (H, d) be a MMS and $A \subseteq H$. Then A is said to be multiplicative open if each point of A is a multiplicative interior point of A.*

Definition 4. *[9] Let (H, d) be a MMS and $A \subseteq H$. A point $k \in H$ is said to be multiplicative limit point of A if $(B_\epsilon(k) \setminus \{k\}) \cap A \neq \emptyset$, for every $\epsilon > 1$. A' denotes collection of multiplicative limit points of A*

Proposition 5. *[9] Let (H, d) be a MMS and $A \subseteq H$. Then A is multiplicative closed if and only if complement of A is multiplicative open. Complement of A is denoted by A^C.*

Proposition 6. *[9] Let (H, d) be a MMS and $A \subseteq H$. Then $\overline{A} = A \cup A'$ is called closure of A, and $\overline{A}$ is a multiplicative closed*

Definition 7. *[9] Let (H, d) be a MMS, $\{k_n\}$ be a sequence in H, and $k \in H$. If for every multiplicative open ball $B_\epsilon(k) = \{l : d(k, l) < \epsilon\}$, $\epsilon > 1$, there exists a natural number $N \in \mathbb{N}$ such that $n > N$ then $k_n \in B_\epsilon(k)$. Then the sequence $\{k_n\}$ is said to be multiplicative converging to k, denoted by $k_n \to k (n \to \infty)$.*

Definition 8. *[9] Let (H, d) be a MMS and $\{k_n\}$ be a sequence in H. The sequence is called a multiplicative Cauchy sequence if it holds that, for all $\epsilon > 1$, there exists $N \in \mathbb{N}$ such that $d(k_n, k_m) < \epsilon, \forall m, n \geq N$.*

Lemma 9. *[9] Let (H, d) be a MMS and $\{k_n\}$ be a sequence in H. Then $\{k_n\}$ is a multiplicative Cauchy sequence if and only if $d(k_n, k_m) \to 1 (m, n \to \infty)$.*

Definition 10. *[9] A MMS (H, d) is said to be multiplicative complete, if every multiplicative Cauchy sequence is multiplicative convergent in H.*

Definition 11. *[9] Let A be a subset of MMS (H, d). If diam $A = \sup\{d(k,l) : k,l \in A\}$ exists, then A is called multiplicative bounded. Moreover A is multiplicative unbounded if diam A does not exist.*

Remark 12. *Closure of a multiplicative bounded set is also a multiplicative bounded set.*

Assumption 13. *Let $\mathbb{B}$ be a local base of multiplicative bounded subsets to the neighbourhoods of a point in a MMS (H, d), where $\mathbb{B}$ is a directed set with respect to inclusion relation. Then $(diam\ B)_{B\in\mathbb{B}} \to 1, \forall B \in \mathbb{B}$*

Convention 14. *Let $\mathbb{B}$ be a filter base in MMS (H, d). If $\mathbb{B}$ satisfies following conditions*

(i) $A \in \mathbb{B} \Rightarrow A$ *is multiplicative open, and nonempty;*

(ii) $A, B \in \mathbb{B} \Rightarrow$ *there exists* $C \in \mathbb{B}$ *such that* $C \subseteq A \cap B$; *and*

(iii) $(diam\ A)_{A\in\mathbb{B}} \to 1$, *where* $\mathbb{B}$ *is a directed set,*

then $\mathbb{B}$ is called one multiplicative converging open filter base.

Definition 15. *Let f be a function on a MMS (H, d) to itself. Let $\mathbb{B}$ be a one multiplicative converging open filter base in (H, d). If for given $B \in \mathbb{B}$, there exists $A \in \mathbb{B}$ such that $f(A) \subseteq B$, then $\mathbb{B}$ is called a fixed filter base of f in H.*

The Cantor's intersection theorem for MMS is proved as Theorem 2.5 in [10]. We use this theorem to prove the following Proposition 16.

Proposition 16. *Let (H, d) be a multiplicative complete MMS. Let f be a self map on H with a fixed filter base $\mathbb{B}$ in H. Let for each $A \in \mathbb{B}$, there is a $B \in \mathbb{B}$ such that $\overline{B} \subseteq A$. Then there is a fixed point $x \in H$ such that $\mathbb{B}$ is a local base for the neighbourhood system of x.*

Proof. Since $diam\ \overline{A} \to 0$ as A varies in $\mathbb{B}$ and H is multiplicative complete, then $\bigcap_{A\in\mathbb{B}} \overline{A} = \{x\}$, for some $x \in H$. The equality $\bigcap_{A\in\mathbb{B}} A = \{x\}$ is also true, because of our assumptions. Since H has a fixed filter base, then for each $A \in \mathbb{B}$, there is a $B \in \mathbb{B}$ such that $f(B) \subseteq A$. Hence $f(x) \in A, \forall A \in \mathbb{B}$. Therefore x is a fixed point of f. Since $diam\ A \to 0$ as A varies in $\mathbb{B}$, $x \in A$ and A is multiplicative open $\forall A \in \mathbb{B}$, and $\bigcap_{A\in\mathbb{B}} A = \{x\}$, then the fixed filter base $\mathbb{B}$ is a local base to the neighbourhood system of x. $\square$

3 Fixed Filter Bases

Assumption 17. *Let (S,d) be a MMS. For $\varrho \in S$ and $k \in (0,1)$, define* $U(\varrho, r^{k^n}) = \{\varpi \in S : d(\varrho,\varpi) < r^{k^n}, r > 1, n = 1,2,...\}$

Assumption 17 is assumed in the following four theorems.

Theorem 18. *Let (S,d) be a MMS. Let f be a function on S to itself such that $d(f(\varrho), f(\varpi)) \leq (d(\varrho,\varpi))^k$, $\forall \varrho, \varpi \in S$, for some $k \in (0,1)$. If $\varrho_{n+1} = f(\varrho_n), n = 1,2,...$, where ϱ_1 is fixed, are defined, then $\left\{ \bigcup\limits_{i \geq n} U(\varrho_i, r^{k^n}) : r > 1, n = 1,2,... \right\}$ is a fixed filter base of f in S.*

Proof. Take $U(\varrho_i, r^{k^n}) = \{\varpi \in S : d(\varrho_i,\varpi) < r^{k^n}, r > 1, n = 1,2,...\}$. Let $U_{n,r^{k^n}} = \bigcup\limits_{i \geq n} U(\varrho_i, r^{k^n})$, where ϱ_1 is fixed and $\varrho_{n+1} = f(\varrho_n), n = 1,2,....$ Then

$$\begin{aligned} d(\varrho_{n+2}, \varrho_{n+1}) &= d(f(\varrho_{n+1}), f(\varrho_n)) \\ &\leq (d(\varrho_{n+1}, \varrho_n))^k \\ &\leq (d(\varrho_n, \varrho_{n-1}))^{k^2} \leq ... \leq (d(\varrho_2, \varrho_1))^{k^n}. \end{aligned}$$

For $m > n$,

$$\begin{aligned} d(\varrho_m, \varrho_n) &\leq d(\varrho_m, \varrho_{m-1}) d(\varrho_{m-1}, \varrho_{m-2}) ... d(\varrho_{n+1}, \varrho_n) \\ &\leq (d(\varrho_2, \varrho_1))^{k^{m-2}} (d(\varrho_2, \varrho_1))^{k^{m-3}} ... (d(\varrho_2, \varrho_1))^{k^{n-1}} \\ &= (d(\varrho_2, \varrho_1))^{k^{n-1}(1+k+...+k^{m-n-1})} \\ &\leq (d(\varrho_2, \varrho_1))^{\left(\frac{k^{n-1}}{1-k}\right)} \end{aligned}$$

Thus $d(\varrho_m, \varrho_n) \to 1$ as $n \to \infty$ with $m > n$. Therefore $\lim\limits_{n\to\infty} diam\{\varrho_n, \varrho_{n+1}, ...\} = 1$. Let $\varrho \in U_{n,r^{k^n}}$, then $\varrho \in U(\varrho_i, r^{k^n})$, for some $i \geq n$,

$$d(f(\varrho), \varrho_{i+1}) = d(f(\varrho), f(\varrho_i)) \leq (d(\varrho, \varrho_i))^k < r^{kk^n} = r^{k^{n+1}}.$$

Therefore $f(\varrho) \in U(\varrho_{i+1}, r^{k^{n+1}})$. Thus, we get $f(U_{n,r^{k^n}}) \subseteq U_{n+1,r^{k^{n+1}}}$. Also,

$$f(U_{n+1,r^{k^{n+1}}}) \subseteq f(U_{n+1,r^{k^n}}) \subseteq f(U_{n,r^{k^n}}) \subseteq U_{n+1,r^{k^{n+1}}} \subseteq U_{n+1,r^{k^n}} \subseteq U_{n,r^{k^n}}.$$

So, $\{U_{n,r^{k^n}} : r > 1, n = 1,2,...\}$ is a fixed filter base of f in S, because $\lim\limits_{n\to\infty} diam\, U_{n,r^{k^n}} = 1$, and $f(U_{n+1,r^{k^{n+1}}}) \subseteq U_{n+1,r^{k^{n+1}}} \subseteq U_{n,r^{k^n}}, n = 1,2,....$ □

Theorem 19. *Let (S,d) be a MMS. Let f be a function on S to itself such that $d(f(\varrho), f(\varpi)) \leq [d(f(\varrho),\varrho) d(f(\varpi),\varpi)]^k$, $\forall \varrho, \varpi \in S$, for some $k \in (0, \frac{1}{4})$. If $\varrho_{n+1} = f(\varrho_n), n = 1,2,...$, where ϱ_1 is fixed, are defined, then $\left\{ \bigcup\limits_{i \geq n} U(\varrho_i, r^{(3q)^n}) : q = (\frac{k}{1-k}), r > 1, n = 1,2,... \right\}$ is a fixed filter base of f in S.*

Proof. Take $U(\varrho_i, r^{(3q)^n}) = \{\varpi \in S : d(\varrho_i, \varpi) < r^{(3q)^n}, r > 1, n = 1, 2, ...\}$. Let $U_{n,r^{(3q)^n}} = \bigcup\limits_{i\geq n} U(\varrho_i, r^{(3q)^n})$, where ϱ_1 is fixed and $\varrho_{n+1} = f(\varrho_n), n = 1, 2, ...$ Then

$$\begin{aligned}
d(\varrho_{n+2}, \varrho_{n+1}) &= d(f(\varrho_{n+1}), f(\varrho_n)) \\
&\leq (d(f(\varrho_{n+1}), \varrho_{n+1})d(f(\varrho_n), \varrho_n))^k \\
&= (d(\varrho_{n+2}, \varrho_{n+1}))^k (d(\varrho_{n+1}, \varrho_n))^k \\
d(\varrho_{n+2}, \varrho_{n+1}) &\leq (d(\varrho_{n+1}, \varrho_n))^{\left(\frac{k}{1-k}\right)} \\
&\leq (d(\varrho_{n+1}, \varrho_n))^q, \text{ where } q = \left(\frac{k}{1-k}\right) \\
&\leq (d(\varrho_n, \varrho_{n-1}))^{q^2} \leq ... \leq (d(\varrho_2, \varrho_1))^{q^n}.
\end{aligned}$$

For $m > n$,

$$\begin{aligned}
d(\varrho_m, \varrho_n) &\leq d(\varrho_m, \varrho_{m-1})d(\varrho_{m-1}, \varrho_{m-2})...d(\varrho_{n+1}, \varrho_n) \\
&\leq (d(\varrho_2, \varrho_1))^{q^{m-2}} (d(\varrho_2, \varrho_1))^{q^{m-3}} ...(d(\varrho_2, \varrho_1))^{q^{n-1}} \\
&= (d(\varrho_2, \varrho_1))^{q^{n-1}(1+q+...+q^{m-n-1})} \\
&\leq (d(\varrho_2, \varrho_1))^{\left(\frac{q^{n-1}}{1-q}\right)}, \text{ where } q \in (0, 1)
\end{aligned}$$

Thus $d(\varrho_m, \varrho_n) \to 1$ as $n \to \infty$ with $m > n$. Therefore $\lim\limits_{n\to\infty} diam\{\varrho_n, \varrho_{n+1}, ...\} = 1$. Let $\varrho \in U_{n,r^{(3q)^n}}$, then $\varrho \in U(\varrho_i, r^{(3q)^n})$, for some $i \geq n$,

$$\begin{aligned}
d(f(\varrho), \varrho_{i+1}) &= d(f(\varrho), f(\varrho_i)) \\
&\leq (d(f(\varrho), \varrho)d(f(\varrho_i), \varrho_i))^k \\
&= (d(f(\varrho), \varrho))^k (d(\varrho_{i+1}, \varrho_i))^k \\
&\leq (d(f(\varrho), \varrho_{i+1}))^k (d(\varrho_{i+1}, \varrho_i))^{2k} (d(\varrho_i, \varrho))^k \\
&\leq (d(f(\varrho), \varrho_{i+1}))^k (d(\varrho_2, \varrho_1))^{2kq^{i-1}} (d(\varrho_i, \varrho))^k \\
d(f(\varrho), \varrho_{i+1}) &\leq (d(\varrho, \varrho_i))^{\left(\frac{k}{1-k}\right)} (d(\varrho_2, \varrho_1))^{2\left(\frac{k}{1-k}\right)q^{i-1}} \\
&< r^{q(3q)^n} r^{2(3q)^{i+3}} \leq r^{3^n q^{n+1} + 2(3q)^{n+3}} = r^{3^n q^{n+1}(1+2(3q)^2)} < r^{(3q)^{n+1}},
\end{aligned}$$

where $q = \frac{k}{1-k}$.
Therefore $f(\varrho) \in U(\varrho_{i+1}, r^{(3q)^{n+1}})$. Thus, we get $f(U_{n,r^{(3q)^n}}) \subseteq U_{n+1,r^{(3q)^{n+1}}}$. Also,

$$f(U_{n+1,r^{(3q)^{n+1}}}) \subseteq f(U_{n+1,r^{(3q)^n}}) \subseteq f(U_{n,r^{(3q)^n}}) \subseteq U_{n+1,r^{(3q)^{n+1}}} \subseteq U_{n+1,r^{(3q)^n}} \subseteq U_{n,r^{(3q)^n}}.$$

So, $\{U_{n,r^{(3q)^n}} : q = (\frac{k}{1-k}), r > 1, n = 1, 2, ...\}$ is a fixed filter base of f in S, because $\lim\limits_{n\to\infty} diam\, U_{n,r^{(3q)^n}} = 1$, and $f(U_{n+1,r^{(3q)^{n+1}}}) \subseteq U_{n+1,r^{(3q)^{n+1}}} \subseteq U_{n,r^{(3q)^n}}, n = 1, 2, ...$ □

Theorem 20. *Let (S,d) be a MMS. Let f be a function on S to itself such that $d(f(\varrho), f(\varpi)) \leq (d(f(\varrho), \varpi)d(f(\varpi), \varrho))^k$, $\forall \varrho, \varpi \in S$, for some $k \in (0, \frac{1}{4})$. If $\varrho_{n+1} = f(\varrho_n), n = 1, 2, ...,$ where ϱ_1 is fixed, are defined, then $\left\{ \bigcup_{i \geq n} U(\varrho_i, r^{(3q)^n}) : q = (\frac{k}{1-k}), r > 1, n = 1, 2, ... \right\}$ is a fixed filter base of f in S.*

Proof. Take $U(\varrho_i, r^{(3q)^n}) = \{\varpi \in S : d(\varrho_i, \varpi) < r^{(3q)^n}, r > 1, n = 1, 2, ...\}$. Let $U_{n, r^{(3q)^n}} = \bigcup_{i \geq n} U(\varrho_i, r^{(3q)^n})$, where ϱ_1 is fixed and $\varrho_{n+1} = f(\varrho_n), n = 1, 2, ...$ Then

$$\begin{aligned}
d(\varrho_{n+2}, \varrho_{n+1}) &= d(f(\varrho_{n+1}), f(\varrho_n)) \\
&\leq (d(f(\varrho_{n+1}), \varrho_n)d(f(\varrho_n), \varrho_{n+1}))^k \\
&= (d(\varrho_{n+2}, \varrho_n))^k (d(\varrho_{n+1}, \varrho_{n+1}))^k \\
&\leq (d(\varrho_{n+2}, \varrho_{n+1}))^k (d(\varrho_{n+1}, \varrho_n))^k \\
d(\varrho_{n+2}, \varrho_{n+1}) &\leq (d(\varrho_{n+1}, \varrho_n))^{\left(\frac{k}{1-k}\right)} \\
&\leq (d(\varrho_{n+1}, \varrho_n))^q, \text{ where } q = \left(\frac{k}{1-k}\right) \\
&\leq (d(\varrho_n, \varrho_{n-1}))^{q^2} \leq ... \leq (d(\varrho_2, \varrho_1))^{q^n}.
\end{aligned}$$

For $m > n$,

$$\begin{aligned}
d(\varrho_m, \varrho_n) &\leq d(\varrho_m, \varrho_{m-1})d(\varrho_{m-1}, \varrho_{m-2})...d(\varrho_{n+1}, \varrho_n) \\
&\leq (d(\varrho_2, \varrho_1))^{q^{m-2}} (d(\varrho_2, \varrho_1))^{q^{m-3}} ...(d(\varrho_2, \varrho_1))^{q^{n-1}} \\
&= (d(\varrho_2, \varrho_1))^{q^{n-1}(1+q+...+q^{m-n-1})} \\
&\leq (d(\varrho_2, \varrho_1))^{\left(\frac{q^{n-1}}{1-q}\right)}
\end{aligned}$$

Thus $d(\varrho_m, \varrho_n) \to 1$ as $n \to \infty$ with $m > n$. Therefore $\lim_{n \to \infty} diam\{\varrho_n, \varrho_{n+1}, ...\} = 1$. Let $\varrho \in U_{n, r^{(3q)^n}}$, then $\varrho \in U(\varrho_i, r^{(3q)^n})$, for some $i \geq n$,

$$\begin{aligned}
d(f(\varrho), \varrho_{i+1}) &= d(f(\varrho), f(\varrho_i)) \\
&\leq (d(f(\varrho), \varrho_i)d(f(\varrho_i), \varrho))^k \\
&= (d(f(\varrho), \varrho_i))^k (d(\varrho_{i+1}, \varrho))^k \\
&\leq (d(f(\varrho), \varrho_{i+1}))^k (d(\varrho_{i+1}, \varrho_i))^{2k} (d(\varrho_i, \varrho))^k \\
&\leq (d(f(\varrho), \varrho_{i+1}))^k (d(\varrho_2, \varrho_1))^{2kq^{i-1}} (d(\varrho_i, \varrho))^k \\
d(f(\varrho), \varrho_{i+1}) &\leq (d(\varrho, \varrho_i))^{\left(\frac{k}{1-k}\right)} (d(\varrho_2, \varrho_1))^{2\left(\frac{k}{1-k}\right)q^{i-1}} \\
&< r^{q(3q)^n} r^{2(3q)^{i+3}} \leq r^{3^n q^{n+1} + 2(3q)^{n+3}} = r^{3^n q^{n+1}(1+2(3q)^2)} < r^{(3q)^{n+1}},
\end{aligned}$$

where $q = \frac{k}{1-k}$.
Therefore $f(\varrho) \in U(\varrho_{i+1}, r^{(3q)^{n+1}})$. Thus, we get $f(U_{n,r^{(3q)^n}}) \subseteq U_{n+1,r^{(3q)^{n+1}}}$. Also,

$$f(U_{n+1,r^{(3q)^{n+1}}}) \subseteq f(U_{n+1,r^{(3q)^n}}) \subseteq f(U_{n,r^{(3q)^n}}) \subseteq U_{n+1,r^{(3q)^{n+1}}} \subseteq U_{n+1,r^{(3q)^n}} \subseteq U_{n,r^{(3q)^n}}.$$

So, $\{U_{n,r^{(3q)^n}} : q = (\frac{k}{1-k}), r > 1, n = 1, 2, ...\}$ is a fixed filter base of f in S, because $\lim_{n\to\infty} diam\, U_{n,r^{(3q)^n}} = 1$, and $f(U_{n+1,r^{(3q)^{n+1}}}) \subseteq U_{n+1,r^{(3q)^{n+1}}} \subseteq U_{n,r^{(3q)^n}}, n = 1, 2, ...$ □

Theorem 21. *Let (S, d) be a MMS. Let f be a function on S to itself such that $d(f(\varrho), f(\varpi)) \leq (d(\varrho, \varpi))^k (d(f(\varrho), \varrho) d(\varpi, f(\varpi)))^l (d(\varrho, f(\varpi)) d(\varpi, f(\varrho)))^p$, $\forall \varrho, \varpi \in S$, for some $k, l, p \in (0, \frac{1}{11})$. If $\varrho_{n+1} = f(\varrho_n), n = 1, 2, ...,$ where ϱ_1 is fixed, are defined, then $\left\{ \bigcup_{i\geq n} U(\varrho_i, r^{(3h)^n}) : h = \frac{k+l+p}{1-l-p}, r > 1, n = 1, 2, ... \right\}$ is a fixed filter base of f in S.*

Proof. Take $U(\varrho_i, r^{(3h)^n}) = \{\varpi \in S : d(\varrho_i, \varpi) < r^{(3h)^n}, r > 1, n = 1, 2, ...\}$. Let $U_{n,r^{(3h)^n}} = \bigcup_{i\geq n} U(\varrho_i, r^{(3h)^n})$, where ϱ_1 is fixed and $\varrho_{n+1} = f(\varrho_n), n = 1, 2, ...$ Then

$$\begin{aligned}
d(\varrho_{n+2}, \varrho_{n+1}) &= d(f(\varrho_{n+1}), f(\varrho_n)) \\
&\leq (d(\varrho_{n+1}, \varrho_n))^k (d(\varrho_{n+1}, f(\varrho_{n+1})) d(f(\varrho_n), \varrho_n))^l \\
&\quad (d(\varrho_{n+1}, f(\varrho_n)) d(f(\varrho_{n+1}), \varrho_n))^p \\
&= (d(\varrho_{n+1}, \varrho_n))^k (d(\varrho_{n+1}, \varrho_{n+2}) d(\varrho_{n+1}, \varrho_n))^l (d(\varrho_{n+1}, \varrho_{n+1}) d(\varrho_{n+2}, \varrho_n))^p \\
&\leq (d(\varrho_{n+1}, \varrho_n))^k (d(\varrho_{n+1}, \varrho_{n+2}) d(\varrho_{n+1}, \varrho_n))^l (d(\varrho_{n+2}, \varrho_{n+1}) d(\varrho_{n+1}, \varrho_n))^p \\
d(\varrho_{n+2}, \varrho_{n+1}) &\leq (d(\varrho_{n+1}, \varrho_n))^{\left(\frac{k+l+p}{1-l-p}\right)} \\
&\leq (d(\varrho_{n+1}, \varrho_n))^q, \text{ where } q = \left(\frac{k+l+p}{1-l-p}\right) \\
&\leq (d(\varrho_n, \varrho_{n-1}))^{q^2} \leq ... \leq (d(\varrho_2, \varrho_1))^{q^n}.
\end{aligned}$$

For $m > n$,

$$\begin{aligned}
d(\varrho_m, \varrho_n) &\leq d(\varrho_m, \varrho_{m-1}) d(\varrho_{m-1}, \varrho_{m-2}) ... d(\varrho_{n+1}, \varrho_n) \\
&\leq (d(\varrho_2, \varrho_1))^{q^{m-2}} (d(\varrho_2, \varrho_1))^{q^{m-3}} ... (d(\varrho_2, \varrho_1))^{q^{n-1}} \\
&= (d(\varrho_2, \varrho_1))^{q^{n-1}(1+q+...+q^{m-n-1})} \\
&\leq (d(\varrho_2, \varrho_1))^{\left(\frac{q^{n-1}}{1-q}\right)}
\end{aligned}$$

Thus $d(\varrho_m, \varrho_n) \to 1$ as $n \to \infty$ with $m > n$. Therefore $\lim_{n\to\infty} diam\{\varrho_n, \varrho_{n+1}, ...\} = 1$.

Let $\varrho \in U_{n,r^{(3h)^n}}$, then $\varrho \in U(\varrho_i, r^{(3h)^n})$, for some $i \geq n$,

$$\begin{aligned} d(f(\varrho), \varrho_{i+1}) &= d(f(\varrho), f(\varrho_i)) \\ &\leq (d(\varrho, \varrho_i))^k (d(\varrho, f(\varrho)) d(\varrho_i, f(\varrho_i)))^l (d(\varrho, f(\varrho_i)) d(\varrho_i, f(\varrho)))^p \\ &= (d(\varrho, \varrho_i))^k (d(\varrho_i, f(\varrho_i)))^l (d(\varrho, f(\varrho)) d(f(\varrho), \varrho_{i+1}) d(\varrho, \varrho_i) d(\varrho, f(\varrho)))^p \\ &\leq (d(f(\varrho), \varrho_{i+1}))^{l+p} (d(\varrho_2, \varrho_1))^{2(l+p)h^{i-1}} (d(\varrho_i, \varrho))^{k+l+p} \\ d(f(\varrho), \varrho_{i+1}) &\leq (d(\varrho, \varrho_i))^{\left(\frac{k+l+p}{1-l-p}\right)} (d(\varrho_2, \varrho_1))^{2\left(\frac{k+l+p}{1-l-p}\right)h^{i-1}} \\ &< r^{h(3h)^n} r^{2(3h)^{i+3}} \leq r^{3^n h^{n+1} + 2(3h)^{n+3}} = r^{3^n h^{n+1}(1+2(3h)^2)} < r^{(3h)^{n+1}}, \end{aligned}$$

where $h = \frac{k+l+p}{1-l-p}$.
Therefore $f(\varrho) \in U(\varrho_{i+1}, r^{(3h)^{n+1}})$. Thus, we get $f(U_{n,r^{(3h)^n}}) \subseteq U_{n+1,r^{(3h)^{n+1}}}$. Also

$$f(U_{n+1,r^{(3h)^{n+1}}}) \subseteq f(U_{n+1,r^{(3h)^n}}) \subseteq f(U_{n,r^{(3h)^n}}) \subseteq U_{n+1,r^{(3h)^{n+1}}} \subseteq U_{n+1,r^{(3h)^n}} \subseteq U_{n,r^{(3h)^n}}.$$

So, $\{U_{n,r^{(3h)^n}} : h = \frac{k+l+p}{1-l-p}, r > 1, n = 1, 2, ...\}$ is a fixed filter base of f in S, because $\lim_{n\to\infty} diam\ U_{n,(3h)^n r} = 1$, and $f(U_{n+1,r^{(3h)^{n+1}}}) \subseteq U_{n+1,r^{(3h)^{n+1}}} \subseteq U_{n,r^{(3h)^n}}, n = 1, 2, ...$ □

Remark 22. *Construction of fixed filter base for different types of multiplicative contractions on MMS is established in theorems 18,19,20 and 21. Since Proposition 16, the existence of fixed filter base implies the existence of a fixed point for multiplicative complete MMS.*

Example 23. *Let* $S = (0, \infty)$ *and* $d : S \times S \to [1, \infty)$ *be defined by* $d(\varrho, \varpi) = \max\{\varrho\varpi^{-1}, \varpi\varrho^{-1}\}$. *Then* (S, d) *is a MMS.*
Define $f : S \to S$ *by* $f(\varrho) = \varrho^{\frac{1}{4}}$. *For* $\varrho, \varpi \in S$, *we have*

$$\begin{aligned} d(G\varrho, G\varpi) &= \max\left\{\left(\frac{\varrho}{\varpi}\right)^{\frac{1}{4}}, \left(\frac{\varpi}{\varrho}\right)^{\frac{1}{4}}\right\} \\ &= (d(\varrho, \varpi))^{\frac{1}{4}} \\ &\leq d(\varrho, \varpi)^k,\ \forall n = 1, 2, 3, ...,\ \textit{where } k \in [\frac{1}{4}, 1). \end{aligned}$$

Then, hypotheses of Theorem 18 are satisfied. Also, $U = \{(1 - \frac{1}{n}, 1 + \frac{1}{n}) : n = 1, 2, 3, ...\}$ *is a fixed filter base of* f *containing fixed point 1.*

4 Fixed Point at Infinity

Next, we modify the concept of a fixed point at infinity [8] from metric space to MMS.

Definition 24. *Let f be a self mapping on multiplicative unbounded MMS (S,d). If for every multiplicative bounded and multiplicative closed set E_1 in S, there is a multiplicative bounded and multiplicative closed set E_2 in S such that $f((E_2)^C) \subseteq (E_1)^C$, then f has a fixed point at infinity.*

Theorem 25. *Let f be a self mapping on multiplicative unbounded MMS (S,d). Then inverse image of any multiplicative bounded subset of S is multiplicative bounded set in S if and only if f has a fixed point at infinity.*

Proof. Suppose, inverse image of any multiplicative bounded set is multiplicative bounded.
Let E_1 be a multiplicative bounded and multiplicative closed set in S and $E_2 = \overline{f^{-1}(E_1)}$. Then E_2 is multiplicative bounded and multiplicative closed set and $f((E_2)^C) \subseteq (E_1)^C$.
Conversely, suppose f has a fixed point at infinity.
Let A be a multiplicative bounded set in S and $E_1 = \overline{A}$. Then E_1 is also a multiplicative bounded. Then there is a multiplicative bounded and multiplicative closed set E_2 in S such that $f((E_2)^C) \subseteq (E_1)^C$. Then $f^{-1}(A) \subseteq f^{-1}(E_1) \subseteq E_2$. Since E_2 is multiplicative bounded, we get $f^{-1}(A)$ is multiplicative bounded. □

Corollary 26. *Let f be a self mapping on an multiplicative unbounded MMS (S,d) such that*

$$d(\varrho,\varpi) \quad \leq \quad (d(f(\varrho),f(\varpi)))^q,\ \forall\ \varrho,\varpi \in S,\ \textit{for some } q > 0.$$

Then f has a fixed point at infinity.

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