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MULTIVALUED MIZOGUCHI-TAKAHASHI TYPE RATIONAL CONTRACTION IN RELATIONAL METRIC SPACES

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Abstract. In this paper we establish a result in the fixed point theory of multivalued mappings. This is done by combining three prevalent trends in fixed point theory. Some consequences of the main theorem are discussed. We also provide an illustrative example. The results are derived on metric spaces with a relation.

1 Introduction

In certain attempts of generalizing the Banach's result, appropriate functions were used to replace the contraction constant [5, 25]. One such attempt was made by Mizoguchi and Takahashi [23] where they introduced a new function, now known as MT-function [9], which has been used in a large number of papers in fixed point theory. Afterwards, this class of functions was utilized to obtain a new class of contractions whose fixed point properties have been investigated in several works like [11, 29, 30].

In the development of the metric fixed point theory, contractive inequalities with rational expressions appeared first in the work of Dass and Gupta [8]. Several results in this domain for mappings satisfying rational inequalities have appeared in the subsequent times of which [8, 13, 20] are some recent references.

A generalization of the Banach's contraction mapping principle was suggested by A. Alam and M. Imdad [1] where the concept of relation theoretic fixed point was used. Relational metric spaces are useful, as fixed point results in such spaces can be derived by imposing the contractive condition for certain pairs of points only, considerations of arbitrary points from the whole space are not required here. This is a new and emerging branch of fixed point theory. The relation theoretic approaches have been used in several fixed point results like [1, 2, 3, 6, 14, 16, 17, 18, 27].

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In this paper, putting the above-mentioned ideas together, we establish a multivalued fixed point result where we use MT-function to postulate a contraction inequality which is assumed to be satisfied by the mapping. We also use some relation-theoretic notions, namely $\mathcal{R}$-completeness of the metric space, $\mathcal{R}$-continuity of the multivalued mapping T and T-closedness of the relation $\mathcal{R}$.

The present study is in the domain of set-valued analysis. Nadler [24] was the first to extend the Banach's contraction mapping principle to the above mentioned domain. After that multivalued fixed point theory developed in several directions which are described in the books like [12, 22, 32].

2 Mathematical preliminaries

We denote $\mathbb{N}$ as the set of natural numbers, $\mathbb{R}$ as the set of real numbers.

Definition 1. *[1] Let X be a nonempty set. A subset $\mathcal{R}$ of $X \times X$ is a binary relation on X. We say that x and y are $\mathcal{R}$-comparative if either $(x,\ y) \in \mathcal{R}$ or $(y,\ x) \in \mathcal{R}$. We denote it by $[x,\ y] \in \mathcal{R}$.*

Definition 2. *[31] Let X be a nonempty set and $\mathcal{R}$ be a binary relation on X. A sequence $\{x_n\} \subset X$ is called $\mathcal{R}$-preserving if $(x_n, x_{n+1}) \in \mathcal{R}$ for all $n \in \mathbb{N}$.*

Definition 3. *[2] Let (X, d) be a metric space and $\mathcal{R}$ be a binary relation on X. We say that (X, d) is $\mathcal{R}$-complete if every $\mathcal{R}$-preserving Cauchy sequence in X is convergent.*

Definition 4. *[1] Let X be a nonempty set and $T : X \to X$ be a self-mapping. A binary relation $\mathcal{R}$ defined on X is called T-closed if for any $x, y \in X$, $(x, y) \in \mathcal{R}$ implies $(Tx, Ty) \in \mathcal{R}$.*

Definition 5. *[2] Let (X, d) be a metric space and $\mathcal{R}$ be a binary relation on X and $x \in X$. A mapping $T : X \to X$ is called $\mathcal{R}$-continuous at x if for any $\mathcal{R}$-preserving sequence $\{x_n\}$, $x_n \to x$ implies $Tx_n \to Tx$.*

T is called $\mathcal{R}$-continuous if it is $\mathcal{R}$-continuous at each point of X.

Definition 6. *[1] Let $(X,\ d)$ be a metric space. A binary relation $\mathcal{R}$ defined on X is called d-self closed if every $\mathcal{R}$-preserving convergent sequence $\{x_n\}$ with limit $x \in X$ has a subsequence $\{x_{n_k}\}$ such that $[x_{n_k}, x] \in \mathcal{R}$ for all $k \in \mathbb{N}$.*

We use following class of functions for our results.

Definition 7. *[9, 10, 23] A function $\phi : [0,\ \infty) \to [0, 1)$ is said to be an MT-function if it satisfies Mizoguchi Takahashi's condition, that is, $\limsup\limits_{s \to t+} \phi(s) < 1$ for all $t \in [0,\ \infty)$.*

We denote the class of all MT-functions by Φ.

Example 8. *$\varphi : [0,\ \infty) \to [0, 1)$, defined by $\varphi(t) = c$, where $c \in [0, 1)$ is an MT-function.*

Example 9. *$\varphi : [0,\ \infty) \to [0, 1)$, defined by*

$$\varphi(t) = \begin{cases} \frac{1}{16} + \frac{15}{16}t^2, & \text{if } t \in [0, 1) \\ \frac{1}{2t}, & \text{otherwise} \end{cases}$$

is an MT-function.

In the next few lines we discuss the notion of set-valued mappings and that of a metric called Hausdorff-Pompeiu metric.

Definition 10. *Let (X, d) be a metric space. Let $CB(X)$ denote the collection of all nonempty closed and bounded subsets of X. The Hausdorff-Pompeiu metric H, between $A, B\ \in CB(X)$ is defined by*

$$H(A, B) = max\{\sup_{x\in A} D(x, B),\ \sup_{y\in B} D(y, A)\},$$

where $D(t, P) := \inf\{d(t, p) : p \in P\},\ P \in CB(X)$.
It is known that, if (X, d) is complete then $(CB(X), H)$ is also complete.

Definition 11. *[24] Let $T : X \to CB(X)$ be a multivalued (set-valued) map. A point $x \in X$ is said to be a fixed point of T if $x \in Tx$.*

Nadler in [24] proved the following result.

Theorem 12 (Nadler)**.** *[24] Let (X, d) be a complete metric space and $T : X \to CB(X)$ be a multivalued contraction map, that is, there exists $k \in [0, 1)$ such that for all $x, y \in X$,*

$$H(Tx, Ty) \leq kd(x, y). \tag{2.1}$$

Then T has a fixed point.

We next discuss a lemma which is useful in proving our results.

Lemma 13. *[26] Let (X, d) be a metric space and $A,\ B\ \in CB(X)$ and $q > 1$. Then for every $x \in A$, there exists $y \in B$ such that $d(x, y) \leq qH(A, B)$.*

Definition 14. *[28] Let (X, d) be a metric space and $T : X \to CB(X)$ be a multivalued mapping. Let $\mathcal{R}$ be a binary relation on X. Then $\mathcal{R}$ is called $T-closed$ if $(x, y) \in \mathcal{R}$ implies $(u, v) \in \mathcal{R}$, for all $u \in Tx,\ v \in Ty$.*

Definition 15. *[15, 19] Let (X, d) be a metric space and $T : X \to CB(X)$ be a multivalued mapping. Let $\mathcal{R}$ be a binary relation on X. Then T is called $\mathcal{R}$-continuous at $x \in X$, if for any $\mathcal{R}$-preserving sequence $\{x_n\}$ in X, $\lim\limits_{n\to\infty} x_n = x$ implies $\lim\limits_{n\to\infty} H(Tx_n, Tx) = 0$.*

T is called $\mathcal{R} - continuous$ if it is $\mathcal{R}$-continuous at each point in X.

Example 16. Let $X = (-\frac{\pi}{2},\ \frac{\pi}{2})$ be equipped with usual metric. Let $T : X \to CB(X)$ be defined by:

$$Tx = \begin{cases} [0, \frac{\sin^2 x}{16}] & \text{if } x \in [0, \frac{\pi}{2}) \\ [-\cos x, 0] & \text{if } x \in (-\frac{\pi}{2}, 0) \end{cases}$$

and a binary relation $\mathcal{R} \subseteq X \times X$ be defined by:

$$(x,\ y) \in \mathcal{R},\ \text{if } 0 \leq x \leq 1,\ 0 \leq y \leq \frac{1}{8}$$

Let $(x,\ y) \in \mathcal{R}$. Then $0 \leq x \leq 1,\ \ 0 \leq y \leq \frac{1}{8}$ and hence $Tx = [0, \frac{\sin^2 x}{16}]$ and $Ty = [0, \frac{\sin^2 y}{16}]$. Let $u \in Tx,\ v \in Ty$. Then $0 \leq u \leq \frac{1}{16}$ and $0 \leq v \leq \frac{1}{16}$. By the definition of $\mathcal{R}$, $(u,\ v) \in \mathcal{R}$. Therefore, we have that $(x, y) \in \mathcal{R}$ implies $(u, v) \in \mathcal{R}$, for all $u \in Tx,\ v \in Ty$. Hence $\mathcal{R}$ is T-closed.

Let $\{x_n\}$ be a $\mathcal{R}$-preserving sequence with $\lim\limits_{n\to\infty} x_n = x$. So, $(x_n, x_{n+1}) \in \mathcal{R}$, for all $n \in \mathbb{N}$. Then $0 \leq x_n \leq \frac{1}{8}$ for all $n > 1$. As T is continuous on $[0, \frac{1}{8}] \subseteq [0, \frac{\pi}{2})$, we have $\lim\limits_{n\to\infty} H(Tx_n, Tx) = 0$. Therefore, T is $\mathcal{R}$-continuous.

Let $\{x_n\}$ be a $\mathcal{R}$-preserving and Cauchy sequence. Then $0 \leq x_n \leq 1$ for all n. Thus $\{x_n\}$ is a Cauchy sequence in $[0,\ 1]$. As $[0,\ 1]$ is complete, $\{x_n\}$ is convergent in $[0,\ 1] \subseteq (-\frac{\pi}{2},\ \frac{\pi}{2})$. Therefore, X is $\mathcal{R}$-complete.

3 Main result

Theorem 17. *Let (X, d) be a metric space, $\mathcal{R}$ be a relation on X, $\phi : [0,\ \infty) \to [0,\ 1)$ be an MT-function and $T : X \to CB(X)$ be a multivalued map such that (i) (X, d) is $\mathcal{R}$-complete, (ii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in \mathcal{R}$, (iii) $\mathcal{R}$ is T-closed, (iv) T is $\mathcal{R} - continuous$, (v) for all $[x, y] \in \mathcal{R}$, the following inequality holds*

$$\begin{aligned} H(Tx,\ Ty) \leq \phi\left(d\left(x, y\right)\right)\ \max\Big\{ & d(x, y), \frac{D(x, Tx) + D(y, Ty)}{2}, \frac{D(x, Ty) + D(y, Tx)}{2}, \\ & \frac{D(x, Tx)D(y, Ty)}{1 + H(Tx, Ty)}, \frac{D(y, Tx)D(x, Ty)}{1 + H(Tx, Ty)}, \frac{D(y, Ty)D(y, Tx)}{1 + H(Tx, Ty)}\Big\}. \end{aligned} \tag{3.1}$$

Then T has a fixed point in X.

Proof. We first note that $\frac{1}{\sqrt{\phi(d(x,y))}} > 1$ for all $x, y \in X$ when $\phi(d(x,y)) \neq 0$.

By assumption (ii), there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0,\ x_1) \in \mathcal{R}$. If $\phi(d(x_0,\ x_1)) = 0$, then from contractive inequality, we have $0 \leq D(x_1,\ Tx_1) \leq H(Tx_0,\ Tx_1) = 0$, which implies that $D(x_1,\ Tx_1) = 0$, that is $x_1 \in \overline{Tx_1} = Tx_1$, where $\overline{Tx_1}$ denotes the closure of Tx_1. So in this case $x_1 \in Tx_1$, that is x_1 is a fixed point of T. So without loss of generality, we assume $\phi(d(x_0, x_1)) \neq 0$. By Lemma 13 for $x_1 \in Tx_0$, there exists $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \leq \frac{1}{\sqrt{\phi(d(x_0, x_1))}} H(Tx_0, Tx_1).$$

Again if $\phi(d(x_1,\ x_2)) = 0$, then from contractive inequality, we have $0 \leq D(x_2,\ Tx_2) \leq H(Tx_1,\ Tx_2) = 0$, which implies that $D(x_2,\ Tx_2) = 0$, that is $x_2 \in \overline{Tx_2} = Tx_2$, where $\overline{Tx_2}$ denotes the closure of Tx_2. So in this case $x_2 \in Tx_2$, that is x_2 is a fixed point of T. So with out loss of generality, we assume $\phi(d(x_1,\ x_2)) \neq 0$. By Lemma 13 for $x_2 \in Tx_1$, there exists $x_3 \in Tx_2$ such that

$$d(x_2,\ x_3) \leq \frac{1}{\sqrt{\phi(d(x_1,\ x_2))}}\ H(Tx_1,\ Tx_2).$$

Continuing this process, we construct a sequence $\{x_n\}$ in X, such that

$$d(x_{n+1}, x_{n+2}) \leq \frac{1}{\sqrt{\phi(d(x_n,\ x_{n+1}))}}\ H(Tx_n, Tx_{n+1}) \quad \text{with} \quad \phi(d(x_n,\ x_{n+1})) \neq 0.$$

As $(x_0,\ x_1) \in \mathcal{R}$ and $\mathcal{R}$ being T-closed, $(x_n, x_{n+1}) \in \mathcal{R}$ for all $n \in \mathbb{N}$. Thus $\{x_n\}$ is $\mathcal{R}$-preserving. Then by (3.1)

$$\begin{aligned}
d(x_{n+1}, x_{n+2}) &\leq \frac{1}{\sqrt{\phi(d(x_n, x_{n+1}))}} H(Tx_n, Tx_{n+1}) \\
&\leq \frac{1}{\sqrt{\phi(d(x_n, x_{n+1}))}} \phi(d(x_n, x_{n+1}))\ \max\Big\{d(x_n, x_{n+1}), \frac{D(x_n, Tx_n) + D(x_{n+1}, Tx_{n+1})}{2}, \\
&\qquad \frac{D(x_n, Tx_{n+1}) + D(x_{n+1}, Tx_n)}{2}, \frac{D(x_n, Tx_n) D(x_{n+1}, Tx_{n+1})}{1 + H(Tx_n, Tx_{n+1})}, \\
&\qquad \frac{D(x_{n+1}, Tx_n) D(x_n, Tx_{n+1})}{1 + H(Tx_n, Tx_{n+1})}, \frac{D(x_{n+1}, Tx_{n+1}) D(x_{n+1}, Tx_n)}{1 + H(Tx_n, Tx_{n+1})}\Big\} \\
&\leq \sqrt{\phi(d(x_n,\ x_{n+1}))}\ \max\Big\{d(x_n, x_{n+1}), \frac{d(x_n,\ x_{n+1}) + d(x_{n+1}, x_{n+2})}{2}, \frac{d(x_n,\ x_{n+2})}{2} \\
&\qquad \frac{d(x_n,\ x_{n+1})\ D(x_{n+1},\ Tx_{n+1})}{1 + D(x_{n+1},\ Tx_{n+1})},\ 0,\ 0\Big\} \\
&\leq \sqrt{\phi(d(x_n,\ x_{n+1}))}\ \max\Big\{d(x_n, x_{n+1}), \frac{d(x_n,\ x_{n+1}) + d(x_{n+1}, x_{n+2})}{2},
\end{aligned}$$

$$\frac{d(x_n,\ x_{n+1}) + d(x_{n+1}, x_{n+2})}{2},\ d(x_n,\ x_{n+1}),\ 0,\ 0\Big\}$$

$$= \sqrt{\phi\left(d(x_n,\ x_{n+1})\right)}\ \max\Big\{d(x_n, x_{n+1}), \frac{d(x_n,\ x_{n+1}) + d(x_{n+1}, x_{n+2})}{2}\Big\}.$$

As $\frac{d(x_n,\ x_{n+1}) + d(x_{n+1}, x_{n+2})}{2} \leq \max\Big\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\Big\}$, we have,

$$d(x_{n+1}, x_{n+2}) \leq \sqrt{\phi\left(d(x_n, x_{n+1})\right)}\ \max\Big\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\Big\}. \tag{3.2}$$

Let

$$\delta_n = d(x_n,\ x_{n+1}),\ \text{for all}\ \ n \in \mathbb{N}. \tag{3.3}$$

By (3.2), we have

$$\delta_{n+1} \leq \sqrt{\phi(\delta_n)}\ \max\{\delta_n,\ \delta_{n+1}\}. \tag{3.4}$$

Suppose that $0 \leq \delta_n < \delta_{n+1}$ for some $n \in \mathbb{N}$, then $\delta_{n+1} > 0$ and hence from (3.4), we have

$$\delta_{n+1} \leq \sqrt{\phi(\delta_n)}\ \max\{\delta_n,\ \delta_{n+1}\} = \sqrt{\phi(\delta_n)}\ \delta_{n+1} < \delta_{n+1}, \Big(\ \text{as}\ \phi(\delta_n) < 1\Big).$$

which is a contradiction. Therefore, $\delta_{n+1} \leq \delta_n$, for all $n \geq 0$, that is, $\{\delta_n\} = \{d(x_n,\ x_{n+1})\}$ is a monotone decreasing sequence of nonnegative real numbers. Hence there exists a real number $r \geq 0$ such that

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} d(x_n,\ x_{n+1}) = r \tag{3.5}$$

and from (3.4), we have

$$\delta_{n+1} \leq \sqrt{\phi(\delta_n)}\ \max\{\delta_n,\ \delta_{n+1}\} = \sqrt{\phi(\delta_n)}\ \delta_n. \tag{3.6}$$

Since $\phi \in \Phi$, we have $\limsup_{t\to r^+} \phi(t) < 1$, which implies that

$$\sqrt{\limsup_{t\to r^+} \phi(t)} = \limsup_{t\to r^+} \sqrt{\phi(t)} < 1.$$

Hence there exists a $p \in [0,\ 1)$ and $\delta > 0$ such that

$$\sqrt{\phi(t)} \leq p,\ \text{for all}\ t \in [r,\ r+\delta). \tag{3.7}$$

From (3.5), there exists $n_0 \in \mathbb{N}$ such that

$$\delta_n < r + \delta,\ \text{for all}\ n \geq n_0. \tag{3.8}$$

Thus from (3.6), (3.7) and (3.8), it follows that

$$\delta_{n+1} \leq p\ \delta_n,\ \text{for all}\ n \geq n_0.$$

So by the above inequality, we have

$$\sum_{n=0}^{\infty}\delta_n=\sum_{n=0}^{n_0-1}\delta_n+\sum_{n=n_0}^{\infty}\delta_n\leq\sum_{n=0}^{n_0-1}\delta_n+\delta_{n_0-1}\sum_{k=1}^{\infty}p^k.$$

As $p\in[0,\ 1)$, $\sum_{k=1}^{\infty}p^k$ is convergent and hence $\sum_{n=0}^{\infty}\delta_n$ is convergent, that is, $\sum_{n=0}^{\infty}\delta_n<\infty$. Thus, we have $\sum d(x_n,\ x_{n+1})<\infty$, which implies that $\{x_n\}$ is a Cauchy sequence in X. As $\{x_n\}$ is $\mathcal{R}$-preserving Cauchy and X is $\mathcal{R}$-complete, there exists $s\in X$ such that

$$\lim_{n\to\infty}x_n=s. \tag{3.9}$$

Since T is $\mathcal{R}$-continuous and $\{x_n\}$ is $\mathcal{R}$-preserving sequence, we have

$$D\left(s,Ts\right)=\lim_{n\to\infty}D(x_{n+1},Ts)\leq\lim_{n\to\infty}H(Tx_n,Ts)=0$$

Thus $D\left(s,Ts\right)=0$, which implies that $s\in\overline{Ts}$, where $\overline{Ts}$ denotes the closure of $\overline{Ts}$. As Ts is a closed set, we have $\overline{Ts}=Ts$, hence $s\in Ts$, that is, s is a fixed point of T. □

We present the following illustrative example in support of Theorem 17 .

Example 18. Let the metric space X, mapping T and relation $\mathcal{R}$ be as in Example 16 and ϕ be as in Example 9.
It is seen that $X=(-\frac{\pi}{2},\ \frac{\pi}{2})$ is $\mathcal{R}$-complete, $\mathcal{R}$ is $T-closed$ and T is $\mathcal{R}$-continuous. (see Example 16).
It is clear that $1,\ \frac{\sin^2 1}{16}\ \in X$ such that $\frac{\sin^2 1}{16}\in T(1)$ and $(1,\frac{sin^2 1}{16})\in\mathcal{R}$.
Let ϕ be as in Example 9. Let $x,\ y\in X$ with $(x,\ y)\in\mathcal{R}$. Then $x\in[0,\ 1]$ and $y\in[0,\ \frac{1}{8}]$. Therefore, it is required to verify the inequality in Theorem 17 for $x\in[0,\ 1]$ and $y\in[0,\ \frac{1}{8}]$. Now, $d(x,\ y)=\mid x-y\mid$ and

$$\begin{aligned}H(Tx,\ Ty)&=\left|\frac{\sin^2 x}{16}-\frac{\sin^2 y}{16}\right|=\frac{1}{16}\ \mid\sin(x-y)\ \sin(x+y)\mid\\&\leq\frac{1}{16}\ \mid\sin(x-y)\mid\leq\frac{\mid x-y\mid}{16}\leq\Big(\frac{1}{16}+\frac{15}{16}\ d(x,y)^2\Big)\ d(x,y)\\&\leq\phi\big(d(x,y)\big)\ \max\Big\{d(x,y),\frac{D(x,Tx)+D(y,Ty)}{2},\frac{D(x,Ty)+D(y,Tx)}{2},\\&\qquad\frac{D(x,Tx)D(y,Ty)}{1+H(Tx,Ty)},\frac{D(y,Tx)D(x,Ty)}{1+H(Tx,Ty)},\frac{D(y,Ty)D(y,Tx)}{1+H(Tx,Ty)}\Big\}.\end{aligned}$$

Then it follows that the inequality in Theorem 17 is satisfied for all $x,\ y\in X$ with $(x,\ y)\in\mathcal{R}$. Hence all the conditions of Theorem 17 are satisfied and 0 is a fixed point of T.

Remark 19. We note that -
(i) The metric space X is not complete.
(ii) The mapping T is not continuous.
(iii) The fixed point is a point of discontinuity of T.

Theorem 20. *Let the condition (iv) of the Theorem 17 be replaced with the following condition-*
(iv.a) $\mathcal{R}$ is d-self closed.
If the other conditions remain same, then T has a fixed point.

Proof. Applying the same argument as in Theorem 17 the equation (3.9) can be reached. In this case, since $\mathcal{R}$ is d-self closed, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $[x_{n_k}, s] \in \mathcal{R}$ for all $k \in \mathbb{N}$.

Using (3.1), we have

$$\begin{aligned} D(x_{n_k+1},\ Ts) &\leq H(Tx_{n_k},\ Ts) \\ &\leq \phi\big(d(x_{n_k},\ s)\big)\ \max\Big\{d(x_{n_k},\ s), \frac{D(x_{n_k},\ Tx_{n_k}) + D(s,\ Ts)}{2}, \\ &\qquad \frac{D(x_{n_k},\ Ts) + D(s,\ Tx_{n_k})}{2}, \frac{D(x_{n_k},\ Tx_{n_k})\ D(s,\ Ts)}{1 + H(Tx_{n_k},\ Ts)}, \\ &\qquad \frac{D(s,\ Tx_{n_k})\ D(x_{n_k},\ Ts)}{1 + H(Tx_{n_k},\ Ts)}, \frac{D(s,\ Ts)\ D(s,\ Tx_{n_k})}{1 + H(Tx_{n_k},\ Ts)}\Big\} \\ &< \max\Big\{d(x_{n_k},\ s), \frac{d(x_{n_k},\ x_{n_k+1}) + D(s,\ Ts)}{2}, \\ &\qquad \frac{D(x_{n_k},\ Ts) + d(s,\ x_{n_k+1})}{2}, \frac{d(x_{n_k},\ x_{n_k+1}) D(s,\ Ts)}{1 + D(x_{n_k+1}, Ts)} \\ &\qquad \frac{d(s,\ x_{n_k+1})\ D(x_{n_k},\ Ts)}{1 + D(x_{n_k+1}, Ts)}, \frac{D(s,\ Ts)\ d(s,\ x_{n_k+1})}{1 + D(x_{n_k+1}, Ts)}\Big\} \end{aligned} \tag{3.10}$$

Taking limsup as $n \to \infty$ in (3.10) and using (3.9), we have

$$\begin{aligned} D(s,\ Ts) &\leq \limsup_{n\to\infty} \max\Big\{0,\ \frac{D(s,\ Ts)}{2}, \frac{D(s,\ Ts)}{2},\ 0,\ 0,\ 0,\ 0\Big\} \\ &= \frac{D(s,\ Ts)}{2}, \end{aligned}$$

which implies that $D(s,\ Ts) = 0$, which implies that $s \in \overline{Ts}$, where $\overline{Ts}$ denotes the closure of $\overline{Ts}$. As Ts is a closed set, we have $\overline{Ts} = Ts$, hence $s \in Ts$, that is, s is a fixed point of T. □

4 Consequences

Now we present a few special cases illustrating the applicability of Theorem 17 and Theorem 20.

Remark 21. *Since every complete metric space is $\mathcal{R}$-complete and every continuous mapping is $\mathcal{R}-$continuous, for any relation $\mathcal{R}$ we have the following corollary from Theorem 17.*

Corollary 22. *Let (X,d) be a complete metric space, $\mathcal{R}$ be a relation on X, $\phi : [0,\ \infty) \to [0,\ 1)$ be an MT-function and $T : X \to CB(X)$ be a continuous map such that (i) There exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in \mathcal{R}$, (ii) $\mathcal{R}$ is T-closed. Then T has a fixed point if for all $[x, y] \in \mathcal{R}$, one the following inequalities holds.*

$$(i)\quad H(Tx,\ Ty) \le \phi\left(d\left(x,y\right)\right)\ d(x,\ y),$$

$$(ii)\quad H(Tx,\ Ty) \le \phi\left(d\left(x,y\right)\right)\ \frac{[D(x,\ Tx) + D(y,\ Ty)]}{2},$$

$$(iii)\quad H(Tx,\ Ty) \le \phi\left(d\left(x,y\right)\right)\ \frac{[D(x,\ Ty) + D(y,\ Tx)]}{2},$$

$$(iv)\quad H(Tx,\ Ty) \le \phi\left(d\left(x,y\right)\right)\ max\left\{d(x,\ y),\ \frac{1}{2}\ [D(x,\ Tx) + D(y,\ Ty)],\ \frac{1}{2}\ [D(x,\ Ty) + D(y,\ Tx)]\right\},$$

$$(v)\quad H(Tx,\ Ty) \le \phi\left(d\left(x,y\right)\right)\ \max\left\{d(x,y), \frac{D(x,Tx)+D(y,Ty)}{2}, \frac{D(x,Ty)+D(y,Tx)}{2}, \frac{D(x,Tx)D(y,Ty)}{1+H(Tx,Ty)}, \frac{D(y,Tx)D(x,Ty)}{1+H(Tx,Ty)}, \frac{D(y,Ty)D(y,Tx)}{1+H(Tx,Ty)}\right\}.$$

Remark 23. *In Theorem 17 and 20, taking $\mathcal{R}$ to be the universal relation and choosing $\phi(t) = k$, for all $t \in [0,\ \infty)$, where $0 \le k < 1$, we have the following corollaries.*

Corollary 24. *Let (X,d) be a complete metric space and $T : X \to CB(X)$ be a mapping. Then T has a fixed point in X if there exists $k \in [0,\ 1)$ such that for $x,\ y \in X$ one of the following inequalities is satisfied.*

$$(i)\quad H(Tx,\ Ty) \le k\ d(x,\ y),$$

$$(ii)\quad H(Tx,\ Ty) \le k\ \frac{[D(x,\ Tx) + D(y,\ Ty)]}{2},$$

$$(iii)\quad H(Tx,\ Ty) \le k\ \frac{[D(x,\ Ty) + D(y,\ Tx)]}{2},$$

$$(iv)\quad H(Tx,\ Ty) \le k\ max\left\{d(x,\ y),\ \frac{1}{2}\ [D(x,\ Tx) + D(y,\ Ty)],\ \frac{1}{2}\ [D(x,\ Ty) + D(y,\ Tx)]\right\},$$

$$(v)\quad H(Tx,\ Ty) \le k\ \max\left\{d(x,y), \frac{D(x,Tx)+D(y,Ty)}{2}, \frac{D(x,Ty)+D(y,Tx)}{2}, \frac{D(x,Tx)D(y,Ty)}{1+H(Tx,Ty)}, \frac{D(y,Tx)D(x,Ty)}{1+H(Tx,Ty)}, \frac{D(y,Ty)D(y,Tx)}{1+H(Tx,Ty)}\right\}.$$

Remark 25. The above corollary shows that our result extends theorem of Nadler [24]. Further (ii) and (iii) indicate that it is also a multivalued extension of Kannan's contraction [21] and C-contraction [7].

Remark 26. *Since every singleton set $\{x\}$ is closed and bounded, we have the following results in case of single valued mapping.*

Corollary 27. *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping. Then T has a fixed point in X if there exists $k \in [0,\ 1)$ such that for $x,\ y \in X$ one of the following inequalities is satisfied.*

(i) $d(Tx,\ Ty) \leq k\ d(x,\ y)$,

(ii) $d(Tx,\ Ty) \leq k\ \dfrac{[d(x,\ Tx) + d(y,\ Ty)]}{2}$,

(iii) $d(Tx,\ Ty) \leq k\ \dfrac{[d(x,\ Ty) + d(y,\ Tx)]}{2}$,

(iv) $d(Tx,\ Ty) \leq k\ max\ \{d(x,\ y),\ \dfrac{1}{2}\ [d(x,\ Tx) + d(y,\ Ty)],\ \dfrac{1}{2}\ [d(x,\ Ty) + d(y,\ Tx)]\}$,

(v) $$d(Tx,\ Ty) \leq k\ \max\Big\{d(x,y), \frac{d(x,Tx)+d(y,Ty)}{2}, \frac{d(x,Ty)+d(y,Tx)}{2}, \frac{d(x,Tx)d(y,Ty)}{1+d(Tx,Ty)}, \frac{d(y,Tx)D(x,Ty)}{1+H(Tx,Ty)}, \frac{d(y,Ty)d(y,Tx)}{1+d(Tx,Ty)}\Big\}.$$

Remark 28. (i) indicates that our theorem is a multivalued extension of Banach's contraction [4].

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