Surveys in Mathematics and its Applications

ISSN 1842-6298 (electronic), 1843-7265 (print)
Volume 18 (2023), 73 – 82

ON BOUNDED COMPLEX JACOBI MATRICES AND RELATED MOMENT PROBLEMS IN THE COMPLEX PLANE

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Abstract. In this paper we consider the following moment problem: find a positive Borel measure μ on $\mathbb{C}$ subject to conditions $\int z^n d\mu = s_n$, $n \in \mathbb{Z}_+$, where s_n are prescribed complex numbers (moments). This moment problem may be viewed (informally) as an extension of the Stieltjes and Hamburger moment problems to the complex plane. A criterion for the moment problem for the existence of a compactly supported solution is given. In particular, such moment problems appear naturally in the domain of complex Jacobi matrices. For every bounded complex Jacobi matrix its associated functional S has the following integral representation: $S(p) = \int_{\mathbb{C}} p(z) d\mu$, with a positive Borel measure μ in the complex plane. An interrelation of the associated to the complex Jacobi matrix operator A_0, acting in l^2 on finitely supported vectors, and the multiplication by z operator in L^2_μ is discussed.

1 Introduction

The theory of real Jacobi matrices is a well-known and classical subject with a lot of applications in various domains of mathematics and other sciences, see the books of Akhiezer and Berezanskii [1],[4]. Complex Jacobi matrices or J-matrices appeared in a context of J-fractions, see Wall's book [14]. They have not attracted so much attention as their real versions. An important recent work on complex Jacobi matrices was done by Beckermann in 2001, who collected and arranged in a nice form basic facts on this subject, see [2]. After Beckermann's paper the study of complex Jacobi matrices became essentially more active. We can mention the following directions of investigations: perturbations and spectral analysis (see [12],[7] and references therein); quadrature rules ([10]); eigenvalue problems ([9]); determinacy questions ([3]). Two-sided Jacobi matrices are also studied intensively: for the real case we refer to [4], and for recent developments see [11],[5] and references therein.

By a complex Jacobi matrix one means a semi-infinite tridiagonal complex matrix

2020 Mathematics Subject Classification: 44A60
Keywords: Complex Jacobi matrix; moment problem; orthogonal polynomials; linear functional

of the following form:

$$J = \begin{pmatrix} b_0 & a_0 & 0 & 0 & \dots \\ a_0 & b_1 & a_1 & 0 & \dots \\ 0 & a_1 & b_2 & a_2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{1.1}$$

where $a_k, b_k \in \mathbb{C} : a_k \neq 0,\ k \in \mathbb{Z}_+$. Let us recall some basic known facts about complex Jacobi matrices which we shall need in what follows. By matrix multiplication the matrix J generates a linear operator A_0 on l^2_{fin}. If

$$|a_k| \leq M, \quad |b_k| \leq M, \qquad \forall k \in \mathbb{Z}_+, \quad \text{for some } M > 0, \tag{1.2}$$

then A_0 is bounded. In this case, by continuity it extends on the whole space l_2 to a bounded operator A [2].

With a complex Jacobi matrix J one associates a system of polynomials $\{p_n(\lambda)\}_{n=0}^{\infty}$, $p_0(\lambda) = 1$, such that

$$a_{n-1}p_{n-1}(\lambda) + b_n p_n(\lambda) + a_n p_{n+1}(\lambda) = \lambda p_n(\lambda), \qquad n \in \mathbb{Z}_+, \tag{1.3}$$

where $a_{-1} := 0$, $p_{-1}(\lambda) := 0$. A linear with respect to the both arguments functional $\sigma(u,v)$, $u, v \in \mathbb{P}$, which satisfies

$$\sigma(p_n(\lambda), p_m(\lambda)) = \delta_{n,m}, \qquad n, m \in \mathbb{Z}_+, \tag{1.4}$$

is said to be *the spectral function* of the difference equation

$$a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n, \qquad n \in \mathbb{Z}_+, \tag{1.5}$$

see [15]. The difference equation (1.5) (and therefore the complex Jacobi matrix J) can be recovered by its spectral function and a sequence of signs, see [8] and also [15] for the details.

Theorem 1. *([15, Theorem 1]) A linear with respect to the both arguments functional $\sigma(u,v)$, $u, v \in \mathbb{P}$, is the spectral function of a difference equation of type (1.5) iff:*

1) $\sigma(\lambda u(\lambda), v(\lambda)) = \sigma(u(\lambda), \lambda v(\lambda)), \qquad u, v \in \mathbb{P}$;

2) $\sigma(1,1) = 1$;

3) For arbitrary polynomial $u_k(\lambda)$ of degree k, there exists a polynomial $\widehat{u}_k(\lambda)$ of degree k such that:

$$\sigma(u_k(\lambda), \widehat{u}_k(\lambda)) \neq 0.$$

By property 1) we see that

$$\sigma(u(\lambda), v(\lambda)) = \sigma(u(\lambda)v(\lambda), 1), \qquad u, v \in \mathbb{P}. \tag{1.6}$$

Consider the following linear functional S, which is said to be *associated to the complex Jacobi matrix* J:

$$S(u) = \sigma(u, 1), \qquad u \in \mathbb{P}. \tag{1.7}$$

By (1.4) it has the following property:

$$S(p_n p_m) = \delta_{n,m}, \qquad n, m \in \mathbb{Z}_+. \tag{1.8}$$

Denote

$$\mathbf{s}_n := S(\lambda^n), \qquad n \in \mathbb{Z}_+. \tag{1.9}$$

The numbers $\{\mathbf{s}_n\}_{n=0}$ are said to be *the moments* of S. Our main objective here is to provide conditions on the moments $\mathbf{s}_n$, which imply the existence of an integral representation of S of the following form:

$$S(p) = \int_{\mathbb{C}} p(z) d\mu, \tag{1.10}$$

with a (non-negative) Borel measure μ. We shall use the following moment problem: find a (non-negative) measure μ on $\mathfrak{B}(\mathbb{C})$ such that

$$\int_{\mathbb{C}} z^k d\mu(z) = s_k, \qquad k \in \mathbb{Z}_+. \tag{1.11}$$

Here $\{s_k\}_{k\in\mathbb{Z}_+}$ is a prescribed set of complex numbers (moments). This moment problem was recently stated in [18]. Necessary and sufficient conditions for the moment problem (1.11) to have compactly supported solutions will be given in Theorem 2. The existence of an integral representation (1.10) with a positive Borel measure μ for a bounded complex Jacobi matrix will appear in Theorem 3. We also have a connection of the operator A_0 with the multiplication by z operator in L^2_μ, see Theorem 4.

Finally, we remark that the operator A is complex symmetric (see e.g. [6] for definitions), and it belongs to the class $C_+(H)$, for $H = l^2$, see [16].

Notations. As usual, we denote by $\mathbb{R}, \mathbb{C}, \mathbb{N}, \mathbb{Z}, \mathbb{Z}_+$ the sets of real numbers, complex numbers, positive integers, integers and non-negative integers, respectively; $\mathbb{T} = \{z \in \mathbb{C} : \ |z| = 1\}$. By $\mathbb{Z}_{k,l}$ we mean all integers r, which satisfy the following inequality: $k \leq r \leq l$. By $\mathbb{P}$ we mean a set of all complex polynomials. By $\mathfrak{B}(M)$ we denote the set of all Borel subsets of a set $M \subseteq \mathbb{C}$. For a measure μ on $\mathfrak{B}(M)$ we denote by $L^2_\mu = L^2_\mu(M)$ the usual space of all (classes of equivalence of) Borel measurable complex-valued functions f on M, such that $\int_M |f|^2 d\mu < +\infty$. The class of the equivalence containing a function f will be denoted by $[f]$. By l^2 we denote the usual space of square-summable complex sequences $\vec{u} = (u_k)_{k=0}^\infty$, $u_k \in \mathbb{C}$, and l^2_{fin} means the subset of all finitely supported vectors from l^2. Moreover $\vec{e}_k$ means a vector from l^2 having 1 in k's place and zeros in other places ($k \in \mathbb{Z}_+$).

If H is a Hilbert space then $(\cdot,\cdot)_H$ and $\|\cdot\|_H$ mean the scalar product and the norm in H, respectively. Indices may be omitted in obvious cases. For a linear operator A in H, we denote by $D(A)$ its domain, by $R(A)$ its range, and A^* means the adjoint operator if it exists. If A is invertible then A^{-1} means its inverse. $\overline{A}$ means the closure of the operator, if the operator is closable. If A is bounded then $\|A\|$ denotes its norm. By E_H we denote the identity operator in H, i.e. $E_H x = x$, $x \in H$. In obvious cases we may omit the index H. If H_1 is a subspace of H, then $P_{H_1} = P_{H_1}^H$ denotes the orthogonal projection of H onto H_1. By $A|_{H_1}$ we mean the restriction of A to the subspace H_1.

2 Moment problems on $\mathbb{C}$ and complex Jacobi matrices.

At first we shall study the moment problem (1.11) which may be viewed as an extension of the Stieltjes moment problem (SMP) and the Hamburger moment problem (HMP). While the extension

$$\text{SMP} \ \rightarrow \text{HMP}$$

is well known, the extension to the complex plane was usually accompanied by adding additional monomials under the integral sign and the corresponding moments (the complex moment problem).

Theorem 2. *Let the moment problem (1.11) be given with some complex moments $\{s_k\}_{k\in\mathbb{Z}_+}$, $s_0 = 1$. This moment problem has a solution μ with a compact support if and only if the following condition holds:*

$$|s_n| \leq R^n, \qquad n \in \mathbb{Z}_+, \tag{2.1}$$

for some $R > 0$.

Proof. The necessity of condition (2.1) is clear from the estimate of the integral. Let us check its sufficiency. At first we shall consider the moment problem (1.11) with moments $\{s_k\}_{k\in\mathbb{Z}_+}$, $s_0 = 1$, which satisfy the following condition:

$$\sum_{n=0}^{\infty} |s_n|^2 < \infty. \tag{2.2}$$

Introduce the following vectors:

$$\vec{g} := \begin{pmatrix} 0 \\ \overline{s_1} \\ \overline{s_2} \\ \overline{s_3} \\ \vdots \end{pmatrix}, \quad \vec{f} := \begin{pmatrix} \overline{s_1} \\ \overline{s_2} - \overline{s_1}^2 \\ \overline{s_3} - \overline{s_2 s_1} \\ \overline{s_4} - \overline{s_3 s_1} \\ \vdots \end{pmatrix} = \begin{pmatrix} \overline{s_1} \\ \overline{s_2} \\ \overline{s_3} \\ \overline{s_4} \\ \vdots \end{pmatrix} - \overline{s_1} \begin{pmatrix} 0 \\ \overline{s_1} \\ \overline{s_2} \\ \overline{s_3} \\ \vdots \end{pmatrix}. \tag{2.3}$$

By condition (2.2) it follows that $\vec{f}, \vec{g} \in l^2$. Let S be the right shift operator on l^2:

$$S\vec{u} = (0, u_0, u_1, ...), \qquad \vec{u} = (u_0, u_1, ...) \in l^2.$$

Consider the following operator B, which is defined on the whole l^2:

$$B = S + (\cdot, \vec{f})_{l^2}\vec{e}_0 - (\cdot, \vec{g})_{l^2}\vec{e}_1. \tag{2.4}$$

The matrix $\mathcal{M}$ of the bounded operator B with respect to the basis $\{\vec{e}_k\}_{k=0}^{\infty}$ has the following form:

$$\mathcal{M} = \begin{pmatrix} s_1 & s_2 - s_1^2 & s_3 - s_2 s_1 & s_4 - s_3 s_1 & \cdots & s_{n+1} - s_n s_1 & \cdots \\ 1 & -s_1 & -s_2 & -s_3 & \cdots & -s_n & \cdots \\ 0 & 1 & 0 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{2.5}$$

Consider the following vectors in l^2 (a similar construction was used in [17]):

$$x_0 = \vec{e}_0, \quad x_j = \vec{e}_j + s_j\vec{e}_0, \qquad j \in \mathbb{N}, \tag{2.6}$$

and the following operator on l^2_{fin}:

$$M\sum_{k=0}^{d} \alpha_k x_k = \sum_{k=0}^{d} \alpha_k x_{k+1}, \qquad \alpha_k \in \mathbb{C},\ d \in \mathbb{Z}_+. \tag{2.7}$$

By induction we conclude that

$$x_n = M^n x_0, \qquad n \in \mathbb{Z}_+, \tag{2.8}$$

and therefore

$$s_n = (x_n, x_0)_{l^2} = (M^n x_0, x_0)_{l^2}, \qquad n \in \mathbb{Z}_+. \tag{2.9}$$

Since

$$\vec{e}_0 = x_0, \quad \vec{e}_j = x_j - s_j x_0, \qquad j \in \mathbb{N},$$

then

$$M\vec{e}_0 = \vec{e}_1 + s_1\vec{e}_0,$$

$$M\vec{e}_j = \vec{e}_{j+1} - s_j\vec{e}_1 + (s_{j+1} - s_j s_1)\vec{e}_0, \qquad j \in \mathbb{N}.$$

By a direct calculation one may verify that the matrix of M with respect to the basis $\{\vec{e}_k\}_{k=0}^{\infty}$ is exactly the matrix $\mathcal{M}$. Therefore the operator M is bounded and it extends on the whole space l^2 to the bounded operator B. Denote $\rho := \|B\|$. Then the operator $\widetilde{B} := \frac{1}{\rho} B$ is a contraction. It has a unitary dilation U in a Hilbert space $\widehat{H} \supseteq l^2$ (see, e.g., Theorem 4.2 on page 13 in [13]):

$$P_{l^2}^{\widehat{H}} U^k|_{l^2} = \widetilde{B}^k, \qquad k \in \mathbb{Z}_+. \tag{2.10}$$

By (2.9),(2.10) we may write:

$$s_n = \rho^n (\widetilde{B}^n x_0, x_0)_{l^2} = \rho^n (U^n x_0, x_0)_{\widetilde{H}} = ((\rho U)^n x_0, x_0)_{\widetilde{H}}, \ n \in \mathbb{Z}_+. \tag{2.11}$$

Since ρU is a bounded normal operator, its spectral resolution $E(\delta)$, $\delta \in \mathfrak{B}(\mathbb{C})$ provides a solution $\mu = (E(\delta) x_0, x_0)$ to the moment problem.

Consider now the moment problem from the assumptions of the theorem. Choose an arbitrary $\tau > R$, and set

$$\widetilde{s}_n := \frac{s_n}{\tau^n}, \qquad n \in \mathbb{Z}_+.$$

Then

$$|s_n| \le \left(\frac{R}{\tau} \right)^n,$$

and therefore condition (2.2) holds for $\widetilde{s}_n$. Thus, by the already proved result we can see from (2.11) that

$$\widetilde{s}_n = (N^n x_0, x_0)_{\widetilde{H}}, \ n \in \mathbb{Z}_+,$$

for a bounded normal operator N in a Hilbert space $\widetilde{H} \supseteq l^2$. Therefore

$$s_n = ((\tau N)^n x_0, x_0)_{\widetilde{H}}, \ n \in \mathbb{Z}_+.$$

The required result now follows from the spectral theorem for the bounded normal operator τN. □

Theorem 3. *Let J be a complex Jacobi matrix (1.1). Suppose that condition (1.2) holds. For the linear functional S, associated to J, the following integral representation holds:*

$$S(p) = \int_{\mathbb{C}} p(\lambda) d\mu, \qquad p \in \mathbb{P}, \tag{2.12}$$

with a positive measure μ on $\mathfrak{B}(\mathbb{C})$, and μ has a compact support. If the associated operator A of J is a contraction, then the measure μ has its support on $\mathbb{T}$.

Proof. In fact, for a complex Jacobi matrix one may write (see [2]):

$$\vec{e}_n = p_n(A) \vec{e}_0, \qquad n \in \mathbb{Z}_+, \tag{2.13}$$

and therefore

$$(p_m(A)\vec{e}_0, p_n(A)\vec{e}_0) = \delta_{m,n}, \qquad m,n \in \mathbb{Z}_+, \tag{2.14}$$

and

$$S(u) = (u(A)\vec{e}_0, \vec{e}_0), \qquad u \in \mathbb{P}. \tag{2.15}$$

Thus, for the moments $\mathbf{s}_n$ of S we have the following estimate:

$$|\mathbf{s}_n| \leq |(A^n\vec{e}_0, \vec{e}_0)| \leq \|A\|^n, \qquad n \in \mathbb{Z}_+. \tag{2.16}$$

By Theorem 2 it follows that there exists a compactly supported Borel measure μ on $\mathbb{C}$ such that

$$S(z^n) = \mathbf{s}_n = \int z^n d\mu, \qquad n \in \mathbb{Z}_+.$$

By linearity we obtain relation (2.12).

If it is additionally known that A is a contraction, we may apply the unitary dilation theorem directly to A. If U is a unitary dilation of A, acting in a Hilbert space $\widehat{H} \supseteq l^2$, then

$$\mathbf{s}_n = (A^n\vec{e}_0, \vec{e}_0)_{l^2} = (U^n\vec{e}_0, \vec{e}_0)_{\widehat{H}}.$$

The result follows from the spectral theorem for U. □

The last theorem shows that it is convenient to normalize the Jacobi matrix in order to get a contraction, as it is usual when dealing with truncated complex Jacobi matrices, see [2].

For a compactly supported measure μ on $\mathfrak{B}(\mathbb{C})$, we shall denote by Λ_μ the operator of multiplication by the independent variable in L^2_μ, and $\mathcal{P} := \{[p],\ p \in \mathbb{P}\}$.

Theorem 4. *Let J be a complex Jacobi matrix (1.1) and $\{p_n\}_{n=0}^\infty$, $p_0 = 1$, be a system of polynomials satisfying (1.3). Suppose that condition (1.2) holds. Let A_0 be the associated to J operator on l^2_{fin}, and*

$$T\sum_{k=0}^d \xi_k\vec{e}_k = \left[\sum_{k=0}^d \xi_k p_k(z)\right], \qquad \xi_k \in \mathbb{C},\ d \in \mathbb{Z}_+, \tag{2.17}$$

maps l^2_{fin} into L^2_μ, while μ is a positive measure provided by Theorem 3. Then the linear operator T is invertible and

$$TA_0T^{-1} = \Lambda_0, \tag{2.18}$$

where $\Lambda_0 = \Lambda_\mu|_{\mathcal{P}}$.

Proof. Notice that

$$TA_0\vec{e}_k = T(b_{k-1}\vec{e}_{k-1} + a_k\vec{e}_k + b_k\vec{e}_{k+1}) = [b_{k-1}p_{k-1}(z) + a_kp_k(z) + b_kp_{k+1}(z)] =$$

$$= [zp_k(z)] = \Lambda_0[p_k] = \Lambda_0T\vec{e}_k, \qquad k \in \mathbb{Z}_+.$$

By the linearity it follows that

$$TA_0 = \Lambda_0 T. \tag{2.19}$$

It remains to check that T is invertible. Suppose to the contrary that there exists a nonzero vector

$$\vec{u} = \sum_{k=0}^{r} \eta_k \vec{e}_k \in l^2_{fin}, \qquad \eta_k \in \mathbb{C},\ \eta_r \neq 0,\ r \in \mathbb{Z}_+,$$

such that

$$T\vec{u} = 0.$$

Then

$$0 = \|T\vec{u}\|^2 = \int \left| \sum_{k=0}^{r} \eta_k p_k(z) \right|^2 d\mu.$$

This means that the measure μ is finitely atomic, with atoms among the zeros of the polynomial $u_r(z) := \sum_{k=0}^{r} \eta_k p_k(z)$, $\deg u_r = r$. By property 3) of Theorem 1 there exists a polynomial $\widehat{u}_r(\lambda)$ of degree r such that:

$$\int u_r(z)\widehat{u}_r(z) d\mu = \sigma(u_r, \widehat{u}_r) \neq 0.$$

However the integral on the left side is equal to zero, a contradiction. The proof is complete. □

Of course, it would be valuable to obtain for the operator A a result similar to that given in Theorem 4 for the operator A_0. However, it is not clear when T is bounded and has a bounded inverse, except for the real matrices. This question will be studied elsewhere. However, formula (2.18) provides a polynomial calculus for A_0, which is useful as well. We met such situations, when even the calculation of a power of an operator leads to serious problems (in the case of operators associated with Jacobi-type pencils, see [19]).

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