**Surveys in Mathematics and its Applications**

**ISSN** 1842-6298 (electronic), 1843-7265 (print)
Volume 18 (2023), 83 – 95

# CONFORMABLE FUNCTIONAL EVOLUTION EQUATIONS WITH NONLOCAL CONDITIONS IN BANACH SPACES

Abderrahmane Boukenkoul and Mohamed Ziane

**Abstract**. In this paper, we study semilinear conformable fractional evolution equations with finite delay subjected to nonlocal initial conditions in an arbitrary Banach space. We prove the existence of mild solutions under compactness type conditions on the nonlinear forcing term. Our result improves and complements several earlier related works. We apply our result to study a functional conformable partial differential equation of transport type.

## 1 Introduction

Conformable fractional calculus introduced in [2, 19] received much attention in recent years. Several authors have studied different aspects for the theory [7, 17, 27]. Conformable derivative is a consistent generalization of the classical integer one and has been used in Newton mechanics [12], anomalous diffusion [26], stochastic process [11], logistic models [3] and so on [22, 25, 28].

Functional differential equations are important applied mathematics tools, since many phenomenons in biological and physical systems are modeled using the history of the system [16, 24]. In such equations, classical and fractional derivatives, with different variants, were considered [1, 4]. But, up to our knowledge, the conformable fractional derivative has not been yet investigated in the literature.

In this paper, we study the following nonlocal evolution equations involving conformable derivative with finite delay

$$\begin{cases} \dfrac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + f(t, x_t); & \text{for } t \in [0, b], \\ x(t) = \varphi(t) + g(x)(t); & \text{for } t \in [-r, 0], \end{cases} \tag{1.1}$$

$A : \mathcal{D}(A) \subset E \to E$ is a densely defined unbounded linear operator generating a $C_0$-semigroup $\{T(t), t \geq 0\}$ on a Banach space $E$, $\dfrac{d^\alpha(\cdot)}{dt^\alpha}$ is a fractional conformable

2020 Mathematics Subject Classification: 26A33; 34A08; 47H08.
Keywords: conformable fractional derivative; mild solutions; measure of noncompactness.

derivative of order $0 < \alpha < 1$, $f : [0, b] \times C([-r, 0], E) \to E$, $r > 0$, $g : C([0, b], E) \to C([-r, 0], E)$, $\varphi \in C([-r, 0], E)$, and for $t \geq 0$, $x_t : [-r, 0] \to E$ is a history function defined by $x_t(\theta) = x(t + \theta)$.

Nonlocal initial conditions have been introduced in [9, 10]. For instance, the nonlocal condition can be of the form $x(0) = g(x)(\cdot)$ where the implicit function $g(x)$ is given by

$$g(x) = \sum_{i=1}^{m} c_i x(\tau_i), \ c_i (i = 1, 2, \ldots, m) \text{ are a given constants } \tau_i \in [0, b]. \tag{1.2}$$

Equation (1.2) allows to take into account multiple measurements at different times $\tau_i$ in the initial condition, leading to a better description of the phenomenon. For example, Deng [13] considered the phenomenon of diffusion of a small amount of gas in a tube and assumed that the diffusion is observed via the surface of the tube. The nonlocal condition allows additional measurement which is more precise than the classical initial condition alone [8].

In [7], the authors studied the nonlocal evolution equation (1.1) without delay. They obtained existence of a solution by means of Schaefer's fixed point theorem, under compactness of the semi-group. This paper improves and extends the results given in [7] to the finite delay case without assuming the compactness of the semigroup nor the Lipschitz continuity of $g$. We use Mönch's fixed point theorem and assume a regularity condition expressed in terms of the Hausdorff measure of noncompactness on the nonlinearity term.

The rest of the paper is organized as follows. In the next section we recall some useful notions and results. In Section 3, we state and prove a new existence result for the problem (1.1). An illustrative example is given in Section 4.

## 2 Preliminaries

Let $E$ be a separable Banach space provided with the norm $\| \cdot \|$. Throughout this work, $\mathcal{L}(E)$ denotes the space of all bounded linear operators from $E$ to itself, $C([0, b], E)$ is the Banach space of all continuous functions from $[0, b]$ to $E$ endowed with the uniform norm topology $\|x\| = \sup_{t \in [0,b]} \|x(t)\|$, and $L^p([0, b], E)$, $1 \leq p < \infty$, is the space of all integrable functions in Bochner sense normed by $\|x\|_{L^p([0,b],E)} = \left( \int_0^b \|x(s)\|^p \, ds \right)^{\frac{1}{p}}$. If $p = \infty$, $L^\infty([0, b], E)$ is the Banach space of all equivalence classes of strongly measurable functions which are essentially bounded on $[0, b]$ normed by

$$\|x\|_{L^\infty([0,b],E)} = \operatorname*{ess\,sup}_{t \in [0,b]} \|x(t)\| = \inf\{M > 0;\ \|x(t)\| \leq M \text{ for a.e. } t \in [0, b]\}.$$

First, let us recall the following definitions from conformable fractional calculus.

**Definition 1.** *[19] The conformable fractional derivative of order $0 < \alpha \leq 1$ for a function $y(\cdot)$ is defined by*

$$\frac{d^\alpha y(t)}{dt^\alpha} = \lim_{\epsilon \to 0} \frac{y(t + \epsilon t^{1-\alpha}) - y(t)}{\epsilon}, \quad t > 0;$$
$$\frac{d^\alpha y(0)}{dt^\alpha} = \lim_{t \to 0} \frac{d^\alpha y(t)}{dt^\alpha},$$

*and the associated fractional integral $I^\alpha(\cdot)$ is defined by*

$$I^\alpha(y)(t) = \int_0^t s^{1-\alpha} y(s) ds,$$

*provided that the previous limit and integral are well defined.*

Henceforth, we denote by $\chi(\Omega)$ the Hausdorff measure of non-compactness (MNC for short) of nonempty bounded set $\Omega \subset E$, defined by

$$\chi(\Omega) = \inf\{\varepsilon > 0;\ \Omega \text{ admits a finite cover by balls of radius} \leq \varepsilon\}.$$

We recall some properties of $\chi$. For more details, the reader can refer to [6, 18].

Let $\Omega_1$ and $\Omega_2$ are two bounded subsets of $E$.

(1) $\chi(\Omega_1) = 0$ if and only if $\overline{\Omega_1}$ is compact.

(2) $\chi(\Omega_1) = \chi(\overline{\Omega_1}) = \chi(\overline{co}\ \Omega_1)$; where $\overline{co}\ \Omega_1$ denotes the closed convex hull of $\Omega_1$.

(3) $\chi(\lambda\Omega_1) = |\lambda|\chi(\Omega_1)$ for every $\lambda \in \mathbb{R}$.

(4) $\chi(\Omega_1) \leq \chi(\Omega_2)$ if $\Omega_1 \subset \Omega_2$.

(5) $\chi(\Omega_1 + \Omega_2) = \chi(\Omega_1) + \chi(\Omega_2)$.

(6) Let $G : [0, b] \to \mathcal{L}(E)$ be a strongly continuous operator valued map. Then

$$\chi_c(\{G(\cdot)x\ :\ x \in \Omega_1\}) \leq \sup_{t \in [0,b]} \|G(t)\|\chi(\Omega_1).$$

Here $\chi_c$ is the Hausdorff MNC in $C([0, b], E)$.

(7) The sequential MNC, generated by $\chi(\cdot)$ defined as

$$\chi_0(B) = \sup\{\chi(\{x_n :\ n \geq 1\}) :\ (x_n) \text{ is a sequence in } B\}.$$

If $E$ is separable, then $\chi_0(\Omega) = \chi(\Omega)$. In arbitrary $E$ we have

$$\chi_0(\Omega) \leq \chi(\Omega) \leq 2\chi_0(\Omega). \tag{2.1}$$

**Lemma 2.** *[6] Let $V \subseteq C([0,b],E)$ be a bounded set. Then $\chi(V(t)) \leq \chi_c(V)$ for all $t \in [0,b]$. Furthermore, if $V$ is equicontinuous on $[0,b]$, then $\chi(V(\cdot))$ is continuous on $[0,b]$ and*

$$\chi_c(V) = \sup_{t\in[0,b]} \chi(V(t)).$$

**Lemma 3.** *[18] Assume that $\{u_n\}_{n=1}^{+\infty} \subset L^1([0,b],E)$ satisfies $\|u_n(t)\| \leq \kappa(t)$ a.e. on $[0,b]$ for all $n \geq 1$ with some $\kappa \in L^1([0,b],\mathbb{R}_+)$. Then, the function $\chi(\{u_n(t)\}_{n=1}^{+\infty})$ belongs to $L^1([0,b],\mathbb{R}_+)$ and*

$$\chi\left(\left\{\int_0^t u_n(s)ds \ : \ n \geq 1\right\}\right) \leq 2\int_0^t \chi(u_n(s) \ : \ n \geq 1)ds. \tag{2.2}$$

**Theorem 4.** *[21, Theorem 2.1] Let $\mathcal{C}$ be a closed convex subset of a Banach space $E$ and $0 \in \mathcal{C}$. Assume that $F : \mathcal{C} \to E$ is a continuous map which satisfies the Mönch's condition; that is, if $M \subseteq \mathcal{C}$ is countable and $M \subseteq \overline{co}(\{0\} \cup F(M)) \Longrightarrow \overline{M}$ is compact. Then $F$ has a fixed point in $\mathcal{C}$.*

## 3 Main Result

In this section, we give some hypotheses to prove the existence of mild solutions of (1.1). We consider the following assumptions.

(H1) The semigroup $(T(t))_{t\geq 0}$ is continuous in the uniform operator topology.

(H2) (i) The function $f(\cdot,\phi) : [0,b] \to E$ is strongly measurable for each $\phi \in C([-r,0],E)$, and $f(t,\cdot) : C([-r,0],E) \to E$ is continuous for a.e. $t \in [0,b]$.
(ii) There exist a function $h \in L^\infty([0,b],\mathbb{R}_+)$ and a nondecreasing continuous function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$f(t,\phi) \leq h(t)\Phi(\|\phi\|), \quad \text{a.e.}\, t \in [0,b] \text{ and all } \phi \in C([-r,0],E).$$

(iii) There exists a function $\xi \in L^\infty([0,b],\mathbb{R}_+)$ such that for any bounded subset $D \subset C([-r,0],E)$, we have

$$\chi(f(t,D)) \leq \xi(t) \sup_{-r\leq\theta\leq 0} \chi(D(\theta)), \quad \text{for a.e. } t \in [0,b],$$

(H3) $g : C([0,b],E) \to C([-r,0],E)$ is continuous and compact. Moreover

$$\|g(\phi)\| \leq c\|\phi\| + d, \quad \text{for all } \phi \in C([0,b],E),$$

for some constants $c, d > 0$.

We recall that the mild solution can be defined by the following [7, Lemma 3.1] and [17, Definition 7].

**Definition 5.** *A function $x \in C([0,b],E)$ is said to be a mild solution of the problem* (1.1) *if $x(t) = \varphi(t) + g(x)(t)$ for $t \in [-r,0]$ and*

$$x(t) = T\left(\frac{t^\alpha}{\alpha}\right)[\varphi(0)+g(x)(0)] + \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f(s,x_s)ds, \quad t \in [0,b]. \tag{3.1}$$

Thanks to the uniform boundedness principle, there exists $M_\alpha > 0$ such that

$$\sup_{0 \le t \le b} \left\| T\left(\frac{t^\alpha}{\alpha}\right) \right\| \le M_\alpha.$$

**Remark 6.** *In general, the linear operator $S(t) := T\left(\frac{t^\alpha}{\alpha}\right)$ for equation* (3.1) *does not satisfy the usual algebraic semigroup property, namely*

$$S(t+s) \neq S(t)S(s) \quad \textit{for some } t,s \ge 0,$$

*which induces some difficulties in obtaining the compactness of the operator solution. We refer to [5] for more details about the intuitive algebraic semigroup property and the generation theorem.*

Recall that the $C_0$-semigroup $T(t)$ is said to be equicontinuous (or continuous in the uniform operator topology or operator-norm continuous) if $t \mapsto \{T(t)x,\ x \in B\}$ is equicontinuous at $t > 0$ for each bounded set $B \subset E$.

**Proposition 7.** *For $t > 0$, if $T(t)$ is equicontinuous then $S(t)$ is also equicontinuous.*

*Proof.* Suppose that $\{T(t)\}_{t \ge 0}$ is equicontinuous for $t > 0$ and let $x \in E$ such that $\|x\| \le 1$. Then we have for any $0 < \bar{t} < t \le b$

$$\begin{aligned} \|S(t)x - S(\bar{t})x\| &= \left\| T\left(\frac{t^\alpha}{\alpha}\right)x - T\left(\frac{\bar{t}^\alpha}{\alpha}\right)x \right\|, \\ &\le \left\| T\left(\frac{t^\alpha - \bar{t}^\alpha}{\alpha}\right) - I \right\|. \end{aligned}$$

It follows that

$$\|S(t) - S(\bar{t})\| \le \left\| T\left(\frac{t^\alpha - \bar{t}^\alpha}{\alpha}\right) - I \right\|.$$

The continuity in the uniform operator topology of $T(t)$ allows us to deduce that

$$\|S(t) - S(\bar{t})\| \to 0 \text{ as } t \to \bar{t}.$$

□

**Theorem 8.** *Assume that the conditions (H1)-(H3) hold. Then, for each $\varphi \in C([-r,0],E)$, the problem (1.1) has at least one mild solution on $[-r,b]$ provided that there exists $\overline{M}>0$ satisfying*

$$M_\alpha\left(\|\varphi\| + c\overline{M} + d + \frac{b^\alpha}{\alpha}\Phi(\overline{M})\|h\|_{L^\infty([0,b],\mathbb{R}_+)}\right) < \overline{M}. \tag{3.2}$$

*Proof.* To find a mild solution of (1.1) in $C([-r,b],E)$, we will apply Theorem (4). First, let us consider the operator solution $\mathcal{K} : C([-r,b],E) \to C([-r,b],E)$ defined by

$$(\mathcal{K}x)(t) := \begin{cases} \underbrace{\varphi(t) + g(x)(t)}_{:=(\mathcal{K}_1 x)(t)}, & \text{for } t \in [-r,0], \\ \underbrace{T\left(\frac{t^\alpha}{\alpha}\right)[\varphi(0)+g(x)(0)] + \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f(s,x_s)ds}_{:=(\mathcal{K}_2 x)(t)}, & \text{for } t \in [0,b]. \end{cases} \tag{3.3}$$

Our goal is to show that the operator $\mathcal{K}$ has a fixed point in the closed ball

$$\overline{B}_{\overline{M}} = \{x \in C([-r,b],E) \,:\, \|x\| \le \overline{M}\}.$$

The proof is accomplished in four steps.

**Step 1.** $\mathcal{K}$ maps $\overline{B}_{\overline{M}}$ into itself. Indeed, for every $x \in \overline{B}_{\overline{M}}$ and every $t \in [-r,0]$, we have

$$\begin{aligned} \|(\mathcal{K}x)(t)\| = \|(\mathcal{K}_1 x)(t)\| &\le \|\varphi(t)\| + \|g(x)(t)\|, \\ &\le \|\varphi\| + c\|x\| + d, \\ &\le \|\varphi\| + c\overline{M} + d. \end{aligned} \tag{3.4}$$

For $t \in [0,b]$ and $x \in \overline{B}_{\overline{M}}$, we have

$$\begin{aligned} \|(\mathcal{K}x)(t)\| &= \|(\mathcal{K}_2 x)(t)\| \\ &\le \left\|T\left(\frac{t^\alpha}{\alpha}\right)\right\| \|\varphi(0)+g(x)(0)\| + \left\|\int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f(s,x_s)ds\right\|, \\ &\le M_\alpha(\|\varphi\| + c\|x\| + d) + M_\alpha \int_0^t s^{\alpha-1} h(s)\Phi(\|x_s\|)ds, \\ &\le M_\alpha(\|\varphi\| + c\|x\| + d) + M_\alpha \Phi(\overline{M}) \int_0^t s^{\alpha-1} h(s)ds, \\ &\le M_\alpha(\|\varphi\| + c\overline{M} + d) + M_\alpha \Phi(\overline{M}) \frac{b^\alpha}{\alpha}\|h\|_{L^\infty([0,b],\mathbb{R}_+)}. \end{aligned} \tag{3.5}$$

Then, the inequalities (3.2)-(3.5) imply that $\mathcal{K}(\overline{B}_{\overline{M}}) \subseteq \overline{B}_{\overline{M}}$.

**Step 2.** The operator $\mathcal{K}$ is continuous on $\overline{B}_{\overline{M}}$.

Let $(x^m)_{m\geq 1}$ be a sequence in $\overline{B}_{\overline{M}}$ such that $\lim_{m\to\infty} \|x^m - x\| = 0$. Then, for $s \in [0,b]$, one has

$$\begin{aligned}
\|x_s^m - x_s\| &= \sup_{-r\leq\theta\leq 0} \|x_s^m(\theta) - x_s(\theta)\|, \\
&= \sup_{-r\leq\theta\leq 0} \|x^m(\theta+s) - x(\theta+s)\|, \\
&= \sup_{-r\leq\overline{\theta}\leq b} \|x^m(\overline{\theta}) - x(\overline{\theta})\|, \\
&= \|x^m - x\|.
\end{aligned}$$

Therefore

$$\lim_{m\to+\infty} \|x_s^m - x_s\| = 0.$$

By assumption (H2)(ii), we get

$$\|s^{\alpha-1}[f(s,x_s^m) - f(s,x_s)]\| \leq 2s^{\alpha-1}h(s)\Phi(\overline{M}), \quad \text{for } s \in [0,b].$$

Thus, by the Lebesgue dominated convergence theorem, we have

$$M_\alpha \int_0^t s^{\alpha-1}\|f(s,x_s^m) - f(s,x_s)\|ds \underset{m\to\infty}{\longrightarrow} 0, \quad \text{for } t \in [0,b].$$

Finally, by (H3) the function $g$ is continuous. Consequently, we conclude that the operator $\mathcal{K}$ is continuous.

**Step 3.** The family $\mathcal{K}(\overline{B}_{\overline{M}})$ is equicontinuous.

Thanks to (H1) and (H3), we only need to prove the equicontinuity for $[0,b]$.

We start by the equicontinuity at 0. For any $x \in \overline{B}_{\overline{M}}$ and $0 = t_1 < t_2 \leq b$, we have

$$\begin{aligned}
&\|(\mathcal{K}_2x)(t_2) - (\mathcal{K}_2x)(0)\| \\
&\quad\leq \left\|T\left(\frac{t_2^\alpha}{\alpha}\right) - Id_E\right\| \|\varphi(0) + g(x)(0)\| + \left\|\int_0^{t_2} s^{\alpha-1}T\left(\frac{t_2^\alpha - s^\alpha}{\alpha}\right) f(s,x_s)ds\right\|.
\end{aligned}$$

Thus,

$$\|(\mathcal{K}_2x)(t_2)-(\mathcal{K}_2x)(0)\| \leq \left\|T\left(\frac{t_2^\alpha}{\alpha}\right) - Id_E\right\| (\|\varphi\|+c\overline{M}+d)+M_\alpha\Phi(\overline{M}) \int_0^{t_2} s^{\alpha-1}h(s)ds.$$

Consequently,

$$\|(\mathcal{K}_2x)(t_2) - (\mathcal{K}_2x)(0)\| \to 0, \qquad \text{as } t_2 \to 0,$$

which shows that $\mathcal{K}(\overline{B}_{\overline{M}})$ is equicontinuous at $t=0$.

Now, for $0 < t_1 < t_2 \leq b$, we have

$$\begin{aligned}
&\|(\mathcal{K}_2x)(t_2) - (\mathcal{K}_2x)(t_1)\| \\
&\leq \left\| T\left(\frac{t_2^\alpha}{\alpha}\right) - T\left(\frac{t_1^\alpha}{\alpha}\right)\right\| \|\varphi(0) + g(x)(0)\| + \int_{t_1}^{t_2} s^{\alpha-1}\|T\left(\frac{t_2^\alpha - s^\alpha}{\alpha}\right) f(s,x_s)ds\| \\
&\quad + \int_0^{t_1} s^{\alpha-1}\left\| T\left(\frac{t_2^\alpha - s^\alpha}{\alpha}\right) - T\left(\frac{t_1^\alpha - s^\alpha}{\alpha}\right)\right\| \|f(s,x_s)\|ds,
\end{aligned}$$

which means that

$$\begin{aligned}
&\|(\mathcal{K}_2x)(t_2) - (\mathcal{K}_2x)(t_1)\| \\
&\quad \leq \left\| T\left(\frac{t_2^\alpha - t_1^\alpha}{\alpha}\right) - Id_E\right\| (\|\varphi\| + c\overline{M} + d) + M_\alpha \Phi(\overline{M}) \int_{t_1}^{t_2} s^{\alpha-1}h(s)ds \\
&\quad + M_\alpha \Phi(\overline{M}) \int_0^{t_1} s^{\alpha-1}\left\| T\left(\frac{t_2^\alpha - s^\alpha}{\alpha}\right) - T\left(\frac{t_1^\alpha - s^\alpha}{\alpha}\right)\right\| h(s)ds, \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

By (H1), it is easy to see that $I_1, I_2 \to 0$ as $t_2 \to t_1$. In view of (H2)(ii), we obtain

$$s^{\alpha-1}\left\| T\left(\frac{t_2^\alpha - s^\alpha}{\alpha}\right) - T\left(\frac{t_1^\alpha - s^\alpha}{\alpha}\right)\right\| h(s) \leq 2M_\alpha s^{\alpha-1}h(s), \quad 0 \leq s \leq t \leq b.$$

Thanks to the continuity in the uniform operator topology of $T(t)$, the Lebesgue dominated convergence theorem ensures that

$$I_3 \to 0, \text{as} \;\; t_2 \to t_1.$$

Thus, we deduce that $\mathcal{K}(\overline{B}_{\overline{M}})$ is equicontinuous.

**Step 4.** The Mönch's type condition holds.

Suppose $\mathcal{B} \subset \overline{B}_{\overline{M}}$ is countable and $\mathcal{B} \subset \overline{co}\,(\{0\} \cup \mathcal{K}(\mathcal{B}))$. We will prove that $\mathcal{B}$ is relatively compact.

We define the MNC for every bounded subset $D \subset C([-r,b],E)$ as

$$\Psi(D) = (\chi(D([-r,0])), \overline{\chi}_c(D)), \tag{3.6}$$

where

$$\overline{\chi}_c(D) = \sup_{t\in[0,b]} e^{-Lt}\chi(D(t)), \quad L > 0. \tag{3.7}$$

For the previous construction of MNC, we refer to [15].

Since $(\cdot)^{1-\alpha}\xi(\cdot) \in L^1([0,b],\mathbb{R})$, so it is possible to choose $L$ such that

$$q(L) := \sup_{t\in[0,b]} 2M_\alpha \int_0^t e^{-L(t-s)} s^{1-\alpha}\xi(s)ds < 1. \tag{3.8}$$

Let $\{y^m\}_{m=1}^{+\infty}$ be the countable set such that $\{y^m\}_{m=1}^{+\infty} \subset \mathcal{K}(\mathcal{B})$ and

$$\overline{\chi}_c(\mathcal{K}(\mathcal{B})) = \overline{\chi}_c(\{y^m\}_{m=1}^{+\infty}). \tag{3.9}$$

Then, there exists a set $\{x^m\}_{m=1}^{+\infty} \subset \mathcal{B}$ such that

$$y^m(t) = (\mathcal{K}x^m)(t), \quad \text{for } m \geq 1 \text{ and } t \in [-r,b].$$

First, using (H3) together with the Arzelà–Ascoli Theorem, we have immediately

$$\chi(\{\mathcal{K}x^m(t),\ t\in[-r,0]\}_{m=1}^{+\infty}) = \chi(\{\varphi(t)+g(x^m)(t),\ t\in[-r,0]\}_{m=1}^{+\infty}) = 0.$$

Now, we estimate the quantity $\overline{\chi}_c(\{y^m\}_{m=1}^{+\infty})$. Using the condition (H2)(iii), for all $s \in [0,t]$ we have

$$\begin{aligned}
\chi(\{s^{1-\alpha}f(s,x_s^m)\}_{m=1}^{+\infty}) &\leq s^{1-\alpha}\xi(s)\chi(\{x_s^m\}_{m=1}^{+\infty}),\\
&\leq s^{1-\alpha}\xi(s) \sup_{-r\leq\theta\leq 0} \chi(\{x^m(s+\theta)\}_{m=1}^{+\infty}),\\
&\leq s^{1-\alpha}\xi(s) \sup_{0\leq\tau\leq s} \chi(\{x^m(\tau)\}_{m=1}^{+\infty}),\\
&\leq e^{Ls}s^{1-\alpha}\xi(s) \sup_{0\leq s\leq t} e^{-Ls} \sup_{0\leq\tau\leq s} \chi(\{x^m(\tau)\}_{m=1}^{+\infty}),\\
&\leq e^{Ls}s^{1-\alpha}\xi(s)\overline{\chi}_c(\{x^m\}_{m=1}^{+\infty}).
\end{aligned}$$

Thus, applying Lemma 3, we get for all $t \in [0,b]$ and $s \leq t$,

$$\chi(\{(\mathcal{K}x^m)(t)\}_{m=1}^{+\infty}) \leq 2M_\alpha\overline{\chi}_c(\{x^m\}_{m=1}^{+\infty}) \int_0^t e^{Ls}s^{1-\alpha}\xi(s)ds.$$

Multiplying both sides by $e^{-Lt}$, we obtain

$$\sup_{t\in[0,b]} e^{-Lt}\chi(\{(\mathcal{K}x^m)(t)\}_{m=1}^{+\infty}) \leq \overline{\chi}_c(\{x^m\}_{m=1}^{+\infty}) \sup_{t\in[0,b]} 2M_\alpha \int_0^t e^{-L(t-s)}s^{1-\alpha}\xi(s)ds.$$

Taking into account the compactness of $g$, we derive

$$\overline{\chi}_c(\{x^m\}_{m=1}^{+\infty}) \leq \overline{\chi}_c(\mathcal{B}) \leq \overline{co}\,(\{0\}\cup\mathcal{K}(\mathcal{B})) \leq \overline{\chi}_c(\{y^m\}_{m=1}^{+\infty}) \leq q(L)\overline{\chi}_c(\{x^m\}_{m=1}^{+\infty}).$$

The inequality (3.8) implies

$$\overline{\chi}_c(\{x^m\}_{m=1}^{+\infty}) = \overline{\chi}_c(\{y^m\}_{m=1}^{+\infty}) = 0.$$

Therefore, $\Psi(\mathcal{B}) = (0,0)$, and thus $\mathcal{B}$ is relatively compact. Finally, by applying Theorem (4), we deduce the existence of at least one fixed point $x \in \overline{B}_{\overline{M}}$ of the solution operator $\mathcal{K}$ which is a mild solution of the problem (1.1).

□

# 4 Application

Let $\widetilde{\Omega}$ be a nonempty bounded open set in $\mathbb{R}^n$ with smooth boundary $\partial\widetilde{\Omega}$. Denote $E = L^p(\widetilde{\Omega})$, with $1 \leq p < \infty$ and $a \in \mathbb{R}^n$. Consider the functional conformable partial (transport) differential equation

$$\begin{cases} \dfrac{d^\alpha u(t,x)}{dt^\alpha} = a \cdot \nabla u(t,x) + \eta(t)\widetilde{f}(u(t-r)(x)), & [0,b] \times \widetilde{\Omega}, \\ u(t,x) = 0, & [0,b] \times \partial\widetilde{\Omega}, \\ u(\theta,x) = \sin\left(\dfrac{t}{2}\right) + \displaystyle\int_{-r}^{\theta} \left( \int_{\widetilde{\Omega}} \Theta(s,x,\lambda,u(s,\lambda))d\lambda \right) ds, & \theta \in [-r,0],\ x \in \widetilde{\Omega}, \end{cases} \tag{4.1}$$

where the partial derivatives are taken in the sense of distributions over $\widetilde{\Omega}$ and $\eta(t) \in L^\infty([0,b], \mathbb{R}_+)$, $\widetilde{f}$ is Lipschitzian with constant $\widetilde{L} > 0$, $\widetilde{f}(0) = 0$. A simple computation (like in [14, Lemma 25]) shows that the function $f(t, u_t(x)) = \eta(t)\widetilde{f}(u(t-r)(x))$ satisfies (H2)(i)-(iii) with

$$\begin{cases} h(\cdot) = \eta(\cdot), \\ \xi(\cdot) = \widetilde{L}\eta(\cdot), \\ \Phi(\|u\|) = \widetilde{L}\|u\|. \end{cases}$$

Denote

$$\begin{cases} D(A) = \{u \in E; \quad a \cdot \nabla u \in E\}, \\ Au = a \cdot \nabla u. \end{cases}$$

From [23, Theorem 4.4.1], $A$ generates a noncompact semigroup $S(t) = T\left(\dfrac{t^\alpha}{\alpha}\right)$ given by

$$S(t)u = u\left(x - \frac{t^\alpha}{\alpha}a\right), \quad \text{for each } u \in E,\ t \in \mathbb{R}.$$

Clearly, the family $S(t)$ is continuous in the uniform operator topology (it is isometry).

For the nonlocal condition of (4.1), we assume that:

(A1) For every $k > 0$, there exists a positive function $m_k$ such that for $\|\sigma\| \leq k$

$$\|\Theta(t,x,\lambda,\sigma) - \Phi(t,y,\lambda,\sigma)\| \leq m_k(t,x,y,\lambda).$$

(A2) There exist two constants $\bar{c}$ and $\bar{d}$ such that

$$\|\Theta(t,x,\lambda,\sigma)\| \leq \bar{c}\|\sigma\| + \bar{d}, \quad \text{where } \|\sigma\| \leq k.$$

(A3) $\lim\limits_{x\to y} \displaystyle\int_{-h}^{a} \left( \int_{\Omega} m_k(t,x,y,\lambda)d\lambda \right) dt = 0$ uniformly on $\widetilde{\Omega}$.

The system (4.1) can be written in the abstract form given by (1.1). All assumptions in Theorem (8) are satisfied (see for instance [20, page 172]). Then, the problem (4.1) has at least one mild solution.

**Acknowledgement.** The authors wishes to thank the referee for his (her) corrections and remarks which helped to improve the paper.

## References

[1] S. Abbas, M. Benchohra, *Advanced Functional Evolution Equations and Inclusions*, Developments in Mathematics, vol. 39. Springer, (2015). MR3381102. Zbl 1390.34002.

[2] T. Abdeljawad; On conformable fractional calculus, *J. Comput. Appl. Math.*, 279 (2015), 57-66. MR3293309. Zbl 1304.26004.

[3] T. Abdeljawad, Q. M. Al-Mdallal, F. Jarad; Fractional logistic models in the frame of fractional operators generated by conformable derivatives, *Chaos Solitons Frac.*, 119 (2019), 94-101. MR3894310. Zbl 1448.34006.

[4] C. T. Anh, , T. D. Ke, On nonlocal problems for retarded fractional differential equations in Banach spaces *Fixed Point Theory* 15(2), 373-392 (2014). MR3235658. Zbl1307.34120 .

[5] S. Al-Sharif, M. Al Horani, R. Khalil, The Hille-Yosida theorem for conformable fractional semi-groups of operators. *Missouri J. Math. Sci.* 33(1):18–26 (2021). MR4278773. Zbl 1471.47027.

[6] J. Banas and K. Goebel, *Measure of Noncompactness in Banach Spaces*, Lectures Notes in Pure and Applied Mathematics, 50, Marcel Dekker, New York, 1980. MR591679. Zbl 0441.47057.

[7] M. Bouaouid, K. Hilal, S. Melliani, Existence of mild solutions for conformable fractional differential equations with nonlocal conditions. *Rocky Mountain J. Math.*, 50(3), 871-879, (2020). MR4132616. Zbl 1479.34009.

[8] M. Burlica, M. Necula, D. Rosu, I. Vrabie, *Delay Differential Evolutions Subject to Nonlocal Initial Conditions* CRC Press Taylor & Francis group, New York, 2016. MR3618778. Zbl 1348.34001.

[9] L. Byszewski, Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem. *J. Math. Anal. Appl.* 162, 494-505 (1991). MR1137634. Zbl 0748.34040.

[10] L. Byszewski, V. Lakshmikantham, Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space, *Appl. Anal.*, 40(1991), 11-19. MR1121321. Zbl 0694.34001.

[11] Y. Cenesiz, A. Kurt, E. Nane, Stochastic solutions of conformable fractional Cauchy problems, *Stat. Probab. Lett.* 124. (2017) 126–13. MR3608222. Zbl 1417.35218.

[12] W. Chung, Fractional newton mechanics with conformable fractional derivative *J. Comput. Appl. Math.*, 290 (2015), 150-158. MR3370399. Zbl 1336.70033.

[13] K. Deng, Exponential decay of solutions of semilinear parabolic equations with nonlocal initial conditions, *J. Math. Anal. Appl.* 179:2 (1993), 630–637. MR1249842. Zbl 0798.35076.

[14] K. Ezzinbi, M. Ziat, Nonlocal Integro-Differential Equations Without the Assumption of Equicontinuity on the Resolvent Operator in Banach Spaces. *Differ Equ Dyn Syst* 30, 315–333 (2022). MR4402120. Zbl 1486.35406.

[15] L. Guedda, A. Hallouz, An existence result for neutral functional differential inclusions in a Banach space. *Electron. J. Qual. Theory Differ. Equ.*. No. 9, 1-23, (2008). MR2385413. Zbl 1183.34104.

[16] J. K. Hale, S. Verduyn Lunel, *Introduction to Functional-Differential Equations*, Appl. Math. Sci., 99, Springer-Verlag, New York, 1993. MR1243878. Zbl 0787.34002.

[17] A. Jaiswal, D. Bahuguna, Semilinear Conformable Fractional Differential Equations in Banach Spaces. *Differ. Equ. Dyn. Syst.* 27, 313–325 (2019). MR3916235. Zbl 1416.34002.

[18] M. Kamenskii, V. Obukhovskii and P. Zecca, *Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces*, De Gruyter, Berlin, 2001. MR1831201. Zbl 0988.34001.

[19] R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh, A new definition of fractional derivative, *J. Comput. Appl. Math.* 264 (2014), 65-70. MR3164103. Zbl 1297.26013.

[20] R. Martin, *Nonlinear Operators and Differential Equations in Banach Spaces*, Wiley, New York. 1976. MR0492671. Zbl 0333.47023.

[21] H. Mönch, Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces. *Nolinear Anal.* 4, 985–999. (1980). MR0586861. Zbl 0462.34041.

[22] J. Rosales-García, J. A. Andrade-Lucio, O. Shulika, Conformable derivative applied to experimental Newton's law of cooling, *Rev. Mexicana de Fís.*, 66 (2020), 224-227. MR4086978.

[23] I.I. Vrabie, $C_0$-semigroups and applications, North-Holland Mathematics Studies 191, North- Holland Publishing Co., Amsterdam, 2003. MR1972224. Zbl 1119.47044.

[24] J. Wu, *Theory and Applications of Partial Functional Differential Equations*. Springer, Berlin (1996). MR1415838. Zbl 0870.35116.

[25] D. Zhao, M. Luo, General conformable fractional derivative and its physical interpretation, *Calcolo* 54:3 (2017), 903–917. MR3694729. Zbl 1375.26020.

[26] H. Zhou, S. Yang, S. Zhang, Conformable derivative approach to anomalous diffusion. *Phys. A.* 2018, 491, 1001–1013. MR3721559. Zbl 07547549.

[27] W. Zhong, L. Wang, Basic theory of initial value problems of conformable fractional differential equations. *Adv. Difference Equ.* 2018, 321:14. MR3853689. Zbl 1448.34033.

[28] S.Yang, H. W. Zhou, T. T. Gao and L. P. Wang, Conformable Advection-Diffusion Equation for Radionuclide Migration in Porous Media, *ISRM International Symposium-EUROCK 2020*. International Society for Rock Mechanics and Rock Engineering, 2020.

Abderrahmane Boukenkoul
Département d'Agronomie, Faculté des sciences de la vie et de l'environnement,
Université Abou Bekr Belkaid, Tlemcen 13000, Algérie.
e-mail: boukenkoula@gmail.com

Mohamed Ziane
Department of Mathematics,
Ibn Khaldoun University, Tiaret. Algeria.
e-mail: mohamed.ziane@univ-tiaret.dz

## License