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AN APPLICATION OF THE PASCAL DISTRIBUTION SERIES FOR A CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS

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Abstract. In the present paper, we determine necessary and sufficient conditions for the subclass $\mathcal{T}(\alpha, b)$ of analytic functions associated with Pascal distribution. Further, we consider properties of a special function related to Pascal distribution series. Several corollaries and consequences of the main results are also considered.

1 Introduction

Let $\mathcal{A}$ denote the class of the normalized functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Further, let $\mathcal{T}$ be a subclass of $\mathcal{A}$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \qquad (z \in \mathbb{U}). \tag{1.2}$$

Let $\mathcal{T}(\alpha, b)$ denote the class of functions $f(z) \in \mathcal{T}$ which satisfy the condition

$$\operatorname{Re}\left\{ f'(z) + \alpha z f''(z) \right\} > 1 - |b| \tag{1.3}$$

for some $\alpha(\alpha \geq 0)$ and $b \in \mathbb{C}$, and for all $z \in \mathbb{U}$. The class $\mathcal{T}(\alpha, b)$ was introduced and studied by Altintaş and Ertekin [2](see also, [9]), further the class $\mathcal{T}(\alpha, 1-\beta)$, $0 \leq \beta < 1$ was introduced by Altintaş [1] and for $\alpha = 0$, the class $\mathcal{T}(0, 1-\beta) = \mathcal{R}(\beta)$, where the functions in $\mathcal{R}(\beta)$ are called functions of bounded turning (see [16]).

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}^{\tau}(A, B)$,$\tau \in \mathbb{C}\backslash\{0\}$, $-1 \leq B < A \leq 1$, if it satisfies the inequality

$$\left| \frac{f'(z) - 1}{(A - B)\tau - B[f'(z) - 1]} \right| < 1, \quad z \in \mathbb{U}.$$

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This class was introduced by Dixit and Pal [8].

A variable X is said to be Pascal distribution if it takes the values $0,1,2,3,\ldots$, with probabilities
$(1-q)^m, \frac{qm(1-q)^m}{1!}, \frac{q^2m(m+1)(1-q)^m}{2!}, \frac{q^3m(m+1)(m+2)(1-q)^m}{3!}, \ldots$, respectively, where q and m are called the parameters, and thus

$$P(X=k)=\binom{k+m-1}{m-1}q^k(1-q)^m, k=0,1,2,3,\ldots.$$

Very recently, El-Deeb et al. [7] (see also, [14, 4]) introduced a power series whose coefficients are probabilities of Pascal distribution, that is

$$\Psi_q^m(z):=z+\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}(1-q)^m z^n,\ z\in\mathbb{U},$$

where $m\geq 1$, $0\leq q\leq 1$, and we note that, by ratio test the radius of convergence of above series is infinity. We also define the series

$$\Phi_q^m(z):=2z-\Psi_q^m(z)=z-\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}(1-q)^m z^n,\ z\in\mathbb{U}. \tag{1.4}$$

Let consider the linear operator $\mathcal{I}_q^m:\mathcal{A}\to\mathcal{A}$ defined by the convolution or Hadamard product

$$\mathcal{I}_q^m f(z):=\Psi_q^m(z)*f(z)=z+\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}(1-q)^m a_n z^n,\ z\in\mathbb{U},$$

where $m\geq 1$ and $0\leq q\leq 1$.

Motivated by several earlier results on connections between various subclasses of analytic and univalent functions, by using hypergeometric functions (see for example, [6, 15, 17, 24, 25]) and by using various distributions such as Yule-Simon distribution, Logarithmic distribution, Poisson distribution, Binomial distribution, Beta-Binomial distribution, Zeta distribution, Geometric distribution and Bernoulli distribution (see for example, [3], [10]-[13], [18]- [21]), in this paper, we determine the necessary and sufficient condition for Φ_q^m to be in the class $\mathcal{T}(\alpha,b)$. Furthermore, we give sufficient conditions for $\mathcal{I}_q^m(\mathcal{R}^\tau(A,B))\subset\mathcal{T}(\alpha,b)$ and finally, we give sufficient condition for the function f such that its image by the integral operator $\mathcal{G}_q^m f(z)=\int_0^z\frac{\Phi_q^m(t)}{t}dt$ belongs to the class $\mathcal{T}(\alpha,b)$.

2 Preliminary lemmas

To establish our main results, we need the following Lemmas.

Lemma 1. *[2] A function $f \in \mathcal{T}$ of the form (1.2) is in the class $\mathcal{T}(\alpha, b)$ if and only if*

$$\sum_{n=2}^{\infty} n(1-\alpha+n\alpha)\left|a_n\right| \leq |b| \quad . \tag{2.1}$$

The result (2.1) is sharp.

Lemma 2. *[8] If $f \in \mathcal{R}^{\tau}(A, B)$ is of the form (1.1) , then*

$$|a_n| \leq (A-B)\frac{|\tau|}{n}, \qquad n \in \mathbb{N}-\{1\}.$$

The result is sharp for the function

$$f(z)=\int_0^z (1+(A-B)\frac{\tau t^{n-1}}{1+Bt^{n-1}})dt, \qquad (z \in \mathbb{U}; n \in \mathbb{N}-\{1\}).$$

3 Necessary and sufficient condition for $\Phi_q^m \in \mathcal{T}(\alpha, b)$

For convenience throughout in the sequel, we use the following identities that hold for $m \geq 1$ and $0 \leq q < 1$:

$$\sum_{n=0}^{\infty}\binom{n+m-1}{m-1}q^n = \frac{1}{(1-q)^m},$$
$$\sum_{n=0}^{\infty}\binom{n+m}{m}q^n = \frac{1}{(1-q)^{m+1}}, \qquad \sum_{n=0}^{\infty}\binom{n+m+1}{m+1}q^n = \frac{1}{(1-q)^{m+2}}.$$

By simple calculations we derive the following relations:

$$\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1} = \sum_{n=0}^{\infty}\binom{n+m-1}{m-1}q^n - 1 = \frac{1}{(1-q)^m} - 1, \tag{3.1}$$

$$\sum_{n=2}^{\infty}(n-1)\binom{n+m-2}{m-1}q^{n-1} = qm\sum_{n=0}^{\infty}\binom{n+m}{m}q^n = \frac{qm}{(1-q)^{m+1}}, \tag{3.2}$$

$$\begin{aligned}
&\sum_{n=3}^{\infty}(n-1)(n-2)\binom{n+m-2}{m-1}q^{n-1} \\
= \;& q^2 m(m+1)\sum_{n=0}^{\infty}\binom{n+m+1}{m+1}q^n \\
= \;& \frac{q^2 m(m+1)}{(1-q)^{m+2}}.
\end{aligned} \tag{3.3}$$

Unless otherwise mentioned, we shall assume in this paper that $\alpha (\alpha \geq 0)$ and $b \in \mathbb{C}$, while $0 \leq q < 1$.

Firstly, we obtain the necessary and sufficient conditions for Φ_q^m to be in the class $\mathcal{T}(\alpha, b)$.

Theorem 3. *Let $m \geq 1$. Then $\Phi_q^m \in \mathcal{T}(\alpha, b)$, if and only if*

$$\alpha \frac{q^2 m(m+1)}{(1-q)^2} + (1+2\alpha) \frac{qm}{(1-q)} + (1-(1-q)^m) \leq |b| \,. \tag{3.4}$$

Proof. Since Φ_q^m is defined by (1.4), in view of Lemma 1 it is sufficient to show that

$$\sum_{n=2}^{\infty} n(1-\alpha+n\alpha) \binom{n+m-2}{m-1} q^{n-1}(1-q)^m \leq |b| \,. \tag{3.5}$$

Writing

$$n = (n-1)+1$$

and

$$n^2 = (n-1)(n-2) + 3(n-1) + 1$$

in (3.5), we have

$$\begin{aligned} & \sum_{n=2}^{\infty} n(1-\alpha+n\alpha) \binom{n+m-2}{m-1} q^{n-1}(1-q)^m \\ = \; & \alpha \sum_{n=3}^{\infty} (n-1)(n-2) \binom{n+m-2}{m-1} q^{n-1}(1-q)^m \\ & + (1+2\alpha) \sum_{n=2}^{\infty} (n-1) \binom{n+m-2}{m-1} q^{n-1}(1-q)^m \\ & + \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} q^{n-1}(1-q)^m \\ = \; & \alpha \frac{q^2 m(m+1)}{(1-q)^2} + (1+2\alpha) \frac{qm}{(1-q)} + (1-(1-q)^m). \end{aligned}$$

but this last expression is upper bounded by $|b|$ if and only if (3.4) holds. □

4 Sufficient condition for $\mathcal{I}_q^m (\mathcal{R}^\tau(A,B)) \subset \mathcal{T}(\alpha, b)$

Making use of Lemma 2, we will study the action of the Pascal distribution series on the class $\mathcal{T}(\alpha, b)$.

Theorem 4. *Let $m \geq 1$. If $f \in \mathcal{R}^{\tau}(A,B)$ and the inequality*

$$(A-B)\left|\tau\right|\left[\alpha\frac{qm}{(1-q)}+(1-(1-q)^{m})\right] \leq |b|\,. \tag{4.1}$$

is satisfied then $\mathcal{I}_q^m f \in \mathcal{T}(\alpha,b)$.

Proof. According to Lemma 1 it is sufficient to show that

$$\sum_{n=2}^{\infty} n(1-\alpha+n\alpha)\binom{n+m-2}{m-1}q^{n-1}(1-q)^{m}\left|a_n\right| \leq |b|\,.$$

Since $f \in \mathcal{R}^{\tau}(A,B)$, using Lemma 2 we have

$$|a_n| \leq \frac{(A-B)\left|\tau\right|}{n},\ n \in \mathbb{N}\setminus\{1\},$$

therefore

$$\begin{aligned}
&\sum_{n=2}^{\infty} n(1-\alpha+n\alpha)\binom{n+m-2}{m-1}q^{n-1}(1-q)^{m}\left|a_n\right| \\
\leq\ & (A-B)\left|\tau\right|\left[\sum_{n=2}^{\infty}(1-\alpha+n\alpha)\binom{n+m-2}{m-1}q^{n-1}(1-q)^{m}\right] \\
=\ & (A-B)\left|\tau\right|\left[\alpha\sum_{n=2}^{\infty}(n-1)\binom{n+m-2}{m-1}q^{n-1}(1-q)^{m}\right. \\
& \left.+\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}(1-q)^{m}\right] \\
=\ & (A-B)\left|\tau\right|\left[\alpha\frac{qm}{(1-q)}+(1-(1-q)^{m})\right].
\end{aligned}$$

but this last expression is upper bounded by $|b|$ if and only if (4.1) holds. □

5 Properties of a special function

Theorem 5. *Let $m \geq 1$. If the function $\mathcal{G}_q^m$ is given by*

$$\mathcal{G}_q^m(z) := \int_0^z \frac{\Phi_q^m(t)}{t}dt,\ z \in \mathbb{U}, \tag{5.1}$$

then $\mathcal{G}_q^m \in \mathcal{T}(\alpha,b)$ if and only if

$$\alpha\frac{qm}{(1-q)}+(1-(1-q)^{m}) \leq |b|\,. \tag{5.2}$$

Proof. According to (1.4) it follows that

$$\mathcal{G}_q^m(z) = z - \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} q^{n-1}(1-q)^m \frac{z^n}{n}, \ z \in \mathbb{U}.$$

Using Lemma 1, the function $\mathcal{G}_q^m(z)$ belongs to $\mathcal{T}(\alpha, b)$ if and only if

$$\sum_{n=2}^{\infty} n(1-\alpha+n\alpha)\frac{1}{n}\binom{n+m-2}{m-1}q^{n-1}(1-q)^m \le |b|\,.$$

By a similar proof like those of Theorem 4 we get that $\mathcal{G}_q^m f \in \mathcal{T}(\alpha, b)$ if and only if (5.2) holds. □

6 Corollaries and consequences

If we take $b = 1-\beta, 0 \le \beta < 1$, in Theorems 3-5, we obtain the following corollaries.

Corollary 6. *Let $m \ge 1$. Then $\Phi_q^m \in \mathcal{T}(\alpha, 1-\beta)$, if and only if*

$$\alpha\frac{q^2m(m+1)}{(1-q)^2} + (1+2\alpha)\frac{qm}{(1-q)} + (1-(1-q)^m) \le 1-\beta. \tag{6.1}$$

Corollary 7. *Let $m \ge 1$. If $f \in \mathcal{R}^\tau(A, B)$ and the inequality*

$$(A-B)\,|\tau|\left[\alpha\frac{qm}{(1-q)} + (1-(1-q)^m)\right] \le 1-\beta. \tag{6.2}$$

is satisfied then $\mathcal{I}_q^m f \in \mathcal{T}(\alpha, 1-\beta)$.

Corollary 8. *Let $m \ge 1$.The function $\mathcal{G}_q^m \in \mathcal{T}(\alpha, 1-\beta)$ if and only if*

$$\alpha\frac{qm}{(1-q)} + (1-(1-q)^m) \le 1-\beta. \tag{6.3}$$

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