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POSITIVE COHEN $p-$NUCLEAR $m-$HOMOGENEOUS POLYNOMIALS

Asma Hammou, Amar Belacel, Amar Bougoutaia and Abdelmoumen Tiaiba

Abstract. In this paper we introduce the concept of positive Cohen $p-$nuclear polynomials between Banach lattice spaces. We give an analogue to Pietsch domination theorem and we study some properties concerning this notion.

1 Introduction

The class of Cohen $p-$nuclear linear operators, $1 < p < \infty$ on Banach spaces was introduced and studied by Cohen in [7] in 1973. This class of linear operators was extended to the classes of Cohen $p-$nuclear multi-linear operators and homogeneous polynomials on Banach spaces by D. Achour, A. Alouani, P. Rueda and E. A. S Pérez in [2].

On other hand, in 2021, the paper by A. Bougoutaia and A. Belacel was another extension in this line of thought on the case linear this notion. Then they introduced positive Cohen $p-$nuclear operators.

The main goal of this paper is to tackle the problem of generalizing from the linear to the polynomial, therefore we introduce and study the concept of positive Cohen $p-$nuclear $m-$homogeneous polynomials for $1 < p < \infty$ between Banach lattice spaces, which extending the definition given in [5] for operators linear, and we show, for instance, that a polynomial is positive Cohen $p-$nuclear if and only if its associated symmetric multilinear mapping is a positive Cohen $p-$ nuclear.

The paper is designed as follows:

In the first section, we recall basic definitions and some important results to be used later.

In the section 2, we present the notion of positive Cohen $p-$nuclear polynomials which generalization of positive Cohen $p-$nuclear linear operators to the polynomial version. Also, we prove that Pietsch domination theorem for positive. Cohen $p-$nuclear polynomials and some interesting properties.

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In Section 3, we give a extension of Kwapien's theorem for the class of positive Cohen $p-$nuclear $m-$homogeneous polynomials. In the final section, we study some relations between our class and the different classes.

2 Notation and background

In the following, we provide a brief review of the notation and terminology that will be relevant to this work, for Banach spaces X and Y over the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), we denote $\mathcal{L}(X,Y)$ to the Banach space of all linear and continuous applications from X into Y.

Let m be a natural number. A map $P : X \to Y$ is an $m-$homogeneous polynomial if there exists a unique symmetric $m-$linear operator $\widehat{P} : X\times\cdots\times X \to Y$ such that

$$P(x) = \widehat{P}(x, \overset{(m)}{\ldots}, x) \text{ for every } \ x \in X.$$

Both are related by the polarization formula

$$\widehat{P}\left(x^1,...,x^m\right) = \frac{1}{m!2^m} \sum_{\epsilon_1,...,\epsilon_m=\pm 1} \epsilon_1 \cdots \epsilon_m P\left(\sum_{j=1}^{m} \epsilon_j x^j\right) \tag{2.1}$$

We denote by $\mathcal{P}(^m X, Y)$ the Banach space of all continuous $m-$homogeneous polynomials of degree m from X in Y with the norm.

$$\begin{aligned} \|P\| &= \sup\{\|P(x)\| : \|x\| \leq 1\} \\ &= \inf\{C : \|P(x)\| \leq C\|x\|^m, x \in X\} \end{aligned}$$

It is known that

$$\|P\| \leq \left\|\widehat{P}\right\| \leq \frac{m^m}{m!} \|P\|. \tag{2.2}$$

We need to recall the adjoint of a continuous homogeneous polynomial has been an important tool in infinite dimensional holomorphy. Given a continuous m-homogeneous polynomial $P \in \mathcal{P}(^m X, Y)$, the adjoint of P is the following continuous linear operator:

$$P^* : Y^* \to \mathcal{P}(^m X), \ P^*(y^*)(x) := y^*(P(x))$$

It is clear that $\|P^*\| = \|P\|$.

Throughout, E and F denote Banach lattices, $E^+ = \{x \in E, x \geq 0\}$ is the positive cone of E and E^* denotes the topological dual of E. An m-homogeneous polynomial operator $P : E \to F$ is said to be:

- positive, if its associated symmetric m-linear operator $\widehat{P}$ is positive that is, if $\widehat{P}(x_1,...,x_m) \in F^+$ whenever that $x_i \in E^+$ for all $i = 1,...,m$.

- regular, if P can be written as the difference of two positive m-homogeneous polynomials, $\mathcal{P}^r(^mE,F)$ denote the space of all regular m-homogeneous polynomials from E to F, which becomes a Banach lattice with the regular norm $\|P\|_r = \||P|\|$ if F is Dedekind complete [6].

- lattice polymorphism, if its associated symmetric m-linear operator $\widehat{P}$ is a lattice m-morphism that is, if $|\widehat{P}(x_1,...,x_m)| = \widehat{P}(|x_1|,...,|x_m|)$ for every $x_1,...,x_m \in E$.

For a Banach lattice E, the positive projective symmetric tensor norm on $\bar{\otimes}_s^m E$ is defined by

$$\|u\|_{s,|\pi|} = \inf\left\{\sum_{i=1}^{n}\|x_i\|^m : \quad x_i \in E^+,\ |u| \leq \sum_{i=1}^{n} x_i \otimes \cdots \otimes x_i\right\}$$

for each $u \in \bar{\otimes}_s^m E$. Then $\|\cdot\|_{s,|\pi|}$ is a lattice norm on $\bar{\otimes}_s^m E$. Let $\widehat{\otimes}_{s,|\pi|}^m E$ denote the completion of $\bar{\otimes}_s^m E$ under the lattice norm $\|\cdot\|_{s,|\pi|}$. Then $\widehat{\otimes}_{s,|\pi|}^m E$ is a Banach lattice, called the m-fold positive projective symmetric tensor product of E. Moreover, if F is Dedekind complete Banach lattice then for any regular m-homogeneous polynomial $P : E \to F$, there exists a unique regular linear operator $P^{\otimes} : \widehat{\otimes}_{s,|\pi|}^m E \to F$, called the linearization of P, such that the following diagram is commutative

$$\begin{tikzcd}
E \arrow[rr, "P"] \arrow[dr, "\otimes_m"'] & & F \\
& \widehat{\otimes}_{s,|\pi|}^m E \arrow[ur, "P^{\otimes}"'] &
\end{tikzcd}$$

that is,

$$P(x) = P^{\otimes}(x \otimes \cdots \otimes x)$$

for all $x \in E$. Moreover, the correspondence $P \mapsto P^{\otimes}$ is isometrically isomorphic and lattice homomorphic between the Banach lattices $\mathcal{P}^r(^mE,F)$ and $\mathcal{L}^r(\widehat{\otimes}_{s,|\pi|}^m E, F)$. For more details see [6, 8]

We continue to provide many standard notations. Let $\ell_p^n(X)$ be the Banach space of all absolutely $p-$summable sequences $(x_i)_{i=1}^n$ in X with the norm

$$\begin{cases} \|(x_i)_{i=1}^n\|_p = \left(\sum_{i=1}^{n}\|x_i\|_X^p\right)^{\frac{1}{p}}, & \text{if} \quad 1 \leq p < +\infty, \\ \|(x_i)_{i=1}^n\|_\infty = \sup_i \|x_i\|_X, & \text{if} \quad p = +\infty \end{cases}$$

We denote by $\ell_{p,w}^n(X)$ the Banach space of all weakly $p-$summable sequences

$(x_i)_{i=1}^n$ in X with the norm

$$\begin{cases} \|(x_i)_{i=1}^n\|_{\ell_{p,weak}^n} = \sup_{x^*\in B_{X^*}} \left(\sum_{i=1}^n |\langle x_i, x^*\rangle|^p\right)^{\frac{1}{p}}, & \text{if} \quad 1\le p<+\infty, \\ \|(x_i)_{i=1}^n\|_{\ell_{\infty,weak}^n} = \sup_{x^*\in B_{X^*}} \sup_i |\langle x_i, x^*\rangle|, & \text{if} \quad p=+\infty \end{cases}$$

Consider the case where X is replaced by a Banach lattice E, and define

$$\ell_{p,|weak|}^n(E) := \left\{(x_i)_{i=1}^n : (|x_i|)_{i=1}^n \in \ell_{p,weak}^n(E)\right\}$$

Moreover, if $B_{E^*}^+ = B_{E^*}\cap E^{*+} = \{\xi \in B_{E^*}, \xi\ge 0\}$, we have

$$\begin{cases} \|(x_i)_{i=1}^n\|_{\ell_{p,|weak|}^n(E)} = \sup_{\xi\in B_{E^*}^+} \left(\sum_{i=1}^n))\langle |x_i|, \xi\rangle^p\right)^{\frac{1}{p}}, & \text{if} \quad 1\le p<+\infty, \\ \|(x_i)_{i=1}^n\|_{\ell_{p,|weak|}^n(E)} = \sup_{\xi\in B_{E^*}^+} \sup_{1\le i\le n} \langle |x_i|, \xi\rangle, & \text{if} \quad p=+\infty \end{cases}$$

Before we start, let us recall some fundamental definitions that will be employed throughout this paper.

Suppose that $1\le p\le\infty$ and that $u: E\to F$ a linear operator between Banach lattices. We say that u is positive Cohen $p-$nuclear if there is a positive constant C such that for all $(x_i)_{i=1}^n \subset E$ and any $(y_i^*)_{i=1}^n\subset F^*$, we have

$$\sum_{i=1}^n |\langle u(x_i), y_i^*\rangle| \le C \sup_{x^*\in B_{E^*+}} \left(\sum_{i=1}^n \langle |x_i|, x^*\rangle^p\right)^{\frac{1}{p}} \|(y_i^*)_{i=1}^n\|_{\ell_{p^*,|weak|}^n(F^*)}. \tag{2.3}$$

The least constant C for which this inequality 2.3 holds is denoted by $n_p^+(u)$. We use $\mathcal{N}_p^+(E,F)$ to denote the set of all positive Cohen $p-$nuclear operators from E into F.

Let X be a Banach space and F be a Banach lattice. We say that a polynomial $P: X\to F$ is called positive Cohen strongly $p-$summing $(1\le p\le\infty)$ if there is a constant $C>0$ such that for any $(x_i)_{i=1}^n\subset X$ and any $(y_i^*)_{i=1}^n\subset F^*$, we have

$$\sum_{i=1}^n |\langle P(x_i), y_i^*\rangle| \le C \left(\sum_{i=1}^n \|x_i\|^{mp}\right)^{\frac{1}{p}} \|(y_i^*)_{i=1}^n\|_{\ell_{p^*,|weak|}^n(F^*)}. \tag{2.4}$$

The least constant C for which this inequality (2.4) holds is denoted by $d_p^{m+}(u)$. We shall write $\mathcal{P}_{coh^+-p}({}^mX,F)$ for the set of all positive Cohen strongly $p-$summing $m-$homogeneous polynomials from X into F. For $m=1$, it is the space of positive strongly p-summing linear operators $(\mathcal{P}_{coh^+-p}({}^1X,F) = \mathcal{D}_p^+(X,F))$.

3 Positive Cohen $p-$nuclear $m-$homogeneous polynomials

In the linear case, the notion of positive Cohen $p-$nuclear operators has been studied by Bougoutaia and Belacel in their work [5]. In this section, we present the definition of positive Cohen $p-$nuclear $m-$homogeneous polynomials.

Definition 1. *We say that an $m-$homogeneous polynomial $P : E \to F$ is positive Cohen $p-$nuclear, $1 < p < \infty$, if there is a constant $C > 0$ such that*

$$\sum_{i=1}^{n} |\langle P(x_i), y_i^* \rangle| \leq C \sup_{x^* \in B_{E^*}^+} \left(\sum_{i=1}^{n} \langle |x_i|, x^* \rangle^{mp} \right)^{\frac{1}{p}} \|(y_i^*)_{i=1}^n\|_{\ell^n_{p^*,|weak|}(F^*)} \tag{3.1}$$

for every $(x_i)_{i=1}^n \subset E$ and $(y_i^*)_{i=1}^n \subset F^*$.

The class of all positive Cohen $p-$nuclear $m-$homogeneous polynomials from E into F, is denoted by $\mathcal{P}_{N-p}^{c^+}(^mE, F)$. Our space is a Banach space with the norm $n_p^{m+}(.)$, which is the smallest constant C such that the inequality (3.1) holds. For $p = \infty$, we have $\mathcal{P}_{N-\infty}^{c^+}(^mE, F) = \mathcal{P}_{coh^+-\infty}(^mE, F)$.

Below, we provide an example of a positive Cohen $p-$nuclear polynomial:

Example 2. Let $1 < p \leq \infty$ and let $u : E \to F$ be an positive Cohen $p-$nuclear operator, where $E,\ F$ are two Banach lattices. For $\varphi \in E^*$ the mapping

$$\begin{aligned} P :& E \longrightarrow F \\ & x \longmapsto P(x) = \varphi^{m-1}(x) u(x), \end{aligned}$$

is positive Cohen $p-$nuclear polynomial, moreover $n_p^{m+}(P) \leq n_p^+(u)\|\varphi\|^{m-1}$. Indeed, for $x_1, ..., x_n \in E$ and $y_1^*, ..., y_n^* \in F^*$, we have :

$$\begin{aligned} \sum_{i=1}^{n} |\langle P(x_i), y_i{}^* \rangle| &= \sum_{i=1}^{n} \left| \left\langle \varphi^{m-1}(x_i) u(x_i), y_i{}^* \right\rangle \right| \\ &= \sum_{i=1}^{n} \left| \left\langle u(\varphi^{m-1}(x_i)(x_i)), y_i{}^* \right\rangle \right| \\ &\leq n_p^+(u) \sup_{x^* \in B_{E^*}^+} \left(\sum_{i=1}^{n} \langle |\varphi^{m-1}(x_i)| |x_i|, x^* \rangle^{mp} \right)^{\frac{1}{p}} \sup_{y^{**} \in B_{F^{**}}^+} \left(\sum_{i=1}^{n} \langle |y_i^*|, y^{**} \rangle^{p^*} \right)^{\frac{1}{p^*}} \\ &\leq n_p^+(u) \sup_{x^* \in B_{E^*}^+} \left(\sum_{i=1}^{n} |\varphi^{m-1}(x_i)| x^*(|x_i|)^{mp} \right)^{\frac{1}{p}} \sup_{y^{**} \in B_{F^{**}}^+} \left(\sum_{i=1}^{n} \langle |y_i^*|, y^{**} \rangle^{p^*} \right)^{\frac{1}{p^*}} \\ &\leq n_p^+(u) \|\varphi\|^{m-1} \sup_{x^* \in B_{E^*}^+} \left(\sum_{i=1}^{n} \langle |x_i|, x^* \rangle^{mp} \right)^{\frac{1}{p}} \sup_{y^{**} \in B_{F^{**}}^+} \left(\sum_{i=1}^{n} \langle |y_i^*|, y^{**} \rangle^{p^*} \right)^{\frac{1}{p^*}} \end{aligned}$$

Then, P is positive Cohen $p-$nuclear, moreover $n_p^{m+}(P) \leq n_p^{+}(u)\|\varphi\|^{m-1}$.

Proposition 3. *Let $P \in \mathcal{P}(^mE, F)$ and let $v : \ell_p^n \to F^*$ be positive linear operator. Then the polynomial P is positive Cohen $p-$nuclear if*

$$\sum_{i=1}^{n} |\langle P(x_i), v(e_i)\rangle| \leq C \left\|(x_i)\right\|^m_{\ell^n_{p,|weak|}} \|v\| . \tag{3.2}$$

Proof. Let $v : \ell_p^n \to F^*$ be positive linear operator such that

$$v = \sum_{i=1}^{n} e_i \otimes y_i^*$$

where e_i is the canonical base of $\ell_{p^*}^n$. Since there is an isometric between the spaces $\ell^n_{p^*,|weak|}(F^*)$ and $\mathcal{L}^r(\ell_p^n, F^*)$, and $\|v\| = \|(y_i^*)\|_{\ell^n_{p^*,|weak|}}$ □

We present our main results. The following theorem investigates the relationship between positive Cohen $p-$nuclear $m-$homogeneous polynomials and $m-$linear positive Cohen $p-$nuclear operators.

Theorem 4. *Let P be an $m-$homogeneous polynomial between Banach lattices E and F. P is positive Cohen $p-$nuclear if, and only if, its associated symmetric $m-$linear operator $\widehat{P} \in \mathcal{L}(^mE; F)$ is positive Cohen $p-$nuclear, and*

$$n_p^{m+}(P) = n_p^{m+}(\widehat{P}).$$

Proof. Let P be positive Cohen $p-$nuclear $m-$homogeneous polynomial, and let $(x_i)_{i=1}^n \subset E$ such that $\|(x_i)_{i=1}^n\|_{mp,|weak|} \leq 1$ and $y_i^* \in F^*(1 \leq i \leq n)$

$$\left\|\left(\epsilon_1 x_i^1 + \cdots + \epsilon_m x_i^m\right)_{i=1}^n\right\|_{mp,|weak|} \leq \left\|\left(\epsilon_1 x_i^1\right)_{i=1}^n\right\|_{mp,|weak|} + \cdots + \left\|\left(\epsilon_m x_i^m\right)_{i=1}^n\right\|_{mp,|weak|} \leq m$$

for every $\epsilon_1, \cdots, \epsilon_m = \pm 1$

Using the polarization formula 2.1, we obtain

$$\begin{aligned}\sum_{i=1}^{n}\left|\left\langle \widehat{P}(x_i^1,...,x_i^m),y_i^*\right\rangle\right| &= \sum_{i=1}^{n}\left|\left\langle \frac{1}{m!2^m}\sum_{\epsilon_1,...,\epsilon_m=\pm1}\epsilon_1...\epsilon_2 P\left(\sum_{j=1}^{m}\epsilon_j x_i^j\right),y_i^*\right\rangle\right| \\ &\le \frac{1}{m!2^m}\sum_{\epsilon_1,...,\epsilon_2=\pm1}\sum_{i=1}^{n}\left|\left\langle P\left(\sum_{j=1}^{m}\epsilon_j x_i^j\right),y_i^*\right\rangle\right| \\ &\le \frac{1}{m!2^m}n_p^{m+}(P)\sum_{\epsilon_1,...,\epsilon_2=\pm1}\left\|\sum_{i=1}^{n}\epsilon_1 x_i^j\right\|_{mp,|weak|}^m \|(y_i^*)_{i=1}^n\|_{p^*,|weak|} \\ &\le \frac{1}{m!2^m}n_p^{m+}(P)\left(\sum_{\epsilon_1,...,\epsilon_2=\pm1}m^m\right)\|(y_i^*)_{i=1}^n\|_{p^*,|weak|} \\ &\le \frac{\left(2^{\frac{1}{p}-1}m\right)^m}{m!}n_p^{m+}(P)\|(y_i^*)_{i=1}^n\|_{p^*,|weak|}\end{aligned}$$

and for $(x_i^j)_{i=1}^n \in \ell_{p,|weak|}^n(E)$ with $x_i^j \neq 0$ and $j=1,...,m$,

$$\sum_{i=1}^{n}\left|\left\langle \widehat{P}\left(\frac{x_i^1}{\|(x_i^1)_{i=1}^n\|_{mp,|weak|}},...,\frac{x_i^m}{\|(x_i^m)_{i=1}^n\|_{mp,|weak|}}\right),y_i^*\right\rangle\right| \le \frac{\left(2^{\frac{1}{p}-1}m\right)^m}{m!}n_p^{m+}(P)\|(y_i^*)_{i=1}^n\|_{p^*,|weak|}.$$

Then

$$\sum_{i=1}^{n}\left|\left\langle \widehat{P}(x_i^1,...,x_i^m),y_i^*\right\rangle\right| \le \frac{\left(2^{\frac{1}{p}-1}m\right)^m}{m!}n_p^{m+}(P)\prod_{j=1}^{m}\left\|(x_i^j)_{i=1}^n\right\|_{mp,|weak|}\|(y_i^*)_{i=1}^n\|_{p^*,|weak|}$$

At last, $\widehat{P}$ is Cohen $p-$nuclear and

$$n_p^{m+}(\widehat{P}) \le \frac{\left(2^{\frac{1}{p}-1}m\right)^m}{m!}n_p^{m+}(P) \le \frac{m^m}{m!}n_p^{m+}(P).$$

Conversely. Suppose that $\widehat{P}$ is Cohen $p-$nuclear $m-$linear operator by definition we have: For all $x_1, ..., x_n \in E$ and $y_1^*, ..., y_n^* \in F^*$ then

$$\begin{aligned}\sum_{i=1}^{n} |\langle P(x_i), y_i^*\rangle| &= \sum_{i=1}^{n} \left|\left\langle \widehat{P}(x_i, \cdots, x_i), y_i^*\right\rangle\right| \\ &\leq n_p^{m+}(\widehat{P}) \sup_{x^* \in B_{E^*}^+} \left(\sum_{i=1}^{n} \prod^{m} \langle |x_i|, x^*\rangle^p\right)^{\frac{1}{p}} \|(y_i^*)_{i=1}^n\|_{p^*,|weak|} \\ &\leq n_p^{m+}(\widehat{P}) \sup_{x^* \in B_{E^*}^+} \left(\sum_{i=1}^{n} \langle |x_i|, x^*\rangle^{mp}\right)^{\frac{1}{p}} \|(y_i^*)_{i=1}^n\|_{p^*,|weak|}\end{aligned}$$

This implies that P is positive Cohen $p-$nuclear polynomial, and $n_p^{m+}(P) \leq n_p^{m+}(\widehat{P})$.
□

We can obtained the next result as an application of Theorem 4 and [5, Proposition 2.3].

Corollary 5. *Let $P \in \mathcal{P}(^m E, F)$, then P is positive Cohen $p-$nuclear $m-$homogeneous polynomial if, and only if, there is a constant $C > 0$, for all $(x_i)_{i=1}^n \subset E^+$ and $(y_i^*)_{i=1}^n \subset F^{*+}$, we have*

$$\sum_{i=1}^{n} |\langle P(x_i), y_i^*\rangle| \leq C \sup_{x^* \in B_{E^*}^+} \left(\sum_{i=1}^{n} \langle x_i, x^*\rangle^{mp}\right)^{\frac{1}{p}} \|(y_i^*)_{i=1}^n\|_{\ell_{p^*,weak}^n} \tag{3.3}$$

Our next result establishes that in the case of positive Cohen $p-$nuclear polynomials, the ideal property also holds.

Proposition 6. *Let $P \in \mathcal{P}(^m E, F)$, R an operator in $\mathcal{L}(F, G)$ and S a positive operator in $\mathcal{L}(G, E)$.*

1. *If P is positive Cohen $p-$nuclear, then $R \circ P$ is positive Cohen p-nuclear from E to G and $n_p^{m+}(R \circ P) \leq n_p^{m+}(P)\|R\|$.*

2. *If P is positive Cohen $p-$nuclear, then $P \circ S$ is positive Cohen p-nuclear from G to F and $n_p^{m+}(P \circ S) \leq n_p^{m+}(P)\|S\|^m$.*

Proof. Let $P \in \mathcal{P}^{c+}_{N-p}(^mE, F)$. Let $(x_i)_{i=1}^n \subset E$ and $(y_i^*)_{i=1}^n \subset F^{*+}$, we have

$$\begin{aligned}
\sum_{i=1}^{n} |\langle R \circ P(x_i), y_i^* \rangle| &= \sum_{i=1}^{n} |\langle P(x_i), R^*(y_i^*) \rangle| \\
&\leq n_p^{m+}(P) \sup_{x^* \in B^+_{E^*}} \left(\sum_{i=1}^{n} \langle |x_i|, x^* \rangle^{mp} \right)^{\frac{1}{p}} \sup_{y^{**} \in B^+_{F^{**}}} \left(\sum_{i=1}^{n} \left\langle R^*(y_i^*), y^{**} \right\rangle^{p^*} \right)^{\frac{1}{p^*}} \\
&\leq n_p^{m+}(P) \, \|R\| \sup_{x^* \in B^+_{E^*}} \left(\sum_{i=1}^{n} \langle |x_i|, x^* \rangle^{mp} \right)^{\frac{1}{p}} \sup_{y^{**} \in B^+_{F^{**}}} \left(\sum_{i=1}^{n} \left\langle y_i^*, \frac{R^{**}(y^{**})}{\|R^{**}\|} \right\rangle^{p^*} \right)^{\frac{1}{p^*}} \\
&\leq n_p^{m+}(P) \, \|R\| \sup_{x^* \in B^+_{E^*}} \left(\sum_{i=1}^{n} \langle |x_i|, x^* \rangle^{mp} \right)^{\frac{1}{p}} \sup_{\varphi \in B^+_{G^{**}}} \left(\sum_{i=1}^{n} \left\langle y_i^*, \varphi \right\rangle^{p^*} \right)^{\frac{1}{p^*}}
\end{aligned}$$

So, $R \circ P \in \mathcal{P}^{c+}_{N-p}(^mE, G)$ and $n_p^{m+}(R \circ P) \leq n_p^{m+}(P)\|R\|$.

$$\begin{aligned}
\sum_{i=1}^{n} |\langle P \circ S(x_i), y_i^* \rangle| &\leq n_p^{m+}(P) \sup_{x^* \in B^+_{E^*}} \left(\sum_{i=1}^{n} \langle (|Sx_i|), x^* \rangle^{mp} \right)^{\frac{1}{p}} \sup_{y^{**} \in B^+_{F^{**}}} \|y_i^*(y^{**})\|_{\ell^n_{p^*}} \\
&\leq n_p^{m+}(P) \sup_{x^* \in B^+_{E^*}} \left(\sum_{i=1}^{n} \langle S(|x_i|), x^* \rangle^{mp} \right)^{\frac{1}{p}} \sup_{y^{**} \in B^+_{F^{**}}} \|y_i^*(y^{**})\|_{\ell^n_{p^*}} \\
&\leq n_p^{m+}(P)\|S\|^m \sup_{x^* \in B^+_{E^*}} \left(\sum_{i=1}^{n} \left\langle |x_i|, \frac{S^*(x^*)}{\|S\|} \right\rangle^{mp} \right)^{\frac{1}{p}} \sup_{y^{**} \in B^+_{F^{**}}} \|y_i^*(y^{**})\|_{l^n_{p^*}} \\
&\leq n_p^{m+}(P)\|S\|^m \sup_{\varphi \in B^+_{G^*}} \left(\sum_{i=1}^{n} \langle |x_i|, \varphi \rangle^{mp} \right)^{\frac{1}{p}} \sup_{y^{**} \in B^+_{F^{**}}} \|y_i^*(y^{**})\|_{\ell^n_{p^*}}.
\end{aligned}$$

Which means that $P \circ S$ is positive Cohen $p-$nuclear $m-$homogeneous polynomial and

$$n_p^{m+}(P \circ S) \leq n_p^{m+}(P)\|S\|^m.$$

□

Next, we present a version of Pietsch's domination theorem, for positive Cohen $p-$nuclear polynomials. We can proved it by applying of Theorem 4 and [5, Theorem 2.6].

Theorem 7. *Let $1 < p < \infty$. The following are equivalent for a polynomial $P \in \mathcal{P}(^m E, F)$:*

(a) P is positive Cohen $p-$nuclear.

(b) There exists a positive constant $C > 0$ and regular Borel probability measures μ_1 on $B^+_{E^*}$, μ_2 on $B^+_{F^{**}}$ and $C > 0$ such that, for all $x \in E^+$ and $y^* \in F^{*+}$, we have

$$|\langle P(x), y^*\rangle| \leq C \left(\int_{B^+_{E^*}} \langle x, x^*\rangle^{mp} d\mu_1(x^*)\right)^{\frac{1}{p}} \left(\int_{B^+_{F^{**}}} \langle y^*, y^{**}\rangle^{p^*} d\mu_2(y^{**})\right)^{\frac{1}{p^*}}. \tag{3.4}$$

4 Kwapien's Factorization Theorem

The main objective of this section is to demonstrate that a polynomial P is positive Cohen $p-$nuclear if, and only if, it can be expressed in the form $P = Q \circ u$, where Q is a positive Cohen strongly $p-$summing $m-$homogeneous polynomial and u is a positive $mp-$summing operator.

In order to prove this result, we will need to utilize the following lemma, which is stated below.

Lemma 8. [3] *The operator $J_{p,0} \circ i_E : E \to i_E(E) \to L^p_0(\mu)$ is positive $p-$summing, and $\pi^+_p(J_{p,0} \circ i_E) \leq 1$, such that for all $x \in E$*

$$\|\langle x, .\rangle\|_{L^p_0(\mu)} = \|J_{p,0} \circ i_E\|_{L^p_0(\mu)}.$$

Where μ is a probability measure on the set $B^+_{E^*}$.

Theorem 9. *Let $1 < p < \infty$. A polynomial $P : E \to F$ is positive Cohen $p-$nuclear if, and only if, there exist Banach space G, positive $mp-$summing linear operator $u \in \mathcal{L}(E, G)$ and a Cohen positive strongly p-summing m-homogeneous polynomial $Q \in \mathcal{P}(^m G, F)$ such that $P = Q \circ u$, (i.e. $\mathcal{P}^{c+}_{N-p} = \mathcal{P}_{coh^+-p} \circ \prod^+_{mp}$, isometrically)*

$$n^{m+}_p(P) = \inf\left\{d^{m+}_p(Q)\left(\pi^+_{mp}(u)\right)^m : \quad P = Q \circ u\right\}$$

Proof. Suppose that $P \in \mathcal{P}^{c+}_{N-p}(^m E, F)$. Then, through Theorem 7, there exist Radon probability measures μ on $B^+_{E^*}$ and λ on $B^+_{F^{**}}$ such that for all $x \in E$ and $y^* \in F^{*+}$.

$$|\langle P(x), y^*\rangle| \leq C \,\||x|\|^m_{L_{mp}(B^+_{E^*},\mu)} \|y^*\|_{L_{p^*}(B^+_{F^{**}},\lambda)}$$

And we consider the diagram

$$\begin{tikzcd}
E \arrow[r, "P"] \arrow[d, "i_E"] & F \\
i_E(E) \arrow[r, "J_{mp,0}"] \arrow[d, hook] & L_0^{mp}(\mu) \arrow[u, "Q"'] \\
C(B_E^{*+}) &
\end{tikzcd}$$

Where $i_E : E \to C(B_{E^*}^+)$ is the canonical injection. If we denote the range of $J_{p,0} \circ i_E$ by G, and the closure of G by $L_0^{mp}(\mu)$, the map $G \to F : J_{mp,0} \circ i_E(x) \mapsto P(x)$ is well defined operator. So, we apply the Lemma 8, we find that the operators $u = J_{mp,0} \circ i_E : E \to i_E(E) \to L_0^{mp}(\mu)$ are positive $mp-$summing, and $\pi_{mp}^+(u) \leq 1$.

The polynomial Q is defined on $u(E)$, where $u(x) = (J_{mp,0} \circ i_E)(x)$, by

$$Q(u(x)) := P(x)$$

this definition makes sense because

$$|\langle Q(u(x)), y^* \rangle| \leq C \, \|u(x)\|_{L_{mp}(\mu)}^m \left(\int_{B_{F^{**}}^+} \langle y^*, y^{**} \rangle^{p^*} d\lambda(y^{**}) \right)^{\frac{1}{p^*}}.$$

Then, from [10, Theorem 1] we deduce Q is a Cohen positive strongly $p-$summing polynomial and $d_p^{m+}(Q) \leq n_p^{m+}(P)$.

$$|\langle Q(u(x)), y^* \rangle| \leq n_p^{m+}(P) \|u(x)\|_{L_{mp}(\mu)}^m \left(\int_{B_{F^{**}}^+} \langle y^*, y^{**} \rangle^{p^*} d\lambda(y^{**}) \right)^{\frac{1}{p^*}}.$$

Conversely, let $x \in E$ and $y^* \in F^{*+}$, we have

$$\begin{aligned}
|\langle Q(u(x)), y^* \rangle| &\leq d_p^{m+}(Q) \|u(x)\|^m \left(\int_{B_{F^{**}}^+} \langle y^*, y^{**} \rangle^{p^*} d\lambda(y^{**}) \right)^{\frac{1}{p^*}} \\
&\leq d_p^{m+}(Q) (\pi_{mp}^+(u))^m \left(\int_{B_{E^*}^+} \langle |x|, x^* \rangle^{mp} d\mu(x^*) \right)^{\frac{1}{p}} \left(\int_{B_{F^{**}}^+} \langle y^*, y^{**} \rangle^{p^*} d\lambda(y^{**}) \right)^{\frac{1}{p^*}}
\end{aligned}$$

This implies that $Q \circ u \in \mathcal{P}_{N-p}^+({}^mE, F)$ and $n_p^{m+}(Q \circ u) \leq d_p^{m+}(Q)(\pi_{mp}^+(u))^m$. $\square$

Theorem 9 yields the following proposition immediately.

Proposition 10. *For* $1 < p < \infty$.

$$\mathcal{P}_{N-p}^c({}^mE, F) \subseteq \mathcal{P}_{N-p}^{c+}({}^mE, F).$$

Proof. Suppose that $P \in \mathcal{P}^{c}_{N-p}(^{m}E,F)$. It is well-known [5, Proposition 3.4] that there exists Banach space G such that $P = Q \circ u$ where $Q \in \mathcal{P}_{coh^{+}-p}(^{m}G,F)$ and $u \in \Pi_{mp}(E,G)$. It follows from [10, Proposition 4] that $Q \in \mathcal{P}_{coh^{+}-p}(^{m}G,F)$ and it follows from [4, Proposition 3] that $u \in \Pi^{+}_{mp}(E,G)$. Theorem 9 clearly implies that $P \in \mathcal{P}^{c^{+}}_{N-p}(^{m}E,F)$. □

5 Relationships between some classes

Theorem 11. *Let $1 < p < \infty$. Let E and F be Banach lattices such that F is Dedekind complete. If $P \in \mathcal{P}^{r}(^{m}E,F)$ is positive Cohen $p-$nuclear polynomial, then their linearization operator $P^{\otimes}$ belongs to $\mathcal{N}^{+}_{p}(\widehat{\otimes}^{m}_{s,|\pi|}E,F)$.*

Proof. Let $P \in \mathcal{P}(^{m}E,F)$. By Theorem 7 we obtain that

$$|\langle P(x),y^{*}\rangle| \leq n^{m+}_{p}(P)\,\|x\|^{m}_{L_{mp}(B^{+}_{E^{*}},\mu)}\,\|y^{*}\|_{L_{p^{*}}(B^{+}_{F^{**}},\lambda)}$$

For all $x \in E^{+}$ and $y^{*} \in F^{*+}$, we take $z \in \widehat{\otimes}^{m}_{s,|\pi|}E$, for all $\epsilon > 0$, z admits a representation $z = \sum\limits_{i=1}^{n} x_i \otimes \overset{(m)}{\cdots} \otimes x_i$ such that

$$\sum_{i=1}^{n} \|x_i\|^{m}_{L_{mp}(B^{+}_{E^{*}},\mu)} \leq \|z\|_{L_{p}(B^{+}_{(\widehat{\otimes}^{m}_{s,|\pi|}E)^{*}},\widehat{\mu})} + \epsilon \tag{5.1}$$

where $\widehat{\mu} = \mu \otimes \overset{(m)}{\cdots} \otimes \mu$. Then

$$\begin{aligned}
\left|\left\langle P^{\otimes}(z),y^{*}\right\rangle\right| &= \left|\left\langle P^{\otimes}\left(\sum_{i=1}^{n} x_i \otimes \overset{(m)}{\cdots} \otimes x_i\right),y^{*}\right\rangle\right| \\
&= \left|\sum_{i=1}^{n}\left\langle P^{\otimes}(x_i \otimes \overset{(m)}{\cdots} \otimes x_i),y^{*}\right\rangle\right| \\
&\leq \sum_{i=1}^{n} |\langle P(x_i),y^{*}\rangle| \\
&\leq n^{m+}_{p}(P)\sum_{i=1}^{n}\|x_i\|^{m}_{L_{mp}(B^{+}_{E^{*}},\mu)}\,\|y^{*}\|_{L_{p^{*}}(B^{+}_{F^{**}},\lambda)} \\
&\leq n^{m+}_{p}(P)\left(\|z\|_{L_{p}(B^{+}_{(\widehat{\otimes}^{m}_{s,|\pi|}E)^{*}},\widehat{\mu})} + \epsilon\right)\|y^{*}\|_{L_{p^{*}}(B^{+}_{F^{**}},\lambda)}
\end{aligned}$$

Thus, $P^{\otimes} \in \mathcal{N}^{+}_{p}(\widehat{\otimes}^{m}_{s,|\pi|}E,F)$ and $n^{m+}_{p}(P) \leq n^{+}_{p}(P^{\otimes})$. □

Problem 12. Does $P^{\otimes} \in \mathcal{L}^{r}(\widehat{\otimes}^{m}_{s,|\pi|}E,F)$ positive Cohen $p-$nuclear imply $P \in \mathcal{P}^{r}(^{m}E,F)$ is positive Cohen $p-$nuclear polynomial?

Proposition 13. *Let $m \in \mathbb{N}$. Assume that $\mathcal{P}^r_{coh^+-p}(^mE,F) \subset \mathcal{P}^{c+r}_{N-p}(^mE,F)$. Then $\mathcal{D}^{+r}_p(^mE,F) \subset \mathcal{N}^{+r}_p(E,F)$.*

Proof. Let $u \in \mathcal{D}^{+r}_p(E,F)$, we show that $u \in \mathcal{N}^{+r}_p(E,F)$. Fix $e \in B_{E^+}$ and $x_0^* \in B^+_{E^*}$ such that $x_0^*(e) = 1$. Define, see [9], the operator

$$\widehat{K}_j : \widehat{\otimes}^{j+1}_{s,|\pi|}E \to \widehat{\otimes}^{j}_{s,|\pi|}E \ (1 \leq j \leq m-1)$$

by

$$\widehat{K}_j\left(\sum_{i=1}^n x_i \otimes \overset{(j+1)}{\cdots} \otimes x_i\right) = \sum_{i=1}^n x^*(x) x_i \otimes \overset{(j)}{\cdots} \otimes x_i.$$

Let $\otimes_m(x) = x \otimes \overset{(m)}{\cdots} \otimes x$. It follows from [10, Proposition 6], we obtain

$$P := u \circ \widehat{K}_1 \circ \cdots \circ \widehat{K}_{m-1} \circ \otimes_m : E \to F$$

Then, P is positive Cohen strongly $p-$summing, it follows from this and our hypotheses that , $P \in \mathcal{P}^{c+r}_{N-p}(^mE,F)$. By the decomposition $P = P^{\otimes} \circ \otimes_m$ we obtain $P^{\otimes} = u \circ \widehat{K}_1 \circ \cdots \circ \widehat{K}_{m-1}$ is positive Cohen $p-$nuclear.

Now, as it has been proven in the proof of [9, Theorem 4.1], that there are operators $\widehat{J}_j : \widehat{\otimes}^{j}_{s,|\pi|}E \to \widehat{\otimes}^{j+1}_{s,|\pi|}E \ (1 \leq j \leq m-1)$ defined in terms of x_0^* and e such that $\widehat{K}_j \circ \widehat{J}_j$ is the identity map on $\widehat{\otimes}^{j}_{s,|\pi|}E$. We obtain

$$u = u \circ \widehat{K}_1 \circ \cdots \circ \widehat{K}_{m-1} \circ \widehat{J}_{m-1} \circ \cdots \circ \widehat{J}_1 : E \to F$$

Thanks to the ideal property, u is positive Cohen $p-$nuclear. □

Theorem 14. *An $m-$homogeneous polynomial $P \in \mathcal{P}^r(^mE,F)$ is positive Cohen $p-$nuclear polynomial imply the adjoint $P^* : F^* \to \mathcal{P}^r(^mE)$ is positive Cohen p^*-nuclear operator.*

Proof. Assume first that $P : E \to F$ is positive Cohen $p-$nuclear polynomial. By Theorem 11, the linearization $P^{\otimes} : \widehat{\otimes}^{m}_{s,|\pi|}E \to F$ is positive Cohen $p-$nuclear, then its adjoint $P^{\otimes *} : F^* \to (\widehat{\otimes}^{m}_{s,|\pi|}E)^*$ is a positive Cohen p^*-nuclear operator. Consider the isometric isomorphism $\triangle_m : \mathcal{P}^r(^mE) \to (\widehat{\otimes}^{m}_{s,|\pi|}E)^*$ given by $\triangle_m(P) = P^*$. Since $P = P^{\otimes} \circ \otimes_m$, by duality we get $P^* = \otimes_m^* \circ P^{\otimes *} = \triangle_m^{-1} \circ P^{\otimes *}$.

$$\begin{tikzcd} F^* \arrow[rr, "P^*"] \arrow[dr, "P^{\otimes *}"'] & & \mathcal{P}^r(^mE) \\ & \left(\widehat{\otimes}^{m}_{s,|\pi|}E\right)^* \arrow[ur, "\Delta_m^{-1}"'] & \end{tikzcd}$$

The ideal property ensures that P^* is positive Cohen p^*-nuclear.

□

Problem 15. Does the adjoint $P^* \in \mathcal{L}^r(F^*, \mathcal{L}^r(^mE))$ positive Cohen p^*-nuclear operator imply $P \in \mathcal{P}^r(^mE, F)$ is positive Cohen $p-$nuclear polynomial ?

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Asma Hammou
Department of Mathematics, M'sila University, Algeria,
Département Mathématiques, Ecole Normale Supérieure de Laghouat, Algérie.
e-mail: asma.hammou@univ-msila.dz, a.hammou@ens-lagh.dz

Amar Belacel
Laboratory of Pure and Applied Mathematics (LPAM), University of Laghouat, Algeria.
e-mail: amarbelacel@yahoo.fr

Amar Bougoutaia
Laboratory of Pure and Applied Mathematics (LPAM), University of Laghouat, Algeria.,
e-mail: amarbou28@gmail.com

Abdelmoumen Tiaiba
Renewable Materials and Energies Laboratory "LMER" of the university of M'sila. Algeria,
e-mail: abdelmoumen.tiaiba@univ-msila.dz

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