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# THEORETICAL AND NUMERICAL RESULTS FOR THE NONLINEAR SHALLOW WATER PROBLEM

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**Abstract**. In this paper, the nonlinear shallow water problem is studied analytically and numerically. This problem has been studied by different authors in the linear case; here we consider a nonlinear one with non constant coefficients. We first start by a mathematical analysis, where the well posedness is proved via the use of traveling wave solutions. The existence and uniqueness of the solution is demonstrated by the means of Schaulder's and Banach's fixed point theorems. Then, we explore the proposed model and investigate the stability conditions and the efficiency numerically.

## 1 Introduction

The shallow water equations, in particular the Saint-Venant system were obtained from depth-integrating the Navier-Stokes problem when the horizontal length scale of the flow is much larger than the vertical fluid one. Many authors have focussed on this type of problem to describe river and lake hydrodynamics. For example in [5], Gerbeau et al derived the Saint-Venant equations for shallow water, including friction and the Coriolis-Boussinesq coefficient, where the hydrostatic approximation was used in their derivation. In [2] Olivier Besson et al derived a model, using some lateral integration of the incompressible Navier-Stokes equations based on certain assumptions. The existence of global weak solutions of a two-dimensional viscous shallow water model with friction term is given in [3]. This type of model has been explored numerically by many authors, we can cite among others Kamboh et al [7] who considered the two-dimensional Saint-Venant equations in open channels. The reader is also referred to [1] and [8].
The considered shallow water model is one-dimensional and has the following form.

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Let $W=(0,T)\times I$ with $I=[0,L]$, $L$ is the length of the flow:

$$\begin{cases} l\frac{\partial h}{\partial t}+\frac{\partial q}{\partial x}=f_1 \quad \text{in} \;\; W, \\ \frac{\partial q}{\partial t}-\nu\frac{\partial^2 q}{\partial x^2}+\beta(h)\frac{\partial h}{\partial x}=f_2 \quad \text{in} \;\; W, \\ q(t,x)=0 \;\; \text{on} \;\; \partial I, \\ h(0,x)=h_0(x), \\ q(0,x)=q_0(x), \end{cases} \tag{1.1}$$

where the unknowns are the height $h$ and the discharge $q$ of the flow, the parameters $l$, $\nu$ are known denoting the width and the viscosity, respectively and $f=(f_1,f_2)$ represents the known external forces.
The authors in [2] gave results on the existence and uniqueness of the solutions of problem (1.1) by assuming that $\beta$ is constant.
Our first goal in this work is to prove the existence and uniqueness when $\beta$ is non-constant and dependent of $h$.
The outline of this paper is organized as follows. In Section 2, we define our model, then we use the traveling wave solution to transform our model to an ordinary differential system. In Section 3, we prove the existence and uniqueness of the solution. In Section 4, we provide numerical discretizations, then stability analysis. In Section 5, we present numerical results and comments. Then, we finish by a conclusion.

## 2 Proposed model

If we consider that $\beta(h)=a+2bh$ such that $a,b$ are positive constants with $a=lU^2+\frac{p_a l}{U^2}$, $b=\frac{gl^2}{2U^2}$, where $g$ is the gravity, $U$ the characteristic velocity and $p_a$ the adimensional atmospheric pressure.
The nonlinear system is written as follows

$$\begin{cases} l\frac{\partial h}{\partial t}+\frac{\partial q}{\partial x}=f_1 \quad \text{in} \;\; W, \\ \frac{\partial q}{\partial t}-\nu\frac{\partial^2 q}{\partial x^2}+a\frac{\partial h}{\partial x}+2bh\frac{\partial h}{\partial x}=f_2 \quad \text{in} \;\; W, \\ h(0,x)=h_0(x) \;\; \forall x\in I, \\ q(0,x)=q_0(x) \;\; \forall x\in I, \end{cases} \tag{2.1}$$

with proper boundary conditions on $h$ and $q$.
Since the height $h$ should be bounded, so there is a constant $M>0$ such that for all $(t,x)\in[0,T]\times I$, we have

$$\sup_{(t,x)\in[0,T]\times I}|h|\le M. \tag{2.2}$$

If we set

$$\begin{cases} h(x,t) = \varphi(x-ct), \\ q(x,t) = \psi(x-ct), \end{cases} \tag{2.3}$$

where $c > 0$ is the traveling wave velocity.
The use of (2.2) implies

$$\sup_{(t,x)\in[0,T]\times I} |\varphi| \leq M. \tag{2.4}$$

Let $\lambda > 0$ and $\eta = x - ct$, by substituting (2.3) in (2.1), we get the following system

$$\begin{cases} -lc\varphi'(\eta) + \psi'(\eta) = f_1, \quad 0 \leq \eta \leq \lambda \\ -c\psi'(\eta) - \nu\psi''(\eta) + (a + 2b\varphi(\eta))\varphi'(\eta) = f_2, \quad 0 \leq \eta \leq \lambda \\ \varphi(\eta)_{t=0} = \varphi_0 \quad \forall x \in I, \\ \psi(\eta)_{t=0} = \psi_0 \quad \forall x \in I. \end{cases} \tag{2.5}$$

By integrating the second equation in (2.5), we obtain

$$\begin{cases} \varphi'(\eta) = \frac{1}{lc}(\psi'(\eta) - f_1), \\ \psi'(\eta) = \psi_1 + \frac{1}{\nu}\int_0^\eta (a\varphi'(\eta) + 2b\varphi(\eta)\varphi'(\eta) - c\psi'(\eta) - f_2)\, d\eta, \quad \psi_1 = \psi'(0), \\ \varphi(\eta)_{t=0} = \varphi_0 \quad \forall x \in I, \\ \psi(\eta)_{t=0} = \psi_0 \quad \forall x \in I. \end{cases} \tag{2.6}$$

Here, we derive the necessary conditions for obtaining at least one solution and if possible its uniqueness using the fixed point theorems.

## 3 Main Results

Let us introduce the basic and essential definitions. Let $\lambda > 0$ such that the Banach space of continuous functions from $[0,\lambda]$ into $\mathbb{R}$ is denoted by $C([0,\lambda],\mathbb{R})$, with the norm

$$\|\varphi\|_\infty = \sup_{\eta\in[0,\lambda]} |\varphi|.$$

Let $u = (\varphi, \psi)$ be the solution of

$$\begin{cases} \varphi'(\eta) = k_1(\eta, u(\eta)), \\ \psi'(\eta) = k_2(\eta, u(\eta)), \end{cases} \tag{3.1}$$

where $k_{1\leq i\leq 2}(\eta, u(\eta))$ represents the right hand side of (2.6).

We use the fixed point theory to prove that at least one solution of (2.6) exists. Let $u = (\varphi, \psi) \in E$, such that $E = [C([0, \lambda], \mathbb{R}_+]^2$ is a Banach space equipped with the norm

$$\|u\|_E = \|\varphi\|_\infty + \|\psi\|_\infty,$$

and let $k = (k_1, k_2)$ such that

$$\begin{cases} k_1(\eta, u(\eta)) = \frac{1}{lc}(\psi'(\eta) - f_1), \\ k_2(\eta, u(\eta)) = \psi_1 + \dfrac{1}{\nu} \int_0^\eta (a\varphi'(\eta) + 2b\varphi\varphi'(\eta) - c\psi'(\eta) - f_2)\, d\eta, \end{cases} \tag{3.2}$$

it is clear that the function $k \in ([0, \lambda] \times E)^2$ is continuous.
By applying the integral to both sides of system (3.2), we obtain

$$\begin{cases} \varphi(\eta) = \varphi(0) + \int_0^\eta k_1(\xi, u(\xi))d\xi, \\ \psi(\eta) = \psi(0) + \int_0^\eta k_2(\xi, u(\xi))d\xi. \end{cases} \tag{3.3}$$

By choosing $u_0 = (u_1, u_2) = (\varphi(0), \psi(0))$, we obtain

$$u(\eta) = u_0 + \int_0^\eta k(\xi, u(\xi))d\xi.$$

In the following, we present the principal results:

**Theorem 1.** *(**Existence**) let $a, b, c, l, M, \nu, \lambda$ and $\mu \in \mathbb{R}_+$, such that*

$$\mu = max\{\frac{1}{lc}, \frac{\lambda}{\nu}(a + 2bM + c)\},$$

*if*

$$\mu < 1. \tag{3.4}$$

*Then, there is at least one solution of* (2.6) *on* $[0, \lambda]$.

*Proof.* Let us transform (2.6) into a fixed point problem written in the form $Au(\eta) = u(\eta)$, with

$$Au(\eta) = (A_1 u(\eta), A_2 u(\eta)),$$

and

$$A_i u(\eta) = u_0 + \int_0^\eta k_i(\xi, u(\xi))d\xi, i = 1, 2. \tag{3.5}$$

We first prove that if $u \in E$, then $(A_i u)_{1 \le i \le 2}$, is a continuous operator, and therefore $Au$ is an element of $E$ with the norm

$$\|Au\|_E = \sum_{i=1}^{2} \|A_i u\|_\infty.$$

Next, we prove that $A$ satisfies the conditions of Schauder's fixed point theorem, through the following steps:

- **Step 1:** $A$ is a nonlinear continuous operator.
  Let $(u_n)_{n\in N} = (\varphi_n, \psi_n)$ be two positive sequences such that $\lim\limits_{x\to+\infty} u_n = u$ in $E$, this implies $\lim\limits_{x\to+\infty} \varphi_n = \varphi$ and $\lim\limits_{x\to+\infty} \psi_n = \psi$ in $E$.
  Then for each $\eta \in [0, \lambda]$, we have:

$$\begin{aligned}
|A_1u_n(\eta) - A_1u(\eta)| &= |\textstyle\int_0^\eta k_1(\xi, u_n(\xi))d\xi - \int_0^\eta k_1(\xi, u(\xi))d\xi|,\\
&= |\textstyle\int_0^\eta \frac{1}{lc}(\psi_n'(\xi) - f_1)d\xi - \int_0^\eta \frac{1}{lc}(\psi'(\xi) - f_1)d\xi|,\\
&= |\frac{1}{lc}(\psi_n(\eta) - \psi_n(0) - \eta f_1) - \frac{1}{lc}(\psi(\eta) - \psi(0) - \eta f_1)|,\\
&= \frac{1}{lc}|(\psi_n(\eta) - \psi(\eta))|,\\
&\leq (\frac{1}{lc})\|u_n - u\|_E.
\end{aligned}$$

  Similarly, we have

$$\begin{aligned}
|A_2u_n(\eta) - A_2u(\eta)| &= |\textstyle\int_0^\eta k_2(\xi, u_n(\xi))d\xi - \int_0^\eta k_2(\xi, u(\xi))d\xi|,\\
&= |\textstyle\int_0^\eta \left(\psi_1 + \frac{1}{\nu}\int_0^\eta a\varphi_n' + 2b\varphi_n\varphi_n' - c\psi_n' - f_2 d\xi\right) d\xi,\\
&- \textstyle\int_0^\eta \left(\psi_1 + \frac{1}{\nu}\int_0^\eta a\varphi' + 2b\varphi\varphi' - c\psi' - f_2 d\xi\right) d\xi|,\\
&= |\frac{1}{\nu}\textstyle\int_0^\eta \left(a\varphi_n + b\varphi_n^2 - c\psi_n\right) d\xi - \frac{1}{\nu}\int_0^\eta \left(a\varphi + b\varphi^2 - c\psi\right) d\xi|,\\
&\leq \frac{1}{\nu}\left(\textstyle\int_0^\eta a|\varphi_n - \varphi|d\xi + b\int_0^\eta |\varphi_n^2 - \varphi^2|d\xi + c\int_0^\eta |\psi_n - \psi_u|d\xi\right),\\
&\leq \frac{1}{\nu}\{\textstyle\int_0^\eta a|\varphi_n - \varphi|d\xi + b\int_0^\eta |\varphi_n(\varphi_n - \varphi) + \varphi_u(\varphi_n - \varphi)|d\xi\\
&+ c\textstyle\int_0^\eta |\psi_n - \psi|d\xi\},\\
\text{by using (2.4)} &\leq \frac{1}{\nu}\{\textstyle\int_0^\eta a|\varphi_n - \varphi|d\xi + 2bM\int_0^\eta |\varphi_n - \varphi_u|d\xi + c\int_0^\eta |\psi_n - \psi|\},\\
&\leq \frac{\lambda}{\nu}(a + 2bM + c)\|u_n - u\|_E,
\end{aligned}$$

  If we assume that $\mu = max\{\frac{1}{lc}, \frac{\lambda}{\nu}(a + 2bM + c)\}$, then we have

$$|A_iu_n(\eta) - A_iu(\eta)| \leq \mu\|u_n - u\|_E,\ \mu > 0,\ i = 1, 2.$$

  Since $u_n \longrightarrow u$ in $E$, consequently A is continuous.

- **Step 2:** $A(E_r) \subset E_r$.
  We set the positive real $r$ to be

$$r \geq \frac{2}{1 - 2\mu}u_i,$$

and define the subset $E_r$ as follows:

$$E_r = \{u \in E : \|u\|_E \leq r\}.$$

Clearly $E_r$ denotes a closed, bounded and convex subset of $E$.
Let $A : E_r \to E$ be the integral operator given by (3.5), thus $A(E_r) \subset E_r$, so that for each $\eta \in [0, \lambda]$, we have :

$$\begin{aligned} |A_1 u(\eta)| &\leq u_1 + \frac{1}{lc} r, \\ |A_2 u(\eta)| &\leq u_2 + \frac{1}{\nu}(\frac{\lambda}{\nu}(a + 2bM + c)r. \end{aligned} \tag{3.6}$$

Hence, in each case, we have

$$\begin{aligned} |A_i u(\eta)| &\leq u_i + \mu r, \\ &\leq \frac{1 - 2\mu}{2} r + \mu r, \\ &\leq \frac{1}{2} r, \quad i = 1, 2, \end{aligned} \tag{3.7}$$

or $(\|A_i u\|_\infty)_{1 \leq i \leq 2} \leq \frac{r}{2}$, then

$$\|Au\|_E = \sum_{i=1}^{2} \|A_i u\|_\infty \leq r$$

. Consequently, $A(E_r) \subset E_r$.

- **Step 3:** $A(E_r)$ is relatively compact.

  Let $\eta_1, \eta_2 \in [0, \lambda], \eta_1 < \eta_2$ and $u \in E_r$. Then we have

$$\begin{aligned} |A_1 u(\eta_1) - A_1 u(\eta_2)| &= |\textstyle\int_0^{\eta_2} k_1(\xi, u(\xi))d\xi - \int_0^{\eta_1} k_1(\xi, u(\xi))d\xi|, \\ &= |\textstyle\int_0^{\eta_2} \frac{1}{lc}(\psi'(\xi) - f_1)d\xi - \int_0^{\eta_1} \frac{1}{lc}(\psi'(\xi) - f_1)|d\xi, \\ &\leq |\frac{1}{lc}(\psi(\eta_2) - \eta_2 f_1 - \psi(\eta_1) + \eta_1 f_1)|, \\ &\leq |\frac{1}{lc}(\psi(\eta_2) - \psi(\eta_1) - (\eta_2 - \eta_1) f_1)|, \\ &\leq |\frac{1}{lc}(|\psi(\eta_2) - \psi(\eta_1)| + |(\eta_2 - \eta_1) f_1|), \end{aligned} \tag{3.8}$$

we know that $\psi$ is continuous, this implies $\psi(\eta_1) \to \psi(\eta_2)$.
It follows from $\eta_1 \to \eta_2$, that the right-hand side of inequality (3.8) tends to zero.

$$|A_1 u(\eta_1) - A_1 u(\eta_2)| \leq 0. \tag{3.9}$$

Similarly, we have

$$\begin{aligned} |A_2u(\eta_1) - A_2u(\eta_2)| &= |\textstyle\int_0^{\eta_2} k_2(\xi, u(\xi))d\xi - \int_0^{\eta_1} k_2(\xi, v(\xi))d\xi|, \\ &= |\textstyle\int_0^{\eta_2}(\psi_1(\xi) + \dfrac{1}{\nu}\int_0^{\eta_2} a\varphi'(\xi) + 2b\varphi\varphi'(\xi) - c\psi'(\xi) - f_2)d\xi \\ &- \textstyle\int_0^{\eta_1}(\psi_1(\xi) + \dfrac{1}{\nu}\int_0^{\eta_1} a\varphi'(\xi) + 2b\varphi(\xi)\varphi'(\xi) - c\psi'(\xi) - f_2)d\xi| \\ &= |(\eta_2 - \eta_1)\psi_1 + \dfrac{1}{\nu}(a(\varphi(\eta_2) - \varphi(\eta_1)) + b(\varphi^2(\eta_2) - \varphi^2(\eta_1)), \\ &- c(\psi(\eta_2) - \psi(\eta_1)) - (\eta_2 - \eta_1)f_2)|. \end{aligned} \tag{3.10}$$

$\psi, \varphi$ are continuous, i.e. $\psi(\eta_1) \to \psi(\eta_2)$, $\varphi(\eta_1) \to \varphi(\eta_2)$.
It follows from $\eta_1 \to \eta_2$, that the right-hand side of inequality (3.10) tends to zero.

$$|A_2u(\eta_1) - A_2u(\eta_2)| \leq 0. \tag{3.11}$$

Then from (3.9) and (3.11) we obtain

$$|A_iu(\eta_1) - A_iu(\eta_2)| \leq 0, \quad i = 1, 2.$$

As a result of steps 1-3, and through Ascoli-Arzela theorem, we can affirm the continuity and compactness of $A : E_r \to E_r$, i.e $A$ satisfies Schauder's fixed point theorem [6] and [4]. Therefore, $A$ has a fixed point solution of (2.6) on $[0, \lambda]$.

We have already proved in Theorem 1., that there is at least one solution of (2.6). Consequently, if Theorem 1. holds for any $(x, t) \in W$, then there is at least one solution of (2.1) under the traveling wave form (2.3).

□

**Theorem 2.** *(**Uniqueness**) let $a, b, c, l, M, \nu, \lambda$ and $\mu \in \mathbb{R}_+$, such that*

$$\mu = max\{\frac{1}{lc}, \frac{\lambda}{\nu}(a + 2bM + c)\},$$

*if*

$$\mu < 1, \tag{3.12}$$

*then, (2.6) admits a unique solution on $[0, \lambda]$.*

*Proof.* In Theorem 1., we have already proceeded by a transformation of (2.6) into a fixed point problem .
Let $u, v \in E$ satisfy (2.6). This implies that:
For all $\eta \in [0, \lambda]$ , we have

$$\begin{aligned} |A_1u(\eta) - A_1v(\eta)| &\leq \tfrac{1}{lc}\|u - v\|_E, \\ |A_2u(\eta) - A_2v(\eta)| &\leq \frac{\lambda}{\nu}(a + 2bM + c)\|u - v\|_E. \end{aligned} \tag{3.13}$$

If we assume that $\mu = max\{\frac{1}{lc}, \frac{\lambda}{\nu}(a + 2bM + c)\}$, then we have

$$\|A_i u - A_i v\|_\infty \leq \mu \|u - v\|_E.$$

Similarly, we can find that:

$$\|Au - Av\|_E \leq \mu \|u - v\|_E. \tag{3.14}$$

According to Theorem 1., and from inequality (3.14), A is a contraction. As a consequence of Banach's contraction principle (see [6]), we conclude that $A$ has only one fixed point which is the unique solution of (2.6) on $[0, \lambda]$.
Based on Theorem 2., there exists a unique solution of (2.1) provided that (3.12) holds. □

## 4 Numerical Discretisation

In this section we discretise (2.1) with $f = 0$, then, we derive a stability condition to ensure the convergence properties of the proposed numerical schemes.

### 4.1 Finite difference approximation

System (2.1) is approximated using an explicit finite difference scheme, where the temporal and spatial derivatives are written in the following form:

$$\frac{\partial u}{\partial t} = \frac{u_i^{n+1} - u_i^n}{\Delta t}, \quad \frac{\partial u}{\partial x} = \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x}. \tag{4.1}$$

Thus, we get

$$l \frac{h_i^{n+1} - h_i^n}{\Delta t} + \frac{q_{i+1}^n - q_{i-1}^n}{2\Delta x} = 0, \tag{4.2}$$

$$\begin{aligned} \frac{q_i^{n+1} - q_i^n}{\Delta t} &= \nu \left( \frac{q_{i+1}^n - 2q_i^n + q_{i-1}^n}{(\Delta x)^2} \right) - a \left( \frac{h_{i+1}^n - h_{i-1}^n}{2\Delta x} \right) \\ &- b h_i^n \left( \frac{h_{i+1}^n - h_{i-1}^n}{2\Delta x} \right), \end{aligned} \tag{4.3}$$

then, we obtain $h_i^{n+1}$and $q_i^{n+1}$:

$$h_i^{n+1} = h_i^n - \frac{\Delta t}{l} \left( \frac{q_{i+1}^n - q_{i-1}^n}{2\Delta x} \right). \tag{4.4}$$

$$\begin{aligned} q_i^{n+1} &= q_i^n - \Delta t \left( -\nu \left( \frac{q_{i+1}^n - 2q_i^n + q_{i-1}^n}{(\Delta x)^2} \right) - a \left( \frac{h_{i+1}^n - h_{i-1}^n}{2\Delta x} \right) \right) \\ &- \Delta t b h_i^n \left( \frac{h_{i+1}^n - h_{i-1}^n}{2\Delta x} \right). \end{aligned} \tag{4.5}$$

Expressions (4.4) and (4.5) can be evaluated explicitly and the finite difference approximations are performed on each grid point.

## 4.2 Stability analysis

In this section, we prove a result related to numerical stability of (4.4) and (4.5).

**Proposition 1.** *The numerical scheme* (4.4) *satisfies the following stability condition*

$$\Delta t \leq 2l\Delta x, \tag{4.6}$$

*Proof.* To prove this result, we use Fourier analysis: we write the solution in the form (see [8]) $h_j^n = \lambda^n e^{i\pi kj}$ and $q_j^n = \lambda^n e^{i\pi kj}$ with $k$ an integer in (4.4) we get:

$$\lambda = \lambda^n e^{i\pi kj} - \frac{\Delta t}{l} \left( \frac{\lambda^n e^{i\pi k(j+1)} - \lambda^n e^{i\pi k(j-1)}}{2\Delta x} \right). \tag{4.7}$$

If we divide by $\lambda^n e^{\pi kj}$ in (4.7), we obtain

$$\lambda = 1 - \frac{\Delta t}{l} \left( \frac{cos(\pi k)}{\Delta x} \right). \tag{4.8}$$

The scheme (4.4) is stable if $|\lambda| \leq 1$, thus we get

$$|1 - \frac{\Delta t}{l} \left( \frac{cos(\pi k)}{\Delta x} \right) | \leq 1. \tag{4.9}$$

This implies

$$|1 - \frac{\Delta t}{l\Delta x}| \leq 1, \tag{4.10}$$

then

$$0 \leq \Delta t \leq 2l\Delta x. \tag{4.11}$$

□

**Proposition 2.** *The numerical scheme* (4.5) *satisfies the following stability condition*

$$\Delta t \leq \frac{2(\Delta x)^2}{4\nu + a\Delta x + bH\Delta x}, \tag{4.12}$$

*with* $H = max(h_j^n)$.

*Proof.* To prove this result, we use the same procedure as in Proposition 1., and get:

$$\lambda = 1 - \Delta t \left( -\nu \frac{2cos(\pi k) - 2}{(\Delta x)^2} + a \frac{cos(\pi k)}{\Delta x} \right) - \Delta t b H \left( \frac{cos(\pi k)}{\Delta x} \right). \tag{4.13}$$

If we use this formula $cos(\pi k) - 1 = -2sin^2\left(\frac{\pi k}{2}\right)$, (4.13) becomes

$$\lambda = 1 - \Delta t \left( \nu \frac{4sin^2\left(\frac{\pi k}{2}\right)}{(\Delta x)^2} + a \frac{cos(\pi k)}{\Delta x} \right) - \Delta t b H \left( \frac{cos(\pi k)}{\Delta x} \right). \tag{4.14}$$

The scheme (4.5) is stable if $|\lambda| \leq 1$, thus we get

$$|1 - \Delta t \left( \nu \frac{4sin^2\left(\frac{\pi k}{2}\right)}{(\Delta x)^2} + a \frac{cos(\pi k)}{\Delta x} \right) - \Delta t b H \left( \frac{cos(\pi k)}{\Delta x} \right)| \leq 1, \tag{4.15}$$

we have $sin^2\left(\frac{\pi k}{2}\right) \leq 1$ and $cos(\pi k) \leq 1$ which implies

$$|1 - \Delta t \left( \frac{4\nu}{(\Delta x)^2} + a \frac{1}{\Delta x} + bH \frac{1}{\Delta x} \right)| \leq 1, \tag{4.16}$$

then

$$0 \leq \Delta t \leq \frac{2(\Delta x)^2}{4\nu + a\Delta x + bH\Delta x}. \tag{4.17}$$

□

## 5 Numerical implementation

To solve (4.4)-(4.5) numerically, we apply the following algorithm :

**Algorithm 3.** *1. Initialisation: input the gravity $g$, the length $L$ and the width $l$ of the flow, the viscosity $\nu$, the coefficients $a$ and $b$, the time step $\Delta t$, the final time $T$, and the initial and boundary conditions for $h$ and $q$.*

*2. solve (4.4) for the height $h$.*

*3. solve (4.5) for the discharge $q$.*

*4. Repeat steps 2 and 3 until the final time $T$.*

### 5.1 Numerical results and comments

The obtained results for the solutions of (1.1) and (2.1) are resumed in Figures 1 and 2, showing the variation in the height and the discharge of the water at different regions of the flow. The discretization parameters are set as $\Delta x = 0.3$ and $\Delta t$ chosen according to the stability CFL conditions (4.6) and (4.12), length $L = 1m$ , width $l = 5cm$, viscosity $\nu = 5 * 10^{-3}$, coefficient $b = l^2 \cdot g \cdot T^2/(2 \cdot L^2)$, coefficient $a = 1$, with the initial condition of the height given by $h_0(x) = e^{((-(x-5)^2/100)}$, whereas the initial condition of the discharge is set as $q_0 = 0$. The goal of the simulation is to determine the variation of the height and the discharge of the flow at different times t.

We observe higher diffusion in the height $h$ at different times $t$ in the linear case ($\beta$ is constant); however, the nonlinear case when $\beta$ is non constant, seems to provide a more precise description of the height. These deductions are similar for the discharge $q$. The obtained results seem to be stable even for the nonlinear case, as shown in Figures 1 and 2.

## 6 Conclusion

In this paper, we studied a new approach based on the traveling wave to solve Saint-Venant system. Our contribution focused on the mathematical and numerical analysis of the derived model. The obtained results on the existence and uniqueness of solution for the nonlinear model, required several basic tools derived from the Schauder's fixed point theorem and the Banach contraction principle. For numerical experimentation, we made use of a simple explicit finite difference method with respect to the stability CFL conditions (4.6) and (4.12). The numerical results regarding both the linear and non linear cases, are resumed in Figures 1 and 2 which are commented in the previous section. Here we present the variation in the height $h$ and discharge $q$ of the water at different regions of the flow and different times $t$. We deduct from these obtained results that the model is stable for both cases.

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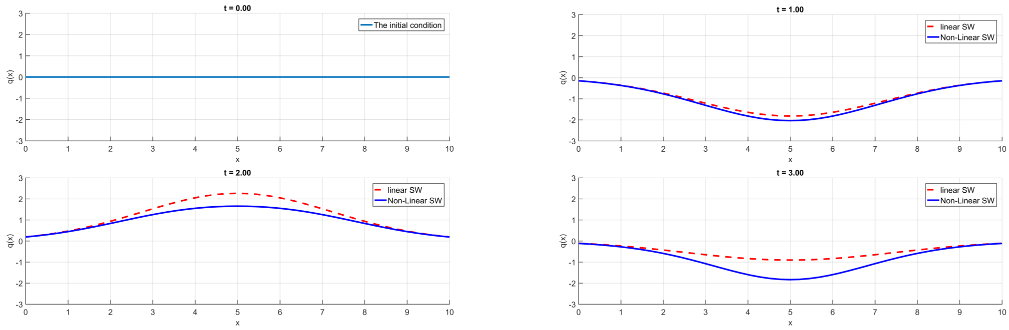


Figure 1: The discharges $q$, solution of (1.1) and (2.1), at $t = 0, 1, 2$ *and* 3.

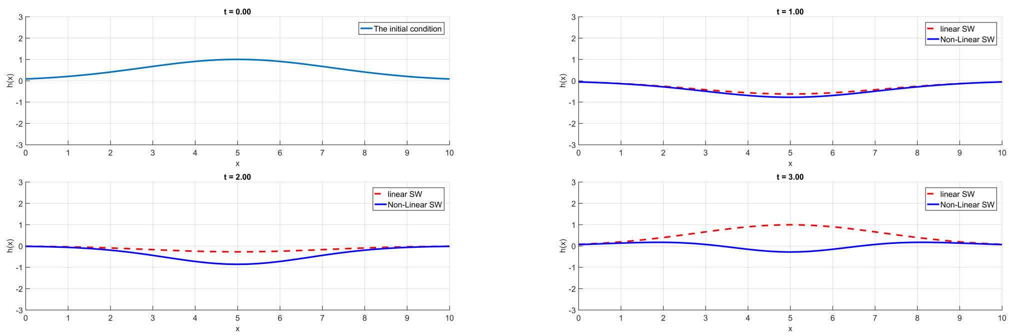


Figure 2: The height $h$, numerical solutions of (1.1) (2.1), at $t = 0, 1, 2$ *and* 3.

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