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S-NOETHERIAN RINGS, MODULES AND THEIR GENERALIZATIONS

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Abstract. Let R be a commutative ring with identity, M an R-module and $S \subseteq R$ a multiplicative set. Then M is called S-finite if there exist an $s \in S$ and a finitely generated submodule N of M such that $sM \subseteq N$. Also, M is called S-Noetherian if each submodule of M is S-finite. A ring R is called S-Noetherian if it is S-Noetherian as an R-module. This paper surveys the most recent developments in describing the structural properties of S-Noetherian rings, S-Noetherian modules and their generalizations. Some interesting constructed examples of S-Noetherian rings and modules are also presented.

1 Introduction

Theory of Noetherian rings and modules played an important role in the developement of structure theory of commutative rings. This theory has a history extending over more than hundred years. Recall that a module over a ring is called Noetherian if it satisfies ascending chain condition on submodules, and a commutative ring is called Noetherian if it is a Noetherian module over itself. One of the most important roots of the theory of Noetherian rings is the Noether's historical article [30] in 1921. Thereafter Noetherian rings and modules were studied continuously and hence became one of the central subjects in the study of ring and module theory. Several attempts have been made to generalize the concept of Noetherian rings and modules in order to extend the structural properties of Noetherian rings and modules. The idea of S-Noetherian rings and modules is one among them.

In 1988, Hamann et al. [12] introduced the notion of almost principal ideal domain. Let D be an integral domain with field of fraction K. An ideal I of $D[X]$ is called almost principal if there exist an $f(X) \in I$ of positive degree and a nonzero $s \in D$ such that $sI \subseteq f(X)D[X]$. A polynomial ring $D[X]$ is called an almost principal ideal domain if all ideals of $D[X]$ with proper extensions to $K[X]$ are

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almost principal. They introduced this notion to study the following questions due to Ratliff, Houston and Arnold:

1. When is $(aX - b)K[X] \cap D[X]$ generated by linear polynomials?
2. When is $f(X)K[X] \cap D[X]$ divisorial?
3. When is an ideal I, which is its own extension-contraction from $D[X]$ to $D[[X]]$ and back, equal to closure of I in the X-adic topology?

In 1995, Anderson et al. [2] introduced the notion of almost Noetherian ring. An ideal I of $D[X]$ is said to be almost finitely generated if there exists a finitely generated ideal N of $D[X]$ and a nonzero $s \in D$ such that $sI \subseteq N \subseteq I$. A polynomial ring $D[X]$ is called almost Noetherian if each nonzero ideal I of $D[X]$ with $IK[X] \neq K[X]$ is almost finitely generated. For a domain D, they proved that $D[X]$ is almost Noetherian if and only if $D[X]$ is an almost PID. They used this concept to study Querre's characterization of divisorial ideals in integrally closed polynomial rings. Later, in 2002, Anderson and Dumitrescu [1] abstracted the notion of almost Noetherian for any commutative ring and called it an S-Noetherian ring. Let R be a commutative ring with identity and S a multiplicatively closed subset of R. Then R is called S-Noetherian if for any ideal I of R, there exists an $s \in S$ and a finitely generated ideal J of R such that $sI \subseteq J \subseteq I$. They have transferred several results on Noetherian rings and modules to S-Noetherian rings and modules. For example, they proved S-version of Hilbert's basis theorem, Eakin-Nagata theorem and Cohen's theorem. In 2015, Hamed [15] studied S-Noetherian rings of the forms $\mathscr{A}[X]$ and $\mathscr{A}[[X]]$, where $\mathscr{A} = (A_n)_{n\geq 0}$ is an ascending chain of commutative rings. He obtained a characterization for the rings $\mathscr{A}[X]$ and $\mathscr{A}[[X]]$ to be S-Noetherian. Non-commutative S-Noetherian rings and S-Noetherian modules were first studied in details by Baeck et al. [6] in 2016. In the same year, Hamed [14] provided a connection between S-Noetherian modules and S-stationary ascending chains and obtained several useful properties of this class of modules. In 2018, Hamed [13] introduced and studied the concept of S-Noetherian spectrum condition as a generalization of S-Noetherian rings and proved Hilbert basis theorem for the S-Noetherian spectrum property. In continuation of the study of this theory, many authors have extended well known results on Noetherian rings and modules to S-Noetherian rings and modules (see [22] [7], [4], [15], [14], [19], [20] and [13], for example). Further, in 2020, Kim and Lim [19] extended the notion of S-Notherian rings to G-graded S-Noetherian rings and obtained sufficient conditions for a G-graded S-Noetherian ring to be an S-Noetherian ring. Motivated by this, in 2023, Ansari and Sharma [4] extended the notion of S-Notherian modules to G-graded S-Noetherian modules and characterized S-Noetherian modules in terms of G-graded S-Noetherian module under a mild condition.

Recently, S-version of many special rings and modules has received much attention; for example see, [4], [21], [23] [15], [14], [19] and [20]. During last 10 years, many research articles have been published on S-Noetherian rings, modules and their generalizations, and it is therefore an area of interest to many algebraists. The aim of this article is to survey the most recent developments in describing the structural properties of S-Noetherian rings, S-Noetherian modules and their generalizations. Throughout the paper, all rings are commutative with identity, all modules are unitary and $\mathbb{N}$ denotes the set of all non-negative integers unless otherwise stated.

2 S-Noetherian Rings and Modules

In this section, we investigate the structural properties of S-Noetherian rings and modules. In 2002, Anderson and Dumitrescu [1] introduced S-Noetherian rings and modules as a generalization of Noetherian rings and modules. This notion was useful for extending many properties of Noetherian rings and modules. We begin this section by introducing the definitions of S-Noetherian rings and modules.

Definition 1. *[1, Definition 1] Let R be a ring and $S \subseteq R$ a multiplicative set. We say that an ideal I of R is S-finite if $sI \subseteq J \subseteq I$ for some finitely generated ideal J of R and some $s \in S$. We say that R is S-Noetherian ring if each ideal of R is S-finite.*

Definition 2. *[14, Definition 2.2] Let R be a ring, $S \subseteq R$ a multiplicative set and M an R-module. We say that M is S-finite if $sM \subseteq N$ for some $s \in S$ and some finitely generated submodule N of M. Also, M is called S-Noetherian if each submodule of M is S-finite.*

It is clear from the definitions that every Noetherian ring is S-Noetherian for any multiplicative subset S. However, an S-Noetherian ring need not be Noetherian. The following examples present S-Noetherian rings which are not Noetherian.

Example 3. *Let R_1 be a non-Noetherian domain and R_2 be a Noetherian domain. Consider $R = R_1 \times R_2$ and a multiplicative set $S = (S_1 \cup \{0\}) \times S_2$ of R, where S_1 is a multiplicative set of R_1 and S_2 is a multiplicative set of R_2. Let $I = I_1 \times I_2$ be an ideal of R, where I_1 is an ideal of R_1 and I_2 is an ideal of R_2. Take $s = (0, s_2) \in S$, where $s_2 \in S_2$. For an ideal $I = I_1 \times I_2 \subseteq R$ we have $sI \subseteq \{0\} \times s_2I_2 \subseteq I$. This implies that I is S-finite, so R is S-Noetherian. However, R is not a Noetherian domain.*

Example 4. *Consider a ring $R = \dfrac{F[X_1, X_2, \ldots, X_n, \ldots]}{\langle X_iX_j; i \neq j\rangle}$, where F is a field. Let $Y_i = \overline{X}_i$ be the image of X_i under the canonical map. Consider the multiplicative set $S = \{Y_1^n : n \in \mathbb{N}\}$ of R. Then $\langle Y_1\rangle \subseteq \langle Y_1, Y_2\rangle \subseteq \cdots \subseteq \langle Y_1, Y_2, \ldots, Y_n\rangle \subseteq \cdots$ is an ascending chain of ideals of R which is not stationary. Consequently, R is* *not a Noetherian ring. Now let I be an ideal of R and $f = \sum_i \sum_j r_{ij} Y_i^j \in I$, and $s = Y_1 \in S$. Write*

$$\begin{aligned} f &= \sum_i \sum_j r_{ij} Y_i^j \\ &= (r_{10} + r_{11}Y_1 + r_{12}Y_1^2 + \cdots + r_{1n_1}Y_1^{n_1}) + (r_{20} + r_{21}Y_2 + r_{22}Y_2^2 + \cdots + r_{2n_2}Y_2^{n_2}) + \cdots \\ &\quad + (r_{m0} + r_{m1}Y_m + r_{m2}Y_m^2 + \cdots + r_{mn_k}Y_m^{n_k}) \\ &= f_1 + f_2 + \cdots + f_m, \end{aligned}$$

where $f_i = r_{i0} + r_{i1}Y_i + r_{i2}Y_i^2 + \cdots + r_{in_i}Y_i^{n_i}$ for $i = 1, 2, \ldots, m$. Then $sf = Y_1 f_1 + Y_1 f_2 + \cdots + Y_1 f_m$. Since $X_1 f_j \in \langle X_i X_j : i \neq j \rangle$ for $j = 2, 3, \ldots, m$, so $Y_1 f_j = 0$ for $j = 2, 3, \ldots, m$. Consequently, $sf = Y_1 f_1$. Now we show that $sR \cong Y_1 F[Y_1]$. For this, define $\phi : R \to F[Y_1]$ by $\phi(f) = f_1$. Clearly, ϕ is a surjective ring homomorphism. Also,

$$\begin{aligned} ker\ \phi &= \{f = f_1 + f_2 + \cdots + f_m \in R : \phi(f) = 0\} \\ &= \{f = f_1 + f_2 + \cdots + f_m \in R : f_1 = 0\} \\ &= \{f_2 + \cdots + f_m : f_i = r_{i0} + r_{i1}Y_i + r_{i2}Y_i^2 + \cdots + r_{in_i}Y_i^{n_i}\}. \end{aligned}$$

Consider the induced map $\phi' : Y_1 R \to Y_1 F[Y_1]$ such that $\phi'(Y_1 f) = \phi(Y_1 f) = \phi(Y_1)\phi(f) = Y_1 f_1$. Since $\phi(Y_1 f) = \phi(Y_1)\phi(f) = Y_1 \phi(f) = Y_1 f_1 = \phi'(Y_1 f)$, so ϕ' is well defined and it can be easily seen that it is a ring homomorphism. Now

$$\begin{aligned} ker\ \phi' &= \{Y_1 f \in Y_1 R : \phi'(Y_1 f) = 0\} \\ &= \{Y_1 f_1 \in Y_1 R : Y_1 f_1 = 0\} \\ &= \{0\}. \end{aligned}$$

Thus ϕ' is an injective ring monomorphism. For surjective, let $Y_1 f_1 \in Y_1 F[Y_1]$, where $f_1 \in F[Y_1]$. Since ϕ is surjective, so there exists $f \in R$ such that $\phi(f) = f_1$. This implies that

$$\begin{aligned} \phi'(Y_1 f) &= \phi(Y_1 f) \\ &= \phi(Y_1)\phi(f) \\ &= Y_1 f_1. \end{aligned}$$

Thus ϕ' is a ring isomorphism. Consequently, $Y_1 R \cong Y_1 F[Y_1]$. Thus $sI = Y_1 I$ is a principal ideal of $Y_1 R$, and so I is S-finite. Hence R is an S-Noetherian ring.

Example 5. *Consider the ring $R = \dfrac{\mathbb{Z}[X_1, X_2, \ldots, X_n, \ldots]}{\langle X_i - X_{i+1}^2, X_1 X_i \rangle}$, where $\mathbb{Z}$ is the ring of integers. Let $Y_i = \overline{X}_i$ be the image of X_i under the canonical map. More precisely $Y_i = X_i + I$, where $I = \langle X_i - X_{i+1}^2, X_1 X_i \rangle$. Consider the multiplicative set $S =$*

$\{Y_1^i : i \in \mathbb{N}\}$ of R. Since $Y_2^2 = Y_1, Y_3^2 = Y_2, Y_4^2 = Y_3, \dots, Y_{n+1}^2 = Y_n, \dots$ Thus the ascending chain of principal ideals $\langle Y_1 \rangle \subseteq \langle Y_2 \rangle \subseteq \langle Y_3 \rangle \subseteq \cdots \subseteq \langle Y_n \rangle \subseteq \cdots$ does not stablize. Consequently, R is not a Noetherian ring. Now we claim that R is an S-Noetherian ring. Let J be an ideal of R. Take $s = Y_1$. Then all the indeterminates belongs to $sJ = Y_1 J$ are omitted except Y_1 since $Y_1 Y_i = 0$ in R. Also, for indeterminate Y_1, we have $Y_1^2 = Y_1 Y_1 = Y_1 Y_2^2 = 0$, so $Y_1^2 = 0$. Thus sJ is a principal ideal, and this shows that every ideal in R is S-finite. Hence R is a S-Noetherian ring.

After introducing the definitions of S-Noetherian rings and modules, Anderson and Dumitrescu [1] proved some basic properties of these classes. In the following result, Anderson and Dumitrescu [1] provided a relationship between S-finiteness and prime ideal as an S-version of the corresponding classical result.

Lemma 6. *[1, Lemma 3] Let R be a ring, $S \subseteq R$ a multiplicative set and M an S-finite R-module. If N is a submodule of M, which is maximal among all non-S-finite submodules of M, then $[N : M] = \{r \in R : rM \subseteq N\}$ is a prime ideal of R.*

Anderson and Dumitrescu [1] used Lemma 6 to prove the following characterization of S-Noetherian modules.

Proposition 7. *[1, Proposition 4] Let R be a ring, $S \subseteq R$ a multiplicative set and M an S-finite R-module. Then M is S-Noetherian if and only if the submodules of the form PM are S-finite for each prime ideal P of R (disjoint from S).*

The next proposition generalizes the following well known result: If R is Noetherian, then so is every finitely generated R-module.

Proposition 8. *[15, Proposition 2.1] Let R be a ring and $S \subseteq R$ be a multiplicative set such that R is an S-Noetherian ring. If M is an S-finite R-module, then M is an S-Noetherian R-module.*

It is well known that a ring is Noetherian if and only if its ideals are finitely generated. Cohen's theorem is a classical result which states that a ring is Noetherian if and only if its prime ideals are finitely generated. Various extensions of this theorem have appeared in the literature. Anderson and Dumitrescu [1] extended Cohen's theorem for S-Noetherian rings.

Theorem 9 (S-Version of Cohen's Theorem)**.** *[1, Corollary 5] Let R be a ring and $S \subseteq R$ a multiplicative set. Then R is S-Noetherian if and only if every prime ideal of R (disjoint from S) is S-finite.*

A subring of a Noetherian ring need not be Noetherian. For instance, let D be a non-Noetherian domain and K be its field of fraction. Then K is a Noetherian ring but its subring D is not Noetherian. In view of this, a natural question arises, which subring of a Noetherian ring is Noetherian. Eakin-Nagatan theorem provides an affirmative answer to this question, which says that if A is a subring of a Noetherian ring B such that B is a finitely generated A-module, then A is a Noetherian ring. This theorem was first proved by Eakin [9] and later independently by Nagata [28]. Anderson and Dumitrescu [1] extended this theorem for S-Noetherian rings.

Theorem 10 (S-Version of Ekin-Nagata Theorem)**.** *[1, Corollary 7] Let $A \subseteq B$ be a ring extension and $S \subseteq A$ a multiplicative set such that B is an S-finite A-module. If B is an S-Noetherian ring, then so is A.*

Hilbert [18] proved the theorem known as Hilbert basis theorem in the course of his proof of finite generation of rings of invariants. This theorem says that a polynomial ring over a Noetherian ring is Noetherian. This theorem plays an important roll in algebraic geometry in studying the set of common roots of finitely many polynomial equations. Anderson and Dumitrescu [1] extended Hilbert basis theorem for S-Noetherian rings. They used anti-Archimedean multiplicative set to prove this theorem. However, it is not known whether this theorem is true for arbitrary multiplicative sets.

Definition 11. *[15, Definition 2.5] Let R be a ring and $S \subseteq R$ a multiplicative set. Then S is said to be anti-Archimedean multiplicative set if $\cap_{n\geq 1} s^n R \cap S \neq \phi$.*

Theorem 12 (S-Version of Hilbert Basis Theorem)**.** *[1, Proposition 9] Let R be a ring and $S \subseteq R$ an anti-Archimedean multiplicative set. If R is S-Noetherian, then the polynomial ring $R[X_1, X_2, \ldots, X_n]$ is also an S-Noetherian ring.*

Proposition 13. *[1, Proposition 10] Let R be a ring and $S \subseteq R$ an anti-Archimedean multiplicative set consisting of nonzero divisors and $X_1, \ldots, X_n$ indeterminates. If R is S-Noetherian, then so is $R[[X_1, \ldots, X_n]]$.*

Proposition 14. *[1, Corollary 11] Let D be an anti-Archimedean domain with quotient field K, $X_1, \ldots, X_n$ indeterminates and set $S = D\backslash\{0\}$. Then $D[[X_1, \ldots, X_n]]$ is S-Noetherian. In particular, $D[[X_1, \ldots, X_n]][K]$ is a Noetherian domain.*

In [25], Liu proved Hilbert basis theorem for the Laurent series.

Theorem 15. *[25, Theorem 3.1] Let R be a ring and $S \subseteq R$ an anti-Archimedean multiplicative set consisting of nonzero divisors and X an indeterminate. If R is S-Noetherian, then so is $R[[X, X^{-1}]]$.*

Let $(M, \leq)$ be a strictly ordered monoid (i.e, $(M, \leq)$ is an ordered monoid satisfying the condition that, if $x_1, x_2, x \in M$ and $x_1 < x_2$, then $x_1 + x < x_2 + x$), and R a ring. Let $[[R^{M,\leq}]]$ be the set of all maps $f : M \to R$ such that $supp(f) = \{x \in M : f(x) \neq 0\}$ is Artinian and narrow. Then $[[R^{M,\leq}]]$ is an abelian group with respect to pointwise addition. For every $x \in M$ and $f, g \in [[R^{M,\leq}]]$, let $X_x(f,g) = \{(y,z) \in M \times M : x = y + z, f(y) \neq 0, g(z) \neq 0\}$. It follows from (4.1) of [32] that $X_x(f,g)$ is finite. This fact allows one to define multiplication on $[[R^{M,\leq}]]$ as $(fg)(x) = \sum_{(y,z)\in X_x(f,g)} f(y)g(z)$. With this operation, and pointwise addition, $[[R^{M,\leq}]]$ becomes a commutative ring, which is called the ring of generalized power series. The elements of $[[R^{M,\leq}]]$ are called generalized power series with coefficients in R and exponents in M. For example, if $M = \mathbb{N}$ and $\leq$ is the usual order, then $[[R^{\mathbb{N},\leq}]] = R[[x]]$, the ring of power series. Examples are given in [26] and [32]. In [8, Theorem 4.3], Ribenboim proved that if $(M, \leq)$ satisfies the condition $0 \leq x$ for every $x \in M$, then $[[R^{M,\leq}]]$ is left Noetherian if and only if R is left Noetherian and M is finitely generated. In [25], Liu extended this result for S-Noetherian rings in the form of following theorem.

Theorem 16. *[25, Theorem 2.3] Let R be a ring and $S \subseteq R$ an anti-Archimedean multiplicative set consisting of nonzero divisors. Let $(M, \leq)$ be a strictly ordered monoid satisfying the condition that $0 \leq x$ for every $x \in M$. Then $[[R^{M,\leq}]]$ is S-Noetherian if and only if R is S-Noetherian and M is finitely generated.*

Corollary 17. *[25, Corollary 2.5] Let R be a ring and $S \subseteq R$ an anti-Archimedean multiplicative set consisting of nonzero divisors. Let M be a submonoid of the additive monoid $\mathbb{N}$ and $\leq$ the usual natural order of $\mathbb{N}$. Then $[[R^{M,\leq}]]$ is S-Noetherian if and only if R is S-Noetherian*

It is well known that there is strong connection between Noetherian modules, stationary ascending chains and maximal element. In fact, Noetherian rings and modules are characterized through stationary ascending chains and maximal element. Consequently, a natural question arises, can we characterize S-Noetherian modules through S-version of stationary ascending chains. Hamed and Sana [14] introduced and studied the concept of S-stationary ascending chains and S-maximal elements as follows:

Definition 18. *[14, Definition 2.1] Let R be a ring, M an R-module and $S \subseteq R$ a multiplicative set.*

1. *An increasing sequence $(N_n)_{n\in\mathbb{N}}$ of submodules of M is called S-stationary if there exist a positive integer k and $s \in S$ such that for each $n \geq k$, $sN_n \subseteq N_k$.*
2. *Let $\mathscr{X}$ be a family of submodules of M. An element $N \in \mathscr{X}$ is said to be S-maximal if there exists an $s \in S$ such that for each $L \in \mathscr{X}$, if $N \subseteq L$ then $sL \subseteq N$.*

Nagata [29] characterized Noetherian modules by their extended submodules. Recall that a submodule N of an R-module M is said to be extended if $N = IM$ for some ideal I of R. Nagata [29] showed that if M is a finitely generated R-module which satisfies the ascending chain condition on extended submodules, then M is a Noetherian R-module. Inspired by this result, Hamed and Sana [14] characterized S-Noetherian modules by their extended submodules.

Theorem 19. *[14, Theorem 2.1] Let R be a ring, $S \subseteq R$ be a multiplicative set, and M an S-finite R-module. We consider the following statements:*

1. *Every nonempty family of extended submodules of M has an S-maximal element.*
2. *Every extended submodule of M is S-finite.*
3. *Every submodule of the form PM is S-finite, where $P \in Spec(R)$ with $P \cap S = \emptyset$.*
4. *M is S-Noetherian.*
5. *Every increasing sequence of extended submodules of M is S-stationary.*

Then $(1) \implies (2) \implies (3) \iff (4) \iff (5)$. *Moreover, if S is finite or S is a countable set and ideals of R are comparable, then* $(5) \implies (1)$.

For finite multiplicative sets, Hamed and Sana [14] proved the following important characterization of S-Noetherian rings.

Theorem 20. *[14, Corollary 2.1] Let R be a ring, $S = \{s_1, s_2, \ldots, s_n\} \subseteq R$ a finite multiplicative set, and M an R-module. Then following statements are equivalent:*

1. *Every nonempty set of ideals of R has an S-maximal element.*
2. *R is an S-Noetherian ring.*
3. *Every increasing sequence of ideals of R is S-stationary.*
4. *Every increasing sequence of S-finite ideals of R is S-stationary.*
5. *Every increasing sequence of finitely generated ideals of R is S-stationary.*

In the above theorem, authors obtained equivalent conditions for being an S-Noetherian ring when S is finite. In 2018, Bilgin et al. [7] extended the above characterizations of a S-Noetherian module and obtained equivalent conditions for a module to be S-Noetherian for an arbitrary multiplicative set. In the following characterization, the hypothesis is weaken to commutative from any ring.

Theorem 21. *[7, Theorem 2.3] Let $S \subseteq R$ be a multiplicative set and M an R-module. Then following are equivalent:*

1. *M is S-Noetherian.*

2. *Every nonempty chain of submodules of M is S-stationary.*

3. *Every nonempty set $\mathscr{F}$ of submodules of M has an S-maximal element.*

Let $\mathscr{R} = (R_n)_{n\in\mathbb{N}}$ be an increasing sequence of unitary rings, $R = \bigcup_{n\in\mathbb{N}} R_n$, and X an indeterminate over R. Let $\mathscr{R}[X]$ (resp., $\mathscr{R}[[X]]$) be the ring of polynomials (resp., power series) with coefficients of degree i in R_i. Then $(\mathscr{R}[[X]], +, .)$ is a subring of the ring of formal power series $R[[X]]$ containing $\mathscr{R}[X]$. Let S be a multiplicative set of R_0. Following [15], the sequence $\mathscr{R} = (R_n)_{n\in\mathbb{N}}$ is said to be S-stationary if there exists a positive integer k, and $s \in S$ such that for all $n \geq k$, $sR_n \subseteq R_k$. In [15], Hamed and Sana have generalized the definition of a Noetherian increasing sequence of commutative rings introduced by Haouat.

Definition 22. *Let $\mathscr{R} = (R_n)_{n\in\mathbb{N}}$ be an increasing sequence of unitary rings and $S \subseteq R_0$ a multiplicative set. We say that $\mathscr{R}$ is S-Noetherian if it satisfies the following conditions:*

1. *R_0 is an S-Noetherian ring.*

2. *The sequence $\mathscr{R} = (R_n)_{n\in\mathbb{N}}$ is S-stationary.*

3. *R_n is an S-finite R_0-module for each $n \in \mathbb{N}$.*

Note: Observe that 0 *belongs to* $\mathbb{N}$.

In [15], Hamed and Sana provided necessary and sufficient conditions for the rings $\mathscr{R}[[X]]$ and $\mathscr{R}[X]$ to be S-Noetherian rings.

Theorem 23. *[15, Theorem 2.1] Let $\mathscr{R} = (R_n)_{n\in\mathbb{N}}$ be an increasing sequence of unitary rings, $R = \bigcup_{n\in\mathbb{N}} R_n$ and $S \subseteq R_0$ an anti-Archimedean multiplicative set, then the following assertions are equivalent:*

1. *The sequence $\mathscr{R} = (R_n)_{n\in\mathbb{N}}$ is S-Noetherian.*

2. *R_0 is an S-Noetherian ring and R is an S-finite R_0-module.*

3. *$\mathscr{R}[X]$ is an S-Noetherian ring.*

Proposition 24. *[15, Proposition 2.3] Let $\mathscr{R} = (R_n)_{n\in\mathbb{N}}$ be an increasing sequence of unitary rings and $S \subseteq R_0$ an anti-Archimedean multiplicative set. If $\mathscr{R} = (R_n)_{n\in\mathbb{N}}$ is an S-Noetherian ring, then the following statements hold:*

1. *For each $n \in \mathbb{N}$, $R_n[X]$ is an S-Notherian ring.*

2. *For each $n \in \mathbb{N}$, $R_0 + XR_n[X]$ is an S-Notherian ring.*

3. *The sequence $(R_n[X])_{n\in\mathbb{N}}$ is S-stationary.*

In [16], Hamed and Malek introduced the notion of S-prime ideals as a generalization of prime ideals and studied many properties of this class of ideals in S-Noetherian rings. Let R be a ring, $S \subseteq R$ a multiplicative set and I an ideal of R disjoint with S. They say that I is an S-prime ideal of R if there exists an $s \in S$ such that for all $a, b \in R$ if $ab \in I$, then $sa \in I$ or $sb \in I$. Note that if S consists of units of R, then notions of S-prime and prime ideal coincide. They studied the basic properties of S-prime ideals and proved S-version of classical results of prime ideals.

The Cohen's theorem is the classic result which states that a ring is Noetherian if and only if its prime ideals are finitely generated. Since this result first appeared in 1950, various extensions have appeared in the literature. Hamed and Malek [16] proved S-version of Cohen's theorem as an extension of this result.

Theorem 25. *[16, Theorem 3] Let R be a ring and $S \subseteq R$ a multiplicative set. Then the following assertions are equivalent:*

1. *R is S-Noetherian.*
2. *Every S-prime ideal of R is S-finite.*
3. *Every prime ideal of R is S-finite.*

Hamed and Malek [16] also characterize the S-Noetherian property in terms of the power series rings in the following result.

Theorem 26. *[16, Theorem 6] Let R be a ring and $S \subseteq R$ a multiplicative set. Then the following assertions are equivalent:*

1. *R is S-Noetherian.*
2. *For each ideal I of R, $sI[[X]] \subseteq IR[[X]] \subseteq I[[X]]$ for some $s \in S$.*

In the particular case when S consists of units of R, Hamed and Malek regain the following well known result.

Corollary 27. *[16, Corollary 5] Let R be a ring. Then R is Noetherian if and only if for each ideal I of R we have $I[[X]] = IR[[X]]$.*

In [24], Lim studied the Nagata ring of S-Noetherian domains and locally S-Noetherian domains. Recall that an integral domain D is said to be locally Noetherian if its localization D_M is Noetherian for each maximal ideals M of D. An integral domain D is said to be locally S-Noetherian if its localization D_M is S-Noetherian for each maximal ideals M of D. An integral domain D is said to be of finite character if each ideal is contained in only finitely many maximal ideals of D. The next result is an S-Noetherian version of the well-known fact that a Noetherian domain is locally Noetherian, and a locally Noetherian domain with finite character is Noetherian [5, Section 7, Exercise 9].

Theorem 28. *[24, Theorem 2] The following statements are true:*

1. *Every S-Noetherian domain is locally S-Noetherian.*

2. *Every locally S-Noetherian domain with finite character is S-Noetherian.*

The converse of Theorem 28(1) does not hold in general. This also indicates the fact that condition with finite character in Theorem 28(2) is essential. For example, if D is an almost Dedekind domain which is not Noetherian, then D is a locally S-Noetherian domain but not S-Noetherian. This is the case when S consists of units in D. Recall that an integral domain D is an almost Dedekind domain if D_M is a Noetherian valuation domain for all maximal ideals M of D.

Let D be a integral domain and $D[X]$ be the polynomial ring over D. For $f \in D[X]$, $c(f)$ denotes the content ideal of f, i.e., the ideal of D generated by the coeffecients of f. Let $T = \{f \in D[X] : c(f) = D\}$. Then T is saturated multiplicative subset of $D[X]$ and the quotient ring $D[X]_N$ is called the Nagata ring of D. It was demonstrated that D is a Noetherian domain if and only if $D[X]$ is a Noetherian domain (see [5, Theorem 7.5]), if and only if $D[X]_T$ is a Noetherian domain ([3, Theorem 2.2(2)]). Lim [24] proved S-Noetherian analogue of these equivalences.

Theorem 29. *[24, Theorem 4] Let D be an integral domain, S an anti-archimedean subset of D, and $T = \{f \in D[X] : c(f) = D\}$. Then the following statements are equivalent:*

1. *D is an S-Noetherian domain.*

2. *$D[X]$ is an S-Noetherian domain.*

3. *$D[X]_T$ is an S-Noetherian domain.*

In the next theorem, Lim [24] also studied locally S-Noetherian domains in terms of Nagata ring.

Theorem 30. *[24, Theorem 7] Let D be an integral domain, S an anti-archimedean subset of D, and $T = \{f \in D[X] : c(f) = D\}$. Then the following statements are equivalent:*

(1) D is a locally S-Noetherian domain.

(2) Nagata ring $D[X]_T$ is a locally S-Noetherian domain.

3 Generalizations of S-Noetherian rings and modules

A ring R is said to have Noetherian spectrum if R satisfies the ascending chain condition (ACC) on radical ideals. This is equivalent to the condition that R satisfies the ACC on prime ideals and each ideal has only finitely many prime ideals minimal over it. Note that every Noetherian ring has Noetherian spectrum and the converse is not true in general (see [10]). For this consider the following example.

Example 31 (A non-Noetherian ring with Noetherian spectrum)**.** *Let F be a field, and $A = F[X_1, X_2, \ldots, X_n, \ldots]$ a polynomial ring over F in countably many indeterminates. Let I be an ideal of A generated by X_1^2 and $X_n - X_{n+1}^2$ for all $n \geq 1$. Write $Y_n = \overline{X}_n$ in $B = A/I$. Then $Y_1^2 = \overline{X}_1^2 = 0$ and $Y_n = Y_{n+1}^2$ for all $n \geq 1$ in B. Now*

$$\begin{array}{c} Y_1 = Y_2^2 \implies Y_1^2 = Y_2^4 \implies Y_2^4 = 0, \\ Y_2 = y_3^2 \implies Y_2^4 = Y_3^8 \implies Y_3^8 = 0 \\ \vdots \end{array}$$

and so $Y_n^{2^n} = 0$ for all $n \geq 2$. If $P = (Y_1, Y_2, \ldots)$, we have $B/P \cong F$, so P is maximal. On the other hand, the generators of P are nilpotent, so P is contained in the nilradical of B, and hence is the unique minimal prime as well. Since all primes contain the maximal ideal of P, P is the only prime of B. Thus Spec(B)=$\{P\}$ is obviously has Noetherian spectrum. But $(Y_1) \subsetneq (Y_2) \subsetneq (Y_3) \subsetneq \cdots$ is an infinite ascending chain of ideals which is not stationary, so B is not Noetherian.

This concept is useful for characterization of Laskerian modules. For instance, a finitely generated module M over a ring R is Laskerian if and only if $R/(AnnM)$ has Noetherian spectrum and for every proper submodule N of M, there is a prime ideal P minimal over $(N : M)$ and an element r in $R \setminus P$ for which the submodule $(N : r)$ is P-primary (see [17, Proposition 2.1]). Motivated by this notion, Hamed [13] generalized the Noetherian spectrum condition by introducing the following definition of rings satisfying the S-Noetherian spectrum condition. This concept is a generalization of S-Noetherian rings.

Definition 32. *[13, Definition 2.1] Let R be a ring and $S \subseteq R$ a multiplicative set. We say that an ideal I of R is radically S-finite if there exist an element $s \in S$ and a finitely generated ideal J of R such that $sI \subseteq \sqrt{J} \subseteq \sqrt{I}$. We also define R to be satisfy the S-Noetherian spectrum property if each ideal of R is radically S-finite.*

Example 33. *[13] Let R be a ring and $S \subseteq R$ a multiplicative set.*

1. *Every S-finite ideal of R is radically S-finite. Indeed, if I is an S-finite ideal of R, then there exists $s \in S$ and a finitely generated ideal J of R such that $sI \subseteq J \subseteq I$. So $sI \subseteq \sqrt{sI} \subseteq \sqrt{J} \subseteq \sqrt{I}$. Hence I is radically S-finite. In particular, every S-Noetherian ring has S-Noetherian spectrum.*

2. *Let I be an ideal of R. If S consists of units of R, then I is radically S-finite if and only if $\sqrt{I} = \sqrt{J}$ for some finitely generated subideal J of I. So R has S-Noetherian spectrum if and only if R has Noetherian spectrum.*

The following example shows that there is a ring R with S-Noetherian spectrum which does not satisfy the Noetherian spectrum property.

Example 34. *Let D be an integral domain whose prime spectrum is not Noetherian and let $S = D \setminus \{0\}$. Then S is a multiplicative subset of D and for each nonzero ideal I of D, $I \cap S \neq \phi$. Thus every nonzero ideal of D is S-finite. So D is an S-Noetherian domain, and hence by the Example 33(1), D has S-Noetherian spectrum.*

In the following theorem, Hamed [13] proved an S-version of the result of Ohm and Pendleton [31] for at most countable multiplicative subset.

Theorem 35. *[13, Theorem 2.1] Let R be a ring and $S \subseteq R$ an at most countable multiplicative set. Then the following statements are equivalent:*

1. *R satisfies the S-Noetherian spectrum property.*
2. *Every increasing sequence of radical ideals of R is S-stationary.*

In the particular case when $S = \{1\}$, Hamed [13] obtained the result of Ohm and Pendleton [31].

Corollary 36. *[13, Corollary 2.1] Let R be a ring. Then the following statements are equivalent:*

1. *Every increasing sequence of radical ideals of R is stationary (R has Noetherian spectrum).*
2. *Every ideal I of R, $\sqrt{I} = \sqrt{J}$ for some finitely generated subideal J of I.*

In the next theorem, Hamed [13] gave the S-invariant of the Cohen-type theorem for rings satisfying the S-Noetherian spectrum property.

Theorem 37. *[13, Theorem 2.2] Let R be a ring and $S \subseteq R$ a multiplicative set. Then the following statements are equivalent:*

1. *R satisfies the S-Noetherian spectrum property.*
2. *Every radical ideal of R is radically S-finite.*
3. *Every prime ideal of R is radically S-finite.*

In [31], Ohm and Pendleton studied the Hilbert basis theorem for a ring to satisfy the Noetherian spectrum condition. They showed that a commutative ring R has Noetherian spectrum if and only if the polynomial ring $R[X]$ has Noetherian spectrum. However, in case of the power series ring, Ribenboim [33] gave an example of a commutative ring R, which has Noetherian spectrum such that $R[[X]]$ has non Noetherian spectrum. In the following result, Hamed [13] proved the Hilbert basis theorem for the property S-Noetherian spectrum.

Theorem 38. *[13, Theorem 3.1] Let R be a ring and $S \subseteq R$ a multiplicative set. Then R has S-Noetherian spectrum if and only if the polynomial ring $R[X]$ has S-Noetherian spectrum.*

Rings and modules satisfying the accr condition were introduced by Lu in [27]: an R-module M satisfies the accr conditions if every ascending chain of submodules of M of the form $(N : J) \subseteq (N : J^2) \subseteq (N : J^3) \subseteq \cdots$ terminates for every submodule N of M and every finitely generated ideal J of R. The ring R satisfies the accr condition if it satisfies (accr) as a module over itself. This concept generalizes the concept of Noetherian rings and modules. In [14], Hamed and Sana generalized the accr condition by introducing the definition of modules and rings satisfying the S-accr condition. This concept is also a generalization of the concept of S-Noetherian rings and modules.

Definition 39. *[14, Definition 3.1] Let R be a ring and $S \subseteq R$ a multiplicative set and M an R-module. Then M is said to satisfy S-accr if the ascending chain of residuals of the form $(N : J) \subseteq (N : J^2) \subseteq (N : J^3) \subseteq \cdots$ is S-stationary for every submodule N of M and every finitely generated ideal J of R. In particular, the ring R satisfies S-accr if it does as an R-module.*

Remark 40. *[14, Notation 3.1] Let M be an R-module, $S \subseteq R$ be a multiplicative set, and J a finitely generated ideal of R such that $S \cap J = \emptyset$. We set $T = \{s - \alpha : s \in S \text{ and } \alpha \in J\}$. Then T is a nonempty multiplicative set of R, and $0^T = \{m \in M : \exists\, t \in T \text{ and } tm = 0\}$ is a submodule of M.*

The following theorem is the S-version of the Krull's intersection theorem.

Theorem 41. *[14, Theorem 3.1] Let R be a ring, $S \subseteq R$ be a multiplicative set, and M an R-module satisfying S-accr. Then $\bigcap_{k\geq 1} J^k M \subseteq 0^T$ for every finitely generated ideal J of R such that $S \cap J = \phi$. In particular, if M is a torsion-free R-module, then $\bigcap_{k\geq 1} J^k M = 0$.*

The following two results provide basic properties of this class of modules.

Theorem 42. *[14, Theorem 3.2] Let N be a submodule of an R-module M and $S \subseteq R$ a multiplicative set. Then M satisfies S-accr if and only if N and M/N satisfy S-accr.*

Theorem 43. *[14, Theorem 3.3] Let R be a ring satisfying S-accr, where $S \subseteq R$ is a multiplicative set. If M is a finitely generated R-module, then M satisfies S-accr.*

In the next result, Hamed and Sana [14] provide a relation between the concept of S-accr and S-Noetherian for finite multiplicative set.

Theorem 44. *[14, Theorem 3.4] Let R be a ring and $S \subseteq R$ a finite multiplicative set. Then the ring $R[X]$ satisfies S-accr if and only if R is an S-Noetherian ring.*

Theory of graded rings and modules extends the theory of rings and modules. Let G be an abelian group with identity element e. Then a ring R is called G-graded if $R = \bigoplus_{g\in G} R_g$ for additive subgroups R_g and $R_gR_h \subseteq R_{gh}$ for every $g, h \in G$. An element of $h(R) = \cup_{g\in G} R_g$ is called the homogeneous element. An R-module M is called G-graded if $M = \bigoplus_{g\in G} M_g$ for additive subgroups M_g and $R_gM_h \subseteq M_{gh}$ for every $g, h \in G$. A submodule N of M is called graded if $N = \bigoplus_{g\in G}(N \cap M_g)$. Similarly, an ideal I of R is called graded if $I = \bigoplus_{g\in G}(I\cap R_g)$. A G-graded R-module M is called G-graded Noetherian if each graded submodule of M is finitely generated. A G-graded ring R is called G-graded Noetherian if it is G-graded Noetherian module over itself. Goto and Yamagishi [11] characterized G-graded Noetherian rings in terms of Noetherian rings. More precisely, they proved that a G-graded ring R is G-graded Noetherian if and only if R is Noethrian, provided G is finitely generated. Inspired by this, Kim and Lim [19] introduced the notion of G-graded S-Noetherian ring as a generalization of both the S-Noetherian rings and G-graded Noetherian rings and extended previous result to this class.

Definition 45. *[19] Let G be an abelian group, $R = \bigoplus_{g\in G} R_g$ a G-graded ring and $S \subseteq R_e$ a multiplicative set. Then R is said to be a G-graded S-Noetherian ring if every graded ideal of R is S-finite.*

Among other results, Kim and Lim [19] characterized S-Noetherian rings in terms of G-graded S-Noetherian rings as a main result in their paper.

Theorem 46. *[19, Theorem 1] Suppose that G is a finitely generated abelian group. Let $R = \bigoplus_{g\in G} R_g$ a G-graded ring and let S be an anti-Archimedean multiplicative subset of R_e. Then the following statements are equivalent.*

1. *R is an S-Noetherian ring.*
2. *R is a graded S-Noetherian ring.*
3. *R_e is an S-Noetherian ring and R is an S-finite R_e-algebra.*

Motivated by the concept of G-graded S-Noetherian rings, Ansari and Sharma [4] introduced the notion of G-graded S-Noetherian module as a generalization of S-Noetherian module.

Definition 47. *[4, Definition 3.1] Let M be a G-graded R-module and $S \subseteq h(R)$ be a multiplicative set. Then M is called S-finite if there exists $s \in S$ and a finitely generated graded submodule F of M such that $sM \subseteq F$. Also, M is called G-graded S-Noetherian if each graded submodule of M is S-finite.*

Every S-Noetherian module is a G-graded S-Noetherian module but converse is not true in general. For this, Ansari and Sharma [4] provided the following example.

Example 48. *[4, Example 3.3] Let $R = K[X_1^{\pm 1}, X_2^{\pm 1}, \ldots, X_n^{\pm 1}, \ldots]$ be a Laurent polynomial ring in infinitely many indeterminates over a field K. Consider the group $G = \bigoplus_{i=1}^{\infty} \mathbb{Z}$ and multiplicative set $S = K \setminus \{0\}$ of R. Then R is a G-graded field with canonical G-grading. Consequently, R is a G-graded S-Noetherian R-module since it has only two graded submodules, namely 0 and R. However, R is not an S-Noetherian R-module.*

Let R be ring, S a multiplicative set of R, and A an R-algebra. Then A is said to be an S-finite R-algebra if there exist $s \in S$ and $a_1, a_2, \ldots, a_n \in A$ such that $sA \subseteq R[a_1, a_2, \ldots, a_n]$. Also, recall from [1] that a multiplicative set S of a ring R is called anti-Archimedean if $\cap_{k=1}^{\infty} s^k R \cap S \neq \emptyset$ for all $s \in S$. Let H be a subgroup of G. Then $R_H = \bigoplus_{h \in H} R_h$ is an H-graded ring. In fact R_H is a G-graded ring. The following theorem provides a connection between G-graded S-Noetherian rings and S-finite algebra.

Theorem 49. *[4, Theorem 3.35] Let R be a G-graded ring, H a subgroup of G such that G/H is finitely generated, and S an anti-Archimedean multiplicative set of $h(R_H)$. If R is G-graded S-Noetherian, then R is an S-finite R_H-algebra.*

In the next result, Ansari and Sharma [4] proved the Hilbert's basis theorem for G-graded S-Noetherian rings. This result is a generalization of [1, Proposition 9].

Proposition 50. *[4, Proposition 3.36] Let R be a G-graded ring and $S \subseteq h(R)$ an anti-Archimedean multiplicative set. If R is G-graded S-Noetherian, then so is the polynomial ring $R[x]$.*

In the following theorem, Ansari and Sharma [4] characterized S-Noetherian modules in terms of G-graded S-Noetherian modules for a countable multiplicative set.

Theorem 51. *[4, Theorem 3.28] Let G be a finitely generated abelian group, M a G-graded R-module and $S \subseteq R_e$ a countable multiplicative set. Then M is a G-graded S-Noetherian R-module if and only if M is an S-Noetherian R-module.*

For arbitrary abelian group G, the next theorem provides another characterization of G-graded S-Noetherian modules in terms of S-Noetherian modules.

Theorem 52. *[4, Theorem 3.32] Let R be a strongly G-graded ring, M a G-graded R-module and $S \subseteq R_e$ a multiplicative set. Then $M = \bigoplus_{g \in G} M_g$ is a G-graded S-Noetherian R-module if and only if M_e is an S-Noetherian R_e-module.*

Corollary 53. *[4, Corollary 3.33] If G is finitely generated, R a strongly G-graded ring, S a countable multiplicative set of R_e and M a G-graded R-module. Then the following are equivalent:*

1. *M is a G-graded S-Noetherian R-module.*
2. *M is an S-Noetherian R-module.*
3. *M_e is an S-Noetherian R_e-module.*

In [4], Ansari and Sharma presented an example which shows that the condition strongly graded in the above theorem is not superflous.

Example 54. *[4, Example 3.34] Let $G = \mathbb{Z}$, $R = \mathbb{Z} = R_0$ and $M = \mathbb{Z}_4^{(\mathbb{N})}$ (Direct sum of countable copies of $\mathbb{Z}_4$) be a naturally G-graded R-module. Take the multiplicative set $S = \{3^n : n \geq 0\}$. Then $M_0 = \mathbb{Z}_4$ is a G-graded S-Noetherian R_0-module but M is not a G-graded S-Noetherian R-module.*

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