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SOME ONE AND TWO PARAMETER ESTIMATORS FOR THE MULTICOLLINEAR GAUSSIAN LINEAR REGRESSION MODEL: SIMULATIONS AND APPLICATIONS

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Abstract. The ordinary least square estimator is inefficient when there exists multicollinearity among regressors in linear regression model. There are many methods available in literature to solve the multicollinearity problem. In this study, we consider some one and two parameter estimators for estimating the regression parameters. We theoretically compared the estimators in terms of smaller mean squared error (MSE) criteria. A Monte Carlo simulation study has been conducted to compare the performance of the estimators numerically. Finally, for illustration purposes, a real-life data is analyzed.

1 Introduction

In linear regression model, we assume that the predictor or explanatory variables are independent. The ordinary least square (OLS) estimator is considered the Best Linear Unbiased Estimator (BLUE) when the independence assumption is valid. However, in practice, there can be strong or near-to-strong linear relationships among the predicted variables. The independent assumption is no longer valid in this case, and it causes the problem of multicollinearity. When there is a multicollinearity between predictor variables, it is not possible to estimate the unique effects of the individual variables in the regression model. Also, the OLS will prove to be inefficient, and the regression coefficients give the wrong signs with large standard error and imprecise confidence intervals (Kibria, 2003 [18]). In this case, multicollinearity becomes one of the most serious issues in linear regression analysis. There are several ways exist to address the multicollinearity problem. For examples, (i) Increase the sample size, (ii) Remove some of the highly correlated independent variables. (iii) Linearly combine the independent variables, (iv) Principal components analysis (v) partial least squares regression and (vi) ridge regression method. However, ridge

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regression method proposed by Hoerl and Kennard (1970) [12] is one of the popular and recommended methods. Several authors have proposed different methods to solve this problem by using different biasing estimators as an alternative to the OLS estimator. To mention a few Hoerl and Kennard (1970) [12], Saleh and Kibria (1993) [7], Kibria (2003) [18], Khalaf and Shukur (2005) [17], Mnuiz et al. (2009) [29], Kibria and Banik (2016) [20]. Kibria and Lukman (2020) [21], James- Stein (1956)[32] , Swindel (1976) [33], Dawoud and Kibria (2020) [5], among others.

The ordinary ridge regression estimator is proposed by Hoerl and Kennard (1970) [12], which is regarded as one of the most widely used among these estimators. They overcome the multicollinearity by using a positive value, k, which is known as the biasing or ridge parameter. A major problem is to choose a biasing parameter k because it plays a very important role in controlling the regression's bias. Several researchers were involved for estimating the biasing parameter k. Among them: Hoerl and Kennard (1970) [12], McDonald and Galarneeau (1975) [28], Hocking et al. (1976) [11], Lawless and Wang (1976) [14], Nomura (1988) [30], Firinguetti (1989) [9], Kibria (2003) [18], Batach et al. (2008) [4], Mansson et al. (2010) [26], and very recently Kibria (2022) [19] are notable.

Liu (1993)[16] proposed another estimator using different biasing parameter d and it combined the stein estimator (1956)[32] and the ordinary ridge regression estimator by Hoerl and Kennard (1970) [12]. It is noted that the ridge regression estimator is a complicated and non-linear function of k, while the Liu estimator is a linear function of d. There are many works have been done on Liu estimators for both linear and non-linear regression models. To mention a few, Kibria et al. (2012) [22], Alheety and Kibria (2013) [1], Mansson et al. (2012 [25], 2017 [27]) among others

There are several researchers, who proposed two-parameter estimators using both biasing parameters k and d to get extra benefit over ridge and Liu estimators. To mentions a few, Lukman et al. (2019) [23], Dorugade (2014) [6], Yang and Chang (2010) [37], Owalabi et. al (2022) [31], and very recently Wu and Asar (2022) [36] and reference there in.

The objective of this paper is to compare the performance of the estimators, namely, OLS, ridge regression, Liu estimator, Modified Liu estimator by Lukman et al. (2020)[24], James-stein estimator, Kibria and Lukman (2020) [21], Dorugade (2014) [6], and Modified ridge type (MRT) using a Monte Carlo simulation and real life data. Based on the smaller MSE criteria, we want to recommend some good estimators for the practitioners.

The organization of this paper is as follows. The statistical methodology is given in section 2 with the theoretical MSE comparison among the estimators. The estimation of the parameters k and d are described in section 3. A simulation study has been conducted in section 4. A real-life data is analyzed in section 5, and a Summary and Concluding remarks are given in section 6.

2 Statistical Methodology

In this section, we discuss the canonical form of linear regression model and different type of estimators, their bias, variance, and MSE. We organize as follows: section 2.1 discuss about different type of existing estimators, in section 2.2, we discuss their bias, variance, and MSE and comparison among those estimators in section 2.3.

2.1 Model and Some Existing Estimators

We consider the following linear regression model (LRM)

$$y = X\beta + \varepsilon, \varepsilon \sim N(0, \sigma^2 I_n) \ , \tag{2.1}$$

where y is a $n \times 1$ vector of the response variable, X is a known $n \times p$ full rank matrix of predictor or explanatory variables, β is an $p \times 1$ vector of unknown regression coefficients and ε is a $n \times 1$ vector of residuals such that $E(\varepsilon) = 0$, and the $V(\varepsilon) = \sigma^2 I_n$, I_n is a $n \times n$ identity matrix. The ordinary least square estimators (OLS) of β in (2.1) is defined as.

$$\hat{\beta}_{OLS} = (S)^{-1} X^{'} y \ , \tag{2.2}$$

where $S = X^{'}X$ is the design matrix.
Let, we have an orthogonal matrix R and $R^{'}X^{'}XR = \Lambda = diag\,(\lambda_1, \lambda_2, \ldots, \lambda_p)$, where λ_i is the i^{th} eigen value of $X^{'}X$. Λ and R are the matrices of eigen values and eigen vectors of $X^{'}X$, respectively and the equation 2.1 can be written as:

$$y = Z\vartheta + \varepsilon \ , \tag{2.3}$$

where $Z = XR$, $\vartheta = R^{'}\beta$ and $Z^{'}Z = \Lambda$.

Now, the OLS estimator of ϑ is

$$\hat{\vartheta}_{OLS} = \Lambda^{-1} Z^{'} y \ , \tag{2.4}$$

Hoerl and Kennard (1970)[12] proposed the ridge estimator (RE) of ϑ as

$$\hat{\vartheta}_{RE}(k) = A_k \hat{\vartheta}_{OLS} \ , \tag{2.5}$$

where $A_k = \Lambda(\Lambda + kI)^{-1}$ and k is the biasing parameter.

Liu (1993)[16] combined Stein estimator with an RE and it is defined as

$$\hat{\vartheta}_{Liu}(d) = (\Lambda + I)^{-1}(\Lambda + dI)\hat{\vartheta}_{OLS} \ , \tag{2.6}$$

Modified Liu estimator (MLiu) is proposed by Kibria and Lukman (2020)[21] and defined as:

$$\hat{\vartheta}_{MLiu}(d) = (\Lambda + I)^{-1}(\Lambda - dI)\hat{\vartheta}_{OLS} , \tag{2.7}$$

James - Stein (1956)[32] estimator (JSE) is also defined as:

$$\hat{\vartheta}_{JSE} = c\hat{\vartheta}_{OLS} , \tag{2.8}$$

where $0 < c < 1$ and for the selection of c, we consider as

$$c = \frac{(\hat{\vartheta}'_{OLS}\hat{\vartheta}_{OLS})}{(\hat{\vartheta}'_{OLS}\hat{\vartheta}_{OLS} + \sigma^2 trace(\Lambda)^{-1})} \tag{2.9}$$

Kibria and Lukman (2020)[21] also introduced an estimator (KL) which is defined as:

$$\hat{\vartheta}_{KL}(k) = W_k M_k \hat{\gamma}_{OLS} , \tag{2.10}$$

where $W_k = (\Lambda + kI)^{-1}$, and $M_k = (\Lambda - kI)$.

Dorugade (2014)[6] also introduced a two parameter estimator (D), which is defined as:

$$\hat{\vartheta}_D(k,d) = D_{(k,d)}\hat{\vartheta}_{OLS} , \tag{2.11}$$

where $D_{(k,d)} = \Lambda(\Lambda + kdI)^{-1}$, k and d are biasing parameters.

And the modified ridge estimator (MRT) by Lukman et al. (2019)[23] is:

$$\hat{\vartheta}_{MRT}(k,d) = M_{(k,d)}\hat{\vartheta}_{OLS} , \tag{2.12}$$

where $M(k,d) = \Lambda(\Lambda + k(1+d)I)^{-1}$, $k > 0$ and $0 < d < 1$.

2.2 Bias, Variance and MSE of the Estimators

The bias, variance and MSE of the OLS estimator are,

$$Bias[\hat{\vartheta}_{OLS}] = 0 \tag{2.13}$$

$$Var[\hat{\vartheta}_{OLS}] = \sigma^2\Lambda^{-1} \tag{2.14}$$

$$MSEM[\hat{\vartheta}_{OLS}] = \sigma^2\Lambda^{-1} \tag{2.15}$$

The bias, variance and MSE of the RE estimator are,

$$Bias[\hat{\vartheta}_{RE}(k)] = (A_k - I)\vartheta \tag{2.16}$$

$$Var[\hat{\vartheta}_{RE}(k)] = \sigma^2 A_k\Lambda^{-1}A_k' \tag{2.17}$$

$$MSEM[\hat{\vartheta}_{RE}(k)] = \sigma^2 A_k\Lambda^{-1}A_k' + (A_k - I)\vartheta\vartheta'(A_k - I)' \tag{2.18}$$

The bias, variance and MSE of the Liu estimator are,

$$Bias[\hat{\vartheta}_{Liu}(d)] = -(\Lambda + I)^{-1}(1-d)\vartheta \tag{2.19}$$

$$Var[\hat{\vartheta}_{Liu}(d)] = \sigma^2(\Lambda + I)^{-1}(\Lambda + dI)\Lambda^{-1}(\Lambda + dI)(\Lambda + I)^{-1} \tag{2.20}$$

$$MSEM[\hat{\vartheta}_{Liu}(d)] = \sigma^2(\Lambda + I)^{-1}(\Lambda + dI)\Lambda^{-1}(\Lambda + dI)(\Lambda + I)^{-1} + (1-d)^2(\Lambda + I)^{-1}\vartheta\vartheta'(\Lambda + I)^{-1} \tag{2.21}$$

The bias, variance and MSE of the MLiu estimator are,

$$Bias[\hat{\vartheta}_{MLiu}(d)] = -(\Lambda + I)^{-1}(1+d)\vartheta \tag{2.22}$$

$$Var[\hat{\vartheta}_{MLiu}(d)] = \sigma^2(\Lambda + I)^{-1}(\Lambda - dI)\Lambda^{-1}(\Lambda - dI)(\Lambda + I)^{-1} \tag{2.23}$$

$$MSEM[\hat{\vartheta}_{MLiu}(d)] = \sigma^2(\Lambda + I)^{-1}(\Lambda - dI)\Lambda^{-1}(\Lambda - dI)(\Lambda + I)^{-1} + (1+d)^2(\Lambda + I)^{-1}\vartheta\vartheta'(\Lambda + I)^{-1} \tag{2.24}$$

The bias, variance and MSE of the JSE estimator are,

$$Bias[\hat{\vartheta}_{JSE}] = (c-1)\vartheta \tag{2.25}$$

$$Var[\hat{\vartheta}_{JSE}] = c^2\sigma^2\Lambda^{-1} \tag{2.26}$$

$$MSEM[\hat{\vartheta}_{JSE}] = c^2\sigma^2\Lambda^{-1} + (c-1)^2\vartheta\vartheta' \tag{2.27}$$

where c can be obtained as:

$$tr(MSE(JSE)) = c^2\sigma^2 tr(S^{-1}) + (c-1)^2\vartheta'\vartheta$$

$$\frac{\partial tr(MSE(JSE))}{\partial c} = 2c\sigma^2 tr(\Lambda^{-1}) + 2(c-1)\vartheta'\vartheta = 0$$

$$\Leftrightarrow c\sigma^2 tr(\Lambda^{-1}) + c\vartheta'\vartheta = \vartheta'\vartheta$$

$$\Leftrightarrow c = \frac{\vartheta'\vartheta}{\vartheta'\vartheta + \sigma^2 tr(\Lambda^{-1})}$$

So,

$$c = \frac{(\hat{\vartheta}'_{OLS}\hat{\vartheta}_{OLS})}{(\hat{\vartheta}'_{OLS}\hat{\vartheta}_{OLS} + \sigma^2 tr(\Lambda^{-1})} \tag{2.28}$$

The bias, variance and MSE of the KL estimator are:

$$Bias[\hat{\vartheta}_{KL}(k)] = (W_k M_k - I)\vartheta \tag{2.29}$$

$$Var[\hat{\vartheta}_{KL}(k)] = \sigma^2 W_k M_k \Lambda^{-1} M'_k W'_k \tag{2.30}$$

$$MSEM[\hat{\vartheta}_{KL}(k)] = \sigma^2 W_k M_k \Lambda^{-1} M'_k W'_k + (W_k M_k - I)\vartheta\vartheta'(W_k M_k - I)' \tag{2.31}$$

The bias, variance and MSE of the D estimator are:

$$Bias[\hat{\vartheta}_D(k,d)] = (D_{(k,d)} - I)\vartheta \quad (2.32)$$

$$Var[\hat{\vartheta}_D(k,d)] = \sigma^2 D_{(k,d)}\Lambda^{-1}D^{'}_{(k,d)} \quad (2.33)$$

$$MSEM[\hat{\vartheta}_D(k,d)] = \sigma^2 D_{(k,d)}\Lambda^{-1}D^{'}_{(k,d)} + (D_{(k,d)} - I)\vartheta\vartheta^{'}(D_{(k,d)} - I)^{'} \quad (2.34)$$

The bias, variance and MSE of the MRT estimator are:

$$Bias[\hat{\vartheta}_{MRT}(k,d)] = (M_{(k,d)} - I)\vartheta \quad (2.35)$$

$$Var[\hat{\vartheta}_{MRT}(k,d)] = \sigma^2 M_{(k,d)}\Lambda^{-1}M^{'}_{(k,d)} \quad (2.36)$$

$$MSEM[\hat{\vartheta}_{MRT}(k,d)] = \sigma^2 M_{(k,d)}\Lambda^{-1}M^{'}_{(k,d)} + (M_{(k,d)} - I)\vartheta\vartheta^{'}(M_{(k,d)} - I)^{'} \quad (2.37)$$

The following notations and lemmas are needful to comparison between these estimators

Lemma 2.1. *Let $n \times n$ matrices $M > 0$, $N > 0$ (or $N \geq 0$), then $M > N$ if and only if $\lambda_i(NM^{-1}) < 1$, where $\lambda_i(NM^{-1})$ is the largest eigenvalue of matrix NM^{-1}.*

Lemma 2.2. *(Farebrother, 1976)[8]: Let M be an $n \times n$ positive definite matrix, let α be a non-zero $n \times 1$ column matrix and let c be a positive scalar. Then $cM - \alpha\alpha^{'} > 0$ iff $\alpha^{'}M^{-1}\alpha < d$, and $cM - \alpha\alpha^{'} \geq 0$ iff $\alpha^{'}M^{-1}\alpha \leq d$.*

Lemma 2.3. *(Trenkler and Toutenburg, 1990)[34]: Let $\hat{\alpha}_i = A_i y, i = 1,2$ be two linear estimators of α. Suppose that $D = Cov(\hat{\alpha}_1) - Cov(\hat{\alpha}_2) > 0$, where $Cov(\hat{\alpha}_i), i = 1,2$ denotes the covariance matrix of $\hat{\alpha}_i$ and $b_i = Bias(\hat{\alpha}_i) = (A_i X - I)\alpha, i = 1,2$. Consequently,*

$$\Delta(\hat{\alpha}_1 - \hat{\alpha}_2) = MSEM(\hat{\alpha}_1) - MSEM(\hat{\alpha}_2) = \sigma^2 D + b_1 b_1^{'} - b_2 b_2^{'} > 0$$

if and only if $b_2^{'}[\sigma^2 D + b_1 b_1^{'}]^{-1} b_2 < 1$, where $MSEM(\hat{\alpha}_i) = Cov(\hat{\alpha}_i) + b_i b_i^{'}$

2.3 Theoretical Comparison among the Estimators

Following Akram (2022)[2], we have made all theoretical comparison in this section.

2.3.1 Comparison between Ridge Regression estimator and OLS estimator

The difference between MSEM($\hat{\vartheta}_{RE}(k)$) and MSEM($\hat{\vartheta}_{OLS}$) is obtained by

$$MSEM[\hat{\vartheta}_{OLS}] - MSEM[\hat{\vartheta}_{RE}(k)] = \sigma^2\Lambda^{-1} - \sigma^2 A_k \Lambda^{-1} A_k^{'} - (A_k - I)\vartheta\vartheta^{'}(A_k - I)^{'} \quad (2.38)$$

Let $k > 0$, then we have the following theorem.

Theorem 2.4. *If $k > 0$, b_k = Bias ($\hat{\vartheta}_{RE}(k)$) is the bias of RE, the estimator $\hat{\vartheta}_{RE}(k)$ is better than that of $\hat{\vartheta}_{OLS}$ using the criterion of MSEM, that is, MSEM[$\hat{\vartheta}_{OLS}$] - MSEM[$\hat{\vartheta}_{RE}(k)$] > 0 if and only if,*

$$b_k^{'}[\sigma^2(\Lambda^{-1} - A_k\Lambda^{-1}A_k^{'})]^{-1}b_k < 1$$

Proof. Using the difference between scalar MSE functions,

$$\begin{aligned} MSEM(\hat{\vartheta}_{OLS}) - MSEM(\hat{\vartheta}_{RE}(k)) &= \sigma^2(\Lambda^{-1} - A_k\Lambda^{-1}A_k^{'}) - b_k b_k^{'} \\ &= \sigma^2 diag\left\{\frac{1}{\lambda_i} - \frac{\lambda_i}{(\lambda_i+k)^2}\right\}_{i=1}^{p} - b_k b_k^{'} \qquad (2.39) \end{aligned}$$

where, $\Lambda^{-1} - A_k\Lambda^{-1}A_k^{'}$ is positive definite if and only if $(\lambda_i + k)^2 - \lambda_i^2 > 0$. For $k > 0$, we observed that $2\lambda_i + k > 0$. So consequently, $\Lambda^{-1} - A_k\Lambda^{-1}A_k^{'}$ is positive definite. By lemma 2.2, the proof is completed. □

2.3.2 Comparison between Liu estimator and OLS estimator

The difference between MSEM($\hat{\vartheta}_{Liu}(d)$) and MSEM($\hat{\vartheta}_{OLS}$) is obtained by

$$\begin{aligned} MSEM[\hat{\vartheta}_{OLS}] - MSEM[\hat{\vartheta}_{Liu}(d)] &= \sigma^2\Lambda^{-1} - \sigma^2(\Lambda+I)^{-1}(\Lambda+dI)\Lambda^{-1}(\Lambda+dI) \\ &\quad (\Lambda+I)^{-1} - (1-d)^2(\Lambda+I)^{-1}\vartheta\vartheta^{'}(\Lambda+I)^{-1} \end{aligned} \qquad (2.40)$$

We have the following theorem.

Theorem 2.5. *If $k > 0$ and $0 < d < 1$, b_{Liu} = Bias($\hat{\vartheta}_{Liu}(d)$) is the bias of Liu estimator, the estimator $\hat{\vartheta}_{Liu}(d)$ is better than to the estimator $\hat{\vartheta}_{OLS}$ using the criterion of MSEM, that is, MSEM[$\hat{\vartheta}_{OLS}$] - MSEM[$\hat{\vartheta}_{Liu}(d)$] > 0 if and only if,*

$$b_{Liu}^{'}[\sigma^2(\Lambda^{-1} - (\Lambda+I)^{-1}(\Lambda+dI)\Lambda^{-1}(\Lambda+dI)(\Lambda+I)^{-1})]b_{Liu} < 1$$

Proof. Using the difference between scalar MSE functions,

$$\begin{aligned} MSEM(\hat{\vartheta}_{OLS}) - MSEM(\hat{\vartheta}_{Liu}(d)) &= \sigma^2(\Lambda^{-1} - (\Lambda+I)^{-1}(\Lambda+dI)\Lambda^{-1}(\Lambda+dI)(\Lambda+I)^{-1}) \\ &\quad - b_{Liu}b_{Liu}^{'} \\ &= \sigma^2 diag\left\{\frac{1}{\lambda_i} - \frac{(\lambda_i+d)^2}{\lambda_i(\lambda_i+1)^2}\right\}_{i=1}^{p} - b_{Liu}b_{Liu}^{'} \qquad (2.41) \end{aligned}$$

where, $(\Lambda^{-1} - (\Lambda+I)^{-1}(\Lambda+dI)\Lambda^{-1}(\Lambda+dI)(\Lambda+I)^{-1})$ is positive definite if and only if $(\lambda_i + 1)^2 - (\lambda_i + d)^2 > 0$. For $k > 0$ and $0 < d < 1$, we observed that $2\lambda_i + (1 + d) > 0$. So, $(\Lambda^{-1} - (\Lambda+I)^{-1}(\Lambda+dI)\Lambda^{-1}(\Lambda+dI)(\Lambda+I)^{-1})$ is positive definite. By lemma 2.2, the proof is completed. □

2.3.3 Comparison between Modified Liu estimator and OLS estimator

The difference between MSEM($\hat{\vartheta}_{Mliu}(d)$) and MSEM($\hat{\vartheta}_{OLS}$) is obtained by

$$MSEM[\hat{\vartheta}_{OLS}] - MSEM[\hat{\vartheta}_{MLiu}(d)] = \sigma^2\Lambda^{-1} - \sigma^2(\Lambda + I)^{-1}(\Lambda - dI)\Lambda^{-1}(\Lambda - dI)$$
$$(\Lambda + I)^{-1} - (1 + d)^2(\Lambda + 1)^{-1}\vartheta\vartheta'(\Lambda + I)^{-1} \tag{2.42}$$

We have the following theorem.

Theorem 2.6. *If $k > 0$ and $0 < d < 1$, b_{MLiu} = Bias ($\hat{\vartheta}_{MLiu}(d)$) is the bias of MLiu estimator, the estimator $\hat{\vartheta}_{MLiu}(d)$ is better than to the estimator $\hat{\vartheta}_{OLS}$ using the criterion of MSEM, that is, $MSEM(\hat{\vartheta}_{OLS}) - MSEM(\hat{\vartheta}_{MLiu}) > 0$ is and only if,*

$$b'_{MLiu}[\sigma^2(\Lambda^{-1} - (\Lambda + I)^{-1}(\Lambda - dI)\Lambda^{-1}(\Lambda - dI)(\Lambda + I)^{-1})]b_{MLiu} < 1$$

Proof. Using the difference between scalar MSE functions,

$$\begin{aligned} MSEM(\hat{\vartheta}_{OLS}) - MSEM(\hat{\vartheta}_{MLiu}(d)) &= \sigma^2(\Lambda^{-1} - (\Lambda + I)^{-1}(\Lambda - dI)\Lambda^{-1}(\Lambda - dI)(\Lambda + I)^{-1}) \\ &\quad - b_{MLiu}b'_{MLiu} \\ &= \sigma^2 diag\left\{\frac{1}{\lambda_i} - \frac{(\lambda_i - d)^2}{\lambda_i(\lambda_i + 1)^2}\right\}_{i=1}^{p} - b_{MLiu}b'_{MLiu} \end{aligned} \tag{2.43}$$

where, $(\Lambda^{-1} - (\Lambda + I)^{-1}(\Lambda - dI)\Lambda^{-1}(\Lambda - dI)(\Lambda + I)^{-1})$ is positive definite if and only if $(\lambda_i + 1)^2 - (\lambda_i - d)^2 > 0$. For $k > 0$ and $0 < d < 1$, we observed that $2\lambda_i + (1 - d) > 0$. So, $(\Lambda^{-1} - (\Lambda + I)^{-1}(\Lambda - dI)\Lambda^{-1}(\Lambda - dI)(\Lambda + I)^{-1})$ is positive definite. By lemma 2.2, it is proved. □

2.3.4 Comparison between James Stein estimator and OLS estimator

The difference between MSEM($\hat{\vartheta}_{JSE}$) and MSEM($\hat{\vartheta}_{OLS}$) is obtained by

$$MSEM[\hat{\vartheta}_{OLS}] - MSEM[\hat{\vartheta}_{JSE}] = \sigma^2\Lambda^{-1} - c^2\sigma^2\Lambda^{-1} - (c - 1)^2\vartheta\vartheta' \tag{2.44}$$

We have the following theorem

Theorem 2.7. *If $k > 0$ and $0 < d < 1$, b_{JSE} = Bias ($\hat{\vartheta}_{JSE}$) is the bias of JSE, the estimator $\hat{\vartheta}_{JSE}$ is better than to the estimator $\hat{\vartheta}_{OLS}$ using the criterion of MSEM, that is, $MSEM(\hat{\vartheta}_{OLS}) - MSEM(\hat{\vartheta}_{JSE}) > 0$ is and only if,*

$$b'_{JSE}[\sigma^2(\Lambda^{-1} - c^2\Lambda^{-1})]b_{JSE} < 1$$

Proof. Using the difference between scalar MSE functions,

$$
\begin{aligned}
MSEM(\hat{\vartheta}_{OLS}) - MSEM(\hat{\vartheta}_{JSE}) &= \sigma^2(\Lambda^{-1} - c^2\Lambda^{-1}) - b_{JSE}b_{JSE}^{'} \\
&= \sigma^2 diag\left\{\frac{1}{\lambda_i} - \frac{c^2}{\lambda_i}\right\}_{i=1}^{p} - b_{JSE}b_{JSE}^{'} \qquad (2.45)
\end{aligned}
$$

where, $\Lambda^{-1} - c^2\Lambda^{-1}$ is positive definite if and only if $(1 - c^2) > 0$. We observed that $\Lambda^{-1} - c^2\Lambda^{-1}$ is positive definite. By lemma 2.2, the proof is completed. □

2.3.5 Comparison between KL estimator and OLS estimator

The difference between MSEM($\hat{\vartheta}_{KL}(k)$) and MSEM($\hat{\vartheta}_{OLS}$) is obtained by

$$
\begin{aligned}
MSEM[\hat{\vartheta}_{OLS}] - MSEM[\hat{\vartheta}_{KL}(k)] &= \sigma^2\Lambda^{-1} - \sigma^2 W_k M_k \Lambda^{-1} M_k^{'} W_k^{'} \\
&\quad -(W_k M_k - I)\vartheta\vartheta^{'}(W_k M_k - I)^{'} \qquad (2.46)
\end{aligned}
$$

We have the following theorem.

Theorem 2.8. *If $k > 0$ and $0 < d < 1$, $b_{KL} = Bias(\hat{\vartheta}_{KL}(k))$ is the bias of KL estimator, the estimator $\hat{\vartheta}_{KL}(k)$ is better than to the estimator $\hat{\vartheta}_{OLS}$ using the criterion of MSEM, that is, $MSEM(\hat{\vartheta}_{OLS}) - MSEM(\hat{\vartheta}_{KL}(k)) > 0$ is and only if,*

$$
b_{KL}^{'}[\sigma^2(\Lambda^{-1} - W_k M_k \Lambda^{-1} M_k^{'} W_k^{'})]b_{KL} < 1
$$

Proof. Using the difference between scalar MSE functions,

$$
\begin{aligned}
MSEM(\hat{\vartheta}_{OLS}) - MSEM(\hat{\vartheta}_{KL}) &= \sigma^2(\Lambda^{-1} - W_k M_k \Lambda^{-1} M_k^{'} W_k^{'}) - b_{KL}b_{KL}^{'} \\
&= \sigma^2 diag\left\{\frac{1}{\lambda_i} - \frac{(\lambda_i - k)^2}{\lambda_i(\lambda_i + k)^2}\right\}_{i=1}^{p} - b_{KL}b_{KL}^{'} \quad (2.47)
\end{aligned}
$$

where, $\Lambda^{-1} - W_k M_k \Lambda^{-1} M_k^{'} W_k^{'}$ is positive definite if and only if $(\lambda_i + k)^2 - (\lambda_i - k)^2 > 0$. For $k > 0$ and $0 < d < 1$, we observed that $(\lambda_i + k)^2 - (\lambda_i - k)^2 = 4\lambda_i k > 0$, so $\Lambda^{-1} - W_k M_k \Lambda^{-1} M_k^{'} W_k^{'}$ is positive definite. By lemma 2.2, it is proved. □

2.3.6 Comparison between D estimator and OLS estimator

The difference between MSEM($\hat{\vartheta}_D(k, d)$) and MSEM($\hat{\vartheta}_{OLS}$) is

$$
\begin{aligned}
MSEM[\hat{\vartheta}_{OLS}] - MSEM[\hat{\vartheta}_D(k, d)] &= \sigma^2\Lambda^{-1} - \sigma^2 D_{(k,d)}\Lambda^{-1}D_{(k,d)} \\
&\quad -(D_{(k,d)} - I)\vartheta\vartheta^{'}(D_{(k,d)} - I)^{'} \qquad (2.48)
\end{aligned}
$$

We have the following theorem.

Theorem 2.9. *If $k > 0$ and $0 < d < 1$, b_D = Bias $(\hat{\vartheta}_D(k,d))$ is the bias of D estimator, the estimator $\hat{\vartheta}_D(k,d)$ is better than to the estimator $\hat{\vartheta}_{OLS}$ using the criterion of MSEM, that is, $MSEM(\hat{\vartheta}_{OLS}) - MSEM(\hat{\vartheta}_D(k,d)) > 0$ is and only if,*

$$b_D{}'[\sigma^2(\Lambda^{-1} - D_{(k,d)}\Lambda^{-1}D^{'}_{(k,d)})]b_D < 1$$

Proof. Using the difference between scalar MSE functions,

$$\begin{aligned} MSEM(\hat{\vartheta}_{OLS}) - MSEM(\hat{\vartheta}_D(k,d)) &= \sigma^2(\Lambda^{-1} - D_{(k,d)}\Lambda^{-1}D^{'}_{(k,d)}) - b_D b^{'}_D \\ &= \sigma^2 diag\left\{\frac{1}{\lambda_i} - \frac{\lambda_i}{(\lambda_i + kd)^2}\right\}_{i=1}^{p} - b_D b^{'}_D \end{aligned} \quad (2.49)$$

where, $(\Lambda^{-1} - D_{(k,d)}\Lambda^{-1}D^{'}_{(k,d)})$ is positive definite if and only if $(\lambda_i + kd)^2 - \lambda_i^2 > 0$. For $k > 0$ and $0 < d < 1$, we observed that $2\lambda_i + kd > 0$, so $(\Lambda^{-1} - D_{(k,d)}\Lambda^{-1}D^{'}_{(k,d)})$ is positive definite. By lemma 2.2, the proof is completed. □

2.3.7 Comparison between MRT estimator and OLS estimator

The difference between MSEM($\hat{\vartheta}_{MRT}(k,d)$) and MSEM($\hat{\vartheta}_{OLS}$) is obtained by

$$\begin{aligned} MSEM[\hat{\vartheta}_{OLS}] - MSEM[\hat{\vartheta}_{MRT}(k,d)] &= \sigma^2\Lambda^{-1} - \sigma^2 M_{(k,d)}\Lambda^{-1}M^{'}_{(k,d)} \\ &\quad -(M_{(k,d)} - I)\vartheta\vartheta^{'}(M_{(k,d)} - I)^{'} \end{aligned} \quad (2.50)$$

We have the following theorem.

Theorem 2.10. *If $k > 0$ and $0 < d < 1$, b_{MRT} = Bias $(\hat{\vartheta}_{MRT}(k,d))$ is the bias of MRT estimator, the estimator $\hat{\vartheta}_{MRT}(k,d)$ is better than to the estimator $\hat{\vartheta}_{OLS}$ using the criterion of MSEM, that is, $MSEM(\hat{\vartheta}_{OLS}) - MSEM(\hat{\vartheta}_{MRT}(k,d)) > 0$ is and only if,*

$$b^{'}_{MRT}[\sigma^2(\Lambda^{-1} - M_{(k,d)}\Lambda^{-1}M^{'}_{(k,d)})]b_{MRT} < 1$$

Proof. Using the difference between scalar MSE functions,

$$\begin{aligned} MSEM(\hat{\vartheta}_{OLS}) - MSEM(\hat{\vartheta}_{MRT}(k,d)) &= \sigma^2(\Lambda^{-1} - M_{(k,d)}\Lambda^{-1}M^{'}_{(k,d)}) - b_{MRT}b^{'}_{MRT} \\ &= \sigma^2 diag\left\{\frac{1}{\lambda_i} - \frac{\lambda_i}{(\lambda_i + k(1+d))^2}\right\}_{i=1}^{p} - b_{MRT}b^{'}_{MRT} \end{aligned} \quad (2.51)$$

where, $(\Lambda^{-1} - M_{(k,d)}\Lambda^{-1}M^{'}_{(k,d)})$ is positive definite if and only if $(\lambda_i + k(1+d))^2 - \lambda_i^2 > 0$. For $k > 0$ and $0 < d < 1$, we observed that $2\lambda_i + k(1+d) > 0$, so $(\Lambda^{-1} - M_{(k,d)}\Lambda^{-1}M^{'}_{(k,d)})$ is positive definite. By lemma 2.2, the proof is completed. □

2.3.8 Comparison between KL estimator and Ridge estimator

The difference between MSEM($\hat{\vartheta}_{KL}(k)$) and MSEM($\hat{\vartheta}_{RE}(k)$) is obtained by

$$MSEM[\hat{\vartheta}_{RE}(k)] - MSEM[\hat{\vartheta}_{KL}(k)] = \sigma^2 A_k \Lambda^{-1} A_k^{'} - \sigma^2 W_k M_k \Lambda^{-1} M_k^{'} W_k^{'} \\ +(A_k - I)\vartheta\vartheta^{'}(A_k - I)^{'} - (W_k M_k - I)\vartheta\vartheta^{'}(W_k M_k - I)^{'} \quad (2.52)$$

We have the following theorem.

Theorem 2.11. *If $k > 0$ and $0 < d < 1$, the estimator $\hat{\vartheta}_{KL}(k)$ is better than to the estimator $\hat{\vartheta}_{RE}(k)$ using the criterion of MSEM, that is, $MSEM(\hat{\vartheta}_{RE}) - MSEM(\hat{\vartheta}_{KL}) > 0$ is and only if,*

$$b_{KL}^{'}[\sigma^2(A_k \Lambda^{-1} A_k^{'} - W_k M_k \Lambda^{-1} M_k^{'} W_k^{'}) + b_k b_k^{'}] b_{KL}$$

Proof. Using the difference between scalar MSE functions,

$$\begin{aligned} MSEM(\hat{\vartheta}_{RE}(k)) - MSEM(\hat{\vartheta}_{KL}(k)) &= \sigma^2(A_k \Lambda^{-1} A_k^{'} - W_k M_k \Lambda^{-1} M_k^{'} W_k^{'}) + b_k b_k^{'} \\ &- b_{KL} b_{KL}^{'} \\ &= \sigma^2 diag\left\{ \frac{\lambda_i}{(\lambda_i + k)^2} - \frac{(\lambda_i - k)^2}{\lambda_i(\lambda_i + k)^2} \right\}_{i=1}^{p} + b_k b_k^{'} - b_{KL} b_{KL}^{'} \end{aligned} \quad (2.53)$$

Since $b_k b_k^{'}$ is non-negative definite, we see that $A_k \Lambda^{-1} A_k^{'} - W_k M_k \Lambda^{-1} M_k^{'} W_k^{'}$ is positive definite if and only if $(\lambda_i + k(1+d))^2 - \lambda_i^2 > 0$. For $k > 0$ and $0 < d < 1$, we observed that $2\lambda_i - k > 0$, so $A_k \Lambda^{-1} A_k^{'} - W_k M_k \Lambda^{-1} M_k^{'} W_k^{'}$ is positive definite. By lemma 2.3, the proof is completed. □

2.3.9 Comparison between MLiu estimator and Ridge estimator

The difference between MSEM($\hat{\vartheta}_{MLiu}(d)$) and MSEM($\hat{\vartheta}_{RE}(k)$) is obtained by

$$MSEM[\hat{\vartheta}_{RE}(k)] - MSEM[\hat{\vartheta}_{MLiu}(d)] = \sigma^2 A_k \Lambda^{-1} A_k^{'} - \sigma^2(\Lambda + I)^{-1}(\Lambda - dI)\Lambda^{-1} \\ (\Lambda - dI)(\Lambda + I)^{-1} + (A_k - I)^{'}\vartheta\vartheta^{'}(A_k - I)^{'} - (d+1)^2(\Lambda + I)^{'}\vartheta\vartheta(\Lambda + I)^{-1} \quad (2.54)$$

We have the following theorem.

Theorem 2.12. *If $k > 0$ and $0 < d < 1$, the estimator $\hat{\vartheta}_{MLiu}(d)$ is better than to the estimator $\hat{\vartheta}_{RE}(k)$ using the criterion of MSEM, that is, $MSEM(\hat{\vartheta}_{RE}(k)) - MSEM(\hat{\vartheta}_{MLiu}(d)) > 0$ is and only if,*

$$b_{MLiu}^{'}[\sigma^2(A_k \Lambda^{-1} A_k^{'} - (\Lambda + I)^{-1}(\Lambda - dI)\Lambda^{-1}(\Lambda - dI)(\Lambda + I)^{-1}) + b_k b_k^{'}] b_{MLiu}$$

Proof. Using the difference between scalar MSE functions,

$$
\begin{aligned}
MSEM(\hat{\vartheta}_{RE}(k)) - MSEM(\hat{\vartheta}_{MLiu}(d)) &= \sigma^2(A_k\Lambda^{-1}A_k^{'} - (\Lambda+I)^{-1}(\Lambda-dI)\Lambda^{-1} \\
&\quad (\Lambda-dI)(\Lambda+I)^{-1}) + b_kb_k^{'} - b_{MLiu}b_{MLiu}^{'} \\
&= \sigma^2 diag\left\{\frac{\lambda_i}{(\lambda_i+k)^2} - \frac{(\lambda_i-d)^2}{\lambda_i(\lambda_i+1)^2}\right\}_{i=1}^{p} + b_kb_k^{'} - b_{MLiu}b_{MLiu}^{'}
\end{aligned} \tag{2.55}
$$

Since $b_kb_k^{'}$ is non-negative definite, we see that $A_k\Lambda^{-1}A_k^{'} - (\Lambda+I)^{-1}(\Lambda-dI)\Lambda^{-1}(\Lambda-dI)(\Lambda+I)^{-1}$ is positive definite if and only if $\lambda_i^2(\lambda_i+1)^2 - (\lambda_i-d)^2(\lambda_i+k)^2 > 0$. For $k > 0$ and $0 < d < 1$, we observed that $\lambda_i(1+d-k)+kd > 0$, so $A_k\Lambda^{-1}A_k^{'} - (\Lambda+I)^{-1}(\Lambda-dI)\Lambda^{-1}(\Lambda-dI)(\Lambda+I)^{-1}$ is positive definite. By lemma 2.3, the proof is completed. □

2.3.10 Comparison between MLiu estimator and KL estimator

The difference between MSEM($\hat{\vartheta}_{MLiu}(d)$) and MSEM($\hat{\vartheta}_{KL}(k)$) is

$$
\begin{aligned}
MSEM[\hat{\vartheta}_{KL}(k)] - MSEM[\hat{\vartheta}_{MLiu}(d)] &= \sigma^2 W_kM_k\Lambda^{-1}M_k^{'}W_k^{'} \\
&\quad - \sigma^2(\Lambda+I)^{-1}(\Lambda-dI)\Lambda^{-1}(\Lambda-dI)(\Lambda+I)^{-1} \\
&\quad + (W_kM_k-I)\vartheta\vartheta^{'}(W_kM_k-I)^{'} - (1+d)^2(\Lambda+I)^{'}\vartheta\vartheta(\Lambda+I)^{-1}
\end{aligned} \tag{2.56}
$$

We have the following theorem.

Theorem 2.13. *If $k > 0$ and $0 < d < 1$, the estimator $\hat{\vartheta}_{MLiu}(d)$ is better than to the estimator $\hat{\vartheta}_{RE}(k)$ using the criterion of MSEM, that is, $MSEM(\hat{\vartheta}_{RE}(k)) - MSEM(\hat{\vartheta}_{MLiu}(d)) > 0$ is and only if,*

$$b_{MLiu}^{'}[\sigma^2(W_kM_k\Lambda^{-1}M_k^{'}W_k^{'} - (\Lambda+I)^{-1}(\Lambda-dI)\Lambda^{-1}(\Lambda-dI)(\Lambda+I)^{-1}) + b_{KL}b_{KL}^{'}]b_{MLiu}$$

Proof. Using the difference between scalar MSE functions,

$$
\begin{aligned}
MSEM(\hat{\vartheta}_{KL}(k)) - MSEM(\hat{\vartheta}_{MLiu}(d)) &= \sigma^2(W_kM_k\Lambda^{-1}M_k^{'}W_k^{'} \\
&\quad - (\Lambda+I)^{-1}(\Lambda-dI)\Lambda^{-1}(\Lambda-dI)(\Lambda+I)^{-1}) + b_{KL}b_{KL}^{'} - b_{MLiu}b_{MLiu}^{'} \\
&= \sigma^2 diag\left\{\frac{(\lambda_i-k)^2}{\lambda_i(\lambda_i+k)^2} - \frac{(\lambda_i-d)^2}{\lambda_i(\lambda_i+1)^2}\right\}_{i=1}^{p} + b_{KL}b_{KL}^{'} - b_{MLiu}b_{MLiu}^{'}
\end{aligned} \tag{2.57}
$$

Since $b_{KL}b_{KL}^{'}$ is non-negative definite, we see that $W_kM_k\Lambda^{-1}M_k^{'}W_k^{'} - (\Lambda+I)^{-1}(\Lambda-dI)\Lambda^{-1}(\Lambda-dI)(\Lambda+I)^{-1}$ is positive definite if and only if $(\lambda_i-k)^2(\lambda_i+1)^2 - (\lambda_i-d)^2(\lambda_i+k)^2 > 0$. For $k > 0$ and $0 < d < 1$, we observed that $\lambda_i(1+d-2k)+k(d-1) > 0$, so $W_kM_k\Lambda^{-1}M_k^{'}W_k^{'} - (\Lambda+I)^{-1}(\Lambda-dI)\Lambda^{-1}(\Lambda-dI)(\Lambda+I)^{-1}$ is positive definite. By lemma 2.3, the proof is completed. □

3 The Selection of Parameters k and d

Following Yang and Chang (2010)[37], the values of k can be obtained for fixed d to be those k that minimizing

$$MSE(\hat{\alpha}(k,d)) = E[(\hat{\alpha}(k,d)-\alpha)^{'}(\hat{\alpha}(k,d)-\alpha)].$$

where, $\hat{\alpha}(k,d) = (\Lambda+I)^{-1}(\Lambda+dI)(\Lambda+kI)^{-1}Z^{'}y$
Now, $h(k,d) = MSE(\hat{\alpha}(k,d)) = tr[MSEM(\hat{\alpha}(k,d))]$, we have:

$$h(k,d) = \sigma^2 \sum_{i=1}^{p} \frac{(\lambda_i+d)^2\lambda_i}{(\lambda_i+1)^2(\lambda_i+k)^2} + \sum_{i=0}^{p} \frac{((k+1-d)\lambda_i+k)^2\alpha_i^2}{(\lambda_i+1)^2(\lambda_i+k)^2}$$

Now, differentiating the $h(k,d)$ with respect to k, we obtain:

$$\frac{\partial h(k,d)}{\partial k} = -2\sigma^2 \sum_{i=1}^{p} \frac{(\lambda_i+d)^2\lambda_i}{(\lambda_i+1)^2(\lambda_i+k)^3} + 2\sum_{i=0}^{p} \frac{((k+1-d)\lambda_i+k)(\lambda_i+d)\lambda_i\alpha_i^2}{(\lambda_i+1)^2(\lambda_i+k)^3}$$

Taking $\frac{\partial h(k,d)}{\partial k} = 0$; we get:

$$k_i = \frac{\sigma^2(\lambda_i+d) - (1-d)\lambda_i\alpha_i^2}{(\lambda_i+1)\alpha_i^2} \tag{3.1}$$

For practical purpose, we use unbiased estimator of $\hat{\sigma}^2$ and $\hat{\alpha}_i$ instead of σ^2 and α_i, and we obtain an estimator for k,

$$\hat{k}_i = \frac{\hat{\sigma}^2(\lambda_i+d) - (1-d)\lambda_i\hat{\alpha}_i^2}{(\lambda_i+1)\hat{\alpha}_i^2} \tag{3.2}$$

Again, when $d=1$, we get $\hat{\alpha}(k,1) = \hat{\alpha}(k)$, RE estimator, $\hat{k}$ would be

$$\hat{k}_i = \frac{\hat{\sigma}^2}{\hat{\alpha}_i^2}$$

which is proposed by Hoerl and Kennard [12].

Kibria (2003)[18] proposed the estimator of k by using arithmetic and geometric means. That is:

$$\hat{k}_{AM} = \frac{1}{p}\sum_{i=0}^{p}\hat{k}_i \qquad and \qquad \hat{k}_{GM} = (\prod_{i=0}^{p}\hat{k}_i)^{\frac{1}{p}} \tag{3.3}$$

where $\hat{k}_i$ is the estimate value of k by Hoerl and Kennard.

Yang and Chang (2010)[37] proposed to obtain the arithmetic and geometric means of $\hat{k}_i$ in (3.2) respectively as follows:

$$\hat{k}_{AM} = \frac{1}{p}\sum_{i=0}^{p} \frac{\hat{\sigma}^2(\lambda_i + d) - (1-d)\lambda_i\hat{\alpha}_i^2}{(\lambda_i + 1)\hat{\alpha}_i^2} \tag{3.4}$$

$$\hat{k}_{GM} = (\prod_{i=0}^{p} \frac{\hat{\sigma}^2(\lambda_i + d) - (1-d)\lambda_i\hat{\alpha}_i^2}{(\lambda_i + 1)\hat{\alpha}_i^2})^{\frac{1}{p}} \tag{3.5}$$

For the optimal value of d, we will minimize $h(k,d)$ for fixed k, derivative with respect to d, we get:

$$\frac{\partial h(k,d)}{\partial d} = 2\sigma^2\sum_{i=1}^{p} \frac{(\lambda_i + d)\lambda_i}{(\lambda_i + 1)^2(\lambda_i + k)^2} - 2\sum_{i=0}^{p} \frac{((k+1-d)\lambda_i + k)\lambda_i\alpha_i^2}{(\lambda_i + 1)^2(\lambda_i + k)^2}$$

Considering $\frac{\partial h(k,d)}{\partial d} = 0$; we get:

$$d_{opt} = \frac{\sum_{i=1}^{p}[((k+1)\lambda_i + k)\lambda_i\alpha_i^2 - \lambda_i^2\sigma^2]/[(\lambda_i + 1)^2(\lambda_i + k)^2]}{\sum_{i=1}^{p}(\sigma^2 + \lambda_i\alpha_i^2)\lambda_i/[(\lambda_i + 1)^2(\lambda_i + k)^2]} \tag{3.6}$$

As, $\sigma^2, \alpha_i, i = 1, \ldots, p$ are unknown, we can replace them by their unbiased estimators, then we get:

$$\hat{d}_{opt} = \frac{\sum_{i=1}^{p}[((k+1)\lambda_i + k)\lambda_i\hat{\alpha}_i^2 - \lambda_i^2\hat{\sigma}^2]/[(\lambda_i + 1)^2(\lambda_i + k)^2]}{\sum_{i=1}^{p}(\hat{\sigma}^2 + \lambda_i\hat{\alpha}_i^2)\lambda_i/[(\lambda_i + 1)^2(\lambda_i + k)^2]} \tag{3.7}$$

And, if $k = 0$, $\hat{\alpha}(k,d)$ becomes Liu estimator as

$$\hat{d}_{opt} = \frac{\sum_{i=1}^{p}(\hat{\alpha}_i^2 - \hat{\sigma}^2)/(\lambda_i + 1)^2}{\sum_{i=1}^{p}(\hat{\sigma}^2 + \lambda_i\hat{\alpha}_i^2)/(\lambda_i + 1)^2\lambda_i},$$

which is known as the estimate of d by Liu.
Again, following Kibria and Lukman (2020)[21] for the value of parameter k, the harmonic version is defined as:

$$\hat{k}_{HMN} = \frac{p\sigma^2}{\sum_{i=1}^{p}[2\hat{\alpha}_i^2 + (\hat{\sigma}^2/\lambda_i)]}$$

And, the minimum version is defined as:

$$\hat{k}_{min} = min\left[\frac{\sigma^2}{2\hat{\alpha}_i^2 + (\hat{\sigma}^2/\lambda_i)}\right]$$

4 Simulation Study

In this section, simulation technique will be discussed in section 4.1 and simulations results will be discussed in section 4.2.

4.1 Simulation Technique

In this section, a Monte Carlo simulation study has been conducted to examine the performances of different estimators. This simulation procedure is following the McDonalds and Galarneau (1975)[28] and Gibbons (1981)[10]; the explanatory variables were given as follows

$$X_{ij} = (1-\rho^2)^{1/2} z_{ij} + \rho z_{i(p+1)}, i = 1, 2, \dots, n, j = 1, 2, \dots, p, \tag{4.1}$$

where z_{ij} is independent standard normal pseudo-random numbers with mean zero and unit variance, ρ is the correlation between any two explanatory variables, and p is the number of explanatory variables. These variables are standarized so that $X^{'}X$ and $X^{'}y$ can be in correlation forms. Different set of values of γ corresponding 0.90, 0.95 and 0.99 were taken. In this study, n observations for the explanatory variables are generated by

$$y_i = \beta_1 X_{i1} + \beta_2 X_{i2} + \cdots + \beta_p X_{ip} + \varepsilon_i, \tag{4.2}$$

where ε_i are independent normal $(0, \sigma^2)$. The sample sizes are taken corresponding 30, 50 and 100 and we used three different values of σ respectively 1, 5 and 10. Then the experiment is replicated by 2000 times. Now, we selected the parameters values $\beta_1, \beta_2, \dots, \beta_p$ as normalized the eigen vectors corresponding the largest eigen values of $X^{'}X$ matrix so that $\beta^{'}\beta = 1$. Now the estimated MSE is calculated as:

$$MSE(\hat{\beta}) = \frac{1}{2000} \sum_{i=1}^{2000} (\hat{\beta}_{ij} - \beta_i)^{'} (\hat{\beta}_{ij} - \beta_i), \tag{4.3}$$

where $\hat{\beta}_{ij}$ denotes the estimate of the i^{th} parametere in the j^{th} replication and β_i is the true parameter values. For different values of n, p, σ and ρ, the estimated MSE of the the estimators are shown in the corresponding tables.

Table 4.1: Estimated MSE values with $n = 30, \rho = 0.90$ and $p = 3$

k	d	σ	OLS	RE	Liu	MLiu	JSE	KL	D	MRT
0.3	0.2	1	0.623	0.515	0.392299	0.305716	0.4562883	0.397339	0.579	0.477
		5	15.914	13.057	9.566057	7.797136	6.833829	10.12374	14.757	12.141
		10	59.486	52.957	37.69796	29.81647	24.68973	41.01607	59.822	49.205
	0.5	1	0.61	0.518	0.481203	0.238898	0.4570952	0.403091	0.553	0.459
		5	14.699	13.097	11.52787	5.976031	6.316263	10.11967	13.842	11.496
		10	62.547	51.199	46.96297	24.97407	23.64792	39.45149	54.265	44.938
	0.8	1	0.625	0.499	0.562271	0.201772	0.454204	0.389007	0.501	0.419
		5	14.629	13.089	13.83378	4.835345	6.870566	10.08251	13.817	10.894
		10	60.053	53.497	55.3413	18.923	25.42258	41.54953	56.779	44.852
0.7	0.2	1	0.596	0.471	0.382728	0.305974	0.461642	0.241842	0.563	0.376
		5	15.726	11.667	9.424653	6.978842	7.018796	6.086345	14.048	9.419
		10	60.645	48.077	39.11647	29.51477	23.58164	24.80282	57.429	38.489
	0.5	1	0.639	0.454	0.481203	0.242719	0.453854	0.238445	0.478	0.332
		5	15.4	11.757	11.82184	6.044264	6.389145	5.855424	12.508	8.275
		10	59.178	45.549	47.28518	24.19757	25.85365	23.52463	49.133	32.898
	0.8	1	0.598	0.483	0.570002	0.190569	0.456718	0.245648	0.423	0.315
		5	15.775	11.911	13.49944	4.724719	6.531076	6.186022	10.648	7.822
		10	60.817	45.449	54.49419	20.16648	23.26371	23.79847	43.619	30.212
0.9	0.2	1	0.617	0.445	0.392699	0.290933	0.459289	0.182271	0.496	0.327
		5	14.912	11.327	9.940168	7.365967	6.4968	4.755175	13.172	8.251
		10	61.001	41.815	39.10206	29.51266	24.95437	17.59523	54.201	33.024
	0.5	1	0.608	0.436	0.468836	0.247938	0.459628	0.184174	0.457	0.287
		5	15.365	11.394	11.465	5.997108	6.252219	4.681804	11.425	7.162
		10	60.731	44.489	45.03494	23.66357	24.77247	18.92259	44.415	30.582
	0.8	1	0.604	0.45	0.552635	0.196655	0.441561	0.18687	0.384	0.255
		5	15.95	10.984	14.19282	4.773286	6.570011	4.69237	10.249	6.46
		10	61.294	43.423	55.86558	19.4529	24.91885	18.06794	38.272	25.874

4.2 Simulation Results Discussion

In this section, the simulation result has been shown and the performance of the estimators has been compared based on the estimated MSE values.

The simulated MSE values for $p = 3$, $\rho = 0.90$, and $n = 30, 50$, and 100 are provided in Tables 4.1,4.2, and 4.3 respectively. We can see that increasing the values of sample size, the values of the MSE decrease for all of the estimators for fixed ρ=0.90 and $p = 3$.

For $p = 5, n = 50$, and $\rho = 0.90$, 0.95, 0.99 are provided in Tables 4.4 to 4.6 respectively. Also from these tables, we can see the effect of the ρ on all estimators. For large values of ρ, the values of the MSE for all the estimators has increased. It is obvious from these tables that the MSE values increase for all estimators for fixed $n = 50$ and $p = 5$.

And, Tables 4.7 to 4.9, provide the simulated values for $n = 100, \rho = 0.95$, and n = 5, 10, 15 respectively. It shows the effect of the number of parameters on the

Table 4.2: Estimated MSE values with $n = 50, \rho = 0.90$ and $p = 3$

k	d	σ	OLS	RE	Liu	Mliu	JSE	KL	D	MRT
0.3	0.2	1	0.350921	0.313852	0.266838	0.233224	0.2845732	0.279266	0.346957	0.302915
		5	8.425408	7.703303	6.508811	5.59715	3.966472	7.139369	8.299742	7.442083
		10	34.57397	31.55419	25.98007	22.81679	14.11873	27.18752	33.1314	30.43103
	0.5	1	0.335758	0.307129	0.297271	0.200078	0.283616	0.261058	0.334917	0.28793
		5	8.414735	7.903776	7.721046	5.210882	4.027343	7.013664	8.058709	7.377789
		10	34.409	30.45539	30.24322	20.73178	14.56487	27.59866	32.55525	28.56849
	0.8	1	0.340578	0.304683	0.328982	0.17474	0.281666	0.270769	0.313494	0.277003
		5	8.470122	7.704104	8.139896	4.466694	3.975499	6.932205	8.085259	7.008785
		10	34.27117	30.81893	32.31394	17.59859	14.20301	27.30341	33.14657	27.99776
0.7	0.2	1	0.340078	0.282068	0.262691	0.225102	0.278727	0.205514	0.330824	0.255267
		5	8.546027	7.171741	6.528633	5.53049	3.802941	5.159514	8.127194	6.435527
		10	33.80123	28.11868	26.22009	22.05476	15.04379	20.46046	32.38227	25.33259
	0.5	1	0.358932	0.277807	0.293807	0.195272	0.285318	0.197172	0.308067	0.236403
		5	8.611018	7.116564	7.054003	5.025438	3.933108	5.03887	8.001114	6.054566
		10	34.15483	28.36294	29.24723	20.68463	14.45864	21.20568	31.34466	23.99647
	0.8	1	0.356484	0.284442	0.314393	0.174378	0.280496	0.214841	0.285496	0.226266
		5	8.864829	7.107305	8.007634	4.431423	3.953649	5.160914	7.093602	5.713413
		10	34.046	28.38868	31.84516	17.17349	14.62196	19.93969	28.12836	22.56145
0.9	0.2	1	0.351782	0.279836	0.26021	0.223506	0.283719	0.178644	0.321845	0.243899
		5	8.691428	7.020977	6.345294	5.603738	3.83362	4.418391	7.981259	6.058243
		10	35.45843	27.64911	26.01748	23.55698	13.57979	17.11038	32.04485	24.14655
	0.5	1	0.345316	0.278073	0.286763	0.202001	0.280569	0.178496	0.293884	0.223388
		5	8.793698	6.813084	7.673222	4.903342	4.034073	4.500085	7.056388	5.519685
		10	36.09258	28.22524	28.10927	20.70033	13.89627	16.83574	29.25276	22.64506
	0.8	1	0.344124	0.277321	0.340721	0.181512	0.283716	0.179448	0.259846	0.207335
		5	8.257713	6.995159	8.212665	4.435822	3.810348	4.261299	6.632682	5.195082
		10	35.20403	27.75463	31.01552	17.91595	13.80544	17.39356	26.31305	20.26095

Table 4.3: Estimated MSE values with $n = 100$, ρ=0.90 and $p = 3$

k	d	σ	OLS	RE	Liu	MLiu	JSE	KL	D	MRT
0.3	0.2	1	0.158301	0.153927	0.14405	0.133933	0.152853	0.140332	0.164387	0.155895
		5	4.039172	3.891924	3.429551	3.323967	2.139648	3.583671	3.87214	3.674665
		10	16.52395	15.5577	14.34346	13.47017	7.059271	15.00242	16.24153	15.40441
	0.5	1	0.160729	0.151017	0.151154	0.127515	0.1487224	0.146265	0.161793	0.153573
		5	4.155749	3.759375	3.777265	2.991551	2.042893	3.733629	3.786602	3.593552
		10	16.83448	15.66096	14.98577	13.11625	6.708239	14.81246	15.49538	14.71392
	0.8	1	0.156839	0.156107	0.164077	0.120348	0.146641	0.140995	0.159831	0.151794
		5	4.027909	3.965695	3.73826	3.164494	2.115832	3.686451	4.120657	3.846143
		10	16.76949	16.35664	16.42103	11.74616	7.047127	14.75539	15.07743	14.10286
0.7	0.2	1	0.163157	0.152528	0.14415	0.130761	0.1424677	0.128136	0.158353	0.140425
		5	4.084396	3.728395	3.576067	3.340194	1.914845	3.16599	3.990068	3.540909
		10	16.65693	15.0603	14.20383	12.92698	6.589153	13.18165	15.93717	14.15434
	0.5	1	0.160574	0.14563	0.151891	0.126181	0.149352	0.12071	0.149312	0.13283
		5	4.081536	3.649962	3.817947	3.159999	2.035546	3.293755	3.719883	3.314779
		10	16.75505	14.71834	15.17849	12.9657	7.100928	12.86146	15.47668	13.78791
	0.8	1	0.163115	0.15115	0.157524	0.122717	0.147426	0.1278	0.148973	0.133053
		5	4.110332	3.762628	3.985109	3.031628	1.934397	3.165959	3.782131	3.371046
		10	16.39027	14.34295	15.98538	11.96661	6.862522	12.93671	15.60494	13.90511
0.9	0.2	1	0.166179	0.137346	0.144068	0.133513	0.146486	0.114122	0.164841	0.141371
		5	3.913792	3.491634	3.555863	3.378064	2.069058	3.02052	4.054951	3.459751
		10	16.41815	14.58909	14.0231	13.41138	6.847603	11.30139	15.73255	13.93458
	0.5	1	0.166186	0.142257	0.148458	0.127364	0.151107	0.116011	0.149864	0.129475
		5	3.889797	3.641534	3.836978	3.190984	1.890721	3.014245	3.798306	3.235382
		10	15.91268	14.45325	14.86891	12.93835	7.346214	11.4093	15.56339	13.11633
	0.8	1	0.166791	0.137137	0.154302	0.118996	0.146862	0.119233	0.148012	0.128983
		5	4.235131	3.597521	4.064584	3.093669	2.020027	3.02052	3.524902	3.209201
		10	15.48294	14.41153	15.89725	12.03344	6.74376	11.30139	13.79673	12.22599

Table 4.4: Estimated MSE values with $n = 50$, ρ=0.90 and $p = 5$

k	d	σ	OLS	RE	Liu	MLiu	JSE	KL	D	MRT
0.3	0.2	1	0.738908	0.640997	0.502141	0.426225	0.469883	0.528846	0.700343	0.611902
		5	17.94916	15.60608	12.65278	10.56391	6.654079	13.03566	17.56119	14.6331
		10	72.55752	62.97132	52.35059	43.39464	24.71512	53.31554	69.9177	61.80035
	0.5	1	0.714563	0.622208	0.576267	0.367763	0.490799	0.529211	0.650202	0.581719
		5	17.60035	15.34095	14.61046	9.543404	6.710435	13.35526	16.29785	14.84039
		10	68.51146	59.55411	57.27408	36.97545	24.87385	53.8132	65.00015	58.07687
	0.8	1	0.695585	0.60454	0.670713	0.326029	0.509084	0.522281	0.628958	0.567497
		5	17.80582	15.45151	16.09734	8.020104	6.837954	13.25689	15.27187	13.57548
		10	70.04344	61.04094	63.70348	32.58255	26.0466	52.56838	62.65746	55.27861
0.7	0.2	1	0.712245	0.524593	0.500713	0.426215	0.495404	0.369789	0.659109	0.507142
		5	17.48725	12.98668	12.95897	10.53954	6.628232	9.315113	16.19456	12.89859
		10	72.15063	53.27607	51.08807	43.09627	23.74002	37.9438	65.80547	51.07254
	0.5	1	0.708633	0.521156	0.586409	0.375472	0.477900	0.372052	0.580428	0.445176
		5	17.87815	13.14899	14.37652	9.238057	6.768094	9.531849	15.21133	11.26013
		10	71.14937	52.77277	58.46649	37.66168	24.63	37.28893	60.24282	45.16332
	0.8	1	0.710372	0.521981	0.67235	0.328524	0.489211	0.362582	0.575415	0.414342
		5	17.13912	12.64476	16.5629	8.055493	6.505055	9.357841	13.54272	10.65786
		10	69.95627	51.62502	64.00623	33.7261	25.98225	37.17321	55.43219	42.29635
0.9	0.2	1	0.72776	0.496071	0.506339	0.431648	0.490098	0.325811	0.64447	0.447918
		5	17.19654	11.84088	12.55357	10.5905	6.797582	7.933321	16.40908	10.97839
		10	70.31771	48.23503	52.7459	42.29955	25.28635	31.26755	64.23393	45.88619
	0.5	1	0.696285	0.477645	0.594612	0.367258	0.487994	0.324355	0.584886	0.430392
		5	17.71301	12.14983	14.6467	9.361346	6.248432	7.984888	14.44506	10.27146
		10	69.5327	48.09603	57.53258	36.91281	24.93478	31.53661	58.18487	41.54285
	0.8	1	0.706797	0.487168	0.667919	0.323	0.497538	0.32585	0.528723	0.390585
		5	18.1779	12.42783	16.06736	7.91452	6.297368	7.815182	12.96195	9.714188
		10	71.0294	48.80776	67.14771	32.4761	24.71313	32.58686	52.53591	39.39756

Table 4.5: Estimated MSE values with $n = 50$, ρ=0.95 and $p = 5$

k	d	σ	OLS	RE	Liu	MLiu	JSE	KL	D	MRT
0.3	0.2	1	1.475918	1.085956	0.831195	0.573379	0.839989	0.83272	1.390911	1.034831
		5	36.43142	28.22961	19.85509	14.21512	13.51124	20.35673	34.32985	25.66647
		10	147.3105	107.1988	83.45549	57.52068	50.32368	81.83913	139.658	100.5377
	0.5	1	1.471098	1.093074	1.042663	0.436885	0.849619	0.816271	1.249667	0.982298
		5	37.89606	27.10547	26.55307	11.06429	14.14599	20.20536	31.94456	25.20156
		10	150.5797	110.1309	104.0565	43.96609	51.93232	77.72807	126.4482	95.67136
	0.8	1	1.509703	1.090195	1.31305	0.327761	0.873618	0.788602	1.174206	0.911001
		5	36.9329	27.56284	31.31789	8.327363	13.91303	20.05783	29.64238	23.16045
		10	146.5347	113.1031	127.6138	32.81082	54.02734	80.25739	116.1902	92.14447
0.7	0.2	1	1.460068	0.826098	0.832153	0.568712	0.836993	0.414256	1.294803	0.768793
		5	37.40622	20.8898	21.14584	14.29581	13.16182	10.49887	32.42248	18.76831
		10	154.4687	83.82373	83.91274	58.30731	50.73703	42.16792	124.4996	74.27014
	0.5	1	1.492325	0.831961	0.992456	0.433009	0.841207	0.407307	1.088809	0.669839
		5	37.8767	20.36346	25.28191	11.09945	12.86599	10.30462	26.68828	16.70266
		10	148.6776	81.75564	100.9317	43.58787	51.71234	42.23981	108.721	67.49479
	0.8	1	1.464343	0.811137	1.233736	0.323596	0.847166	0.409894	0.910669	0.606397
		5	35.63259	20.45294	31.84445	8.231111	12.9096	10.02872	23.04763	14.81312
		10	144.7511	79.8453	127.5229	32.70959	52.06494	40.80515	88.22215	60.19097
0.9	0.2	1	1.486307	0.722521	0.799219	0.590485	0.820145	0.316214	1.217109	0.652373
		5	36.41885	18.12749	19.54069	14.37797	13.84418	7.754396	31.42252	16.26495
		10	148.6088	71.91197	81.7553	56.71826	51.62708	31.1817	127.6607	65.54272
	0.5	1	1.452686	0.726092	1.065859	0.444985	0.901623	0.310748	1.007094	0.581524
		5	37.5231	17.58772	25.20578	10.91106	12.97379	7.891245	23.54436	14.27043
		10	147.8248	73.1818	103.0503	43.44996	53.20537	31.69794	95.56856	58.47253
	0.8	1	1.488678	0.727361	1.304522	0.341377	0.853441	0.320508	0.795612	0.506965
		5	37.01129	18.80297	32.16358	8.154084	13.7945	7.994307	19.50122	12.32119
		10	148.0069	71.38372	131.8551	33.69246	51.7605	31.78837	80.39684	50.83567

Table 4.6: Estimated MSE values with $n = 50, \rho = 0.99$ and $p = 5$

k	d	σ	OLS	RE	Liu	MLiu	JSE	KL	D	MRT
0.3	0.2	1	7.945044	2.586075	1.660963	0.462552	3.420907	1.007246	5.70578	2.295263
		5	201.4201	64.77576	41.00998	11.18026	73.81714	24.67874	142.0096	56.43075
		10	813.2191	267.3663	165.132	47.28645	299.7719	103.9592	580.731	231.358
	0.5	1	8.100756	2.59286	3.485365	0.449092	3.560758	1.038646	3.928178	1.937163
		5	202.1039	66.68234	85.73234	11.51313	72.96393	25.87135	101.9026	47.5402
		10	819.4305	255.4763	351.7043	45.34282	302.0075	101.0502	394.7235	193.395
	0.8	1	8.283599	2.675455	5.829675	1.295327	3.441452	1.028941	3.066357	1.648753
		5	199.1407	64.20915	156.3006	30.01523	71.21083	25.07403	75.68856	40.4579
		10	785.8877	260.4461	588.7987	124.0302	310.9686	98.01018	300.38	162.3423
0.7	0.2	1	8.022116	1.262168	1.671312	0.456169	3.505512	1.549303	4.192828	1.073432
		5	200.4173	32.7885	40.68437	11.28761	75.83642	37.32448	106.855	27.04174
		10	818.5277	130.2208	159.8291	44.90037	290.3	154.9431	410.4559	108.2883
	0.5	1	8.289664	1.293416	3.435118	0.453875	3.473249	1.5298	2.320317	0.851798
		5	209.3487	32.13945	81.07628	11.46746	69.11927	37.76752	57.81393	21.55081
		10	788.993	128.9752	343.1233	46.34876	285.4147	158.2449	237.6525	84.13866
	0.8	1	8.43709	1.30307	6.151803	1.169698	3.532968	1.526377	1.582107	0.667526
		5	202.3478	32.31816	147.2042	31.19085	78.05969	37.34659	41.48987	16.59723
		10	788.171	128.0728	594.3267	122.4445	303.9434	163.3371	160.7409	67.35474
0.9	0.2	1	8.151017	0.977708	1.666484	0.44272	3.130876	1.965134	3.581037	0.820379
		5	205.4597	24.58885	41.45669	11.2067	70.26319	47.46004	89.43609	20.24794
		10	802.9421	100.7072	164.013	44.87138	273.697	200.0391	356.5292	79.77605
	0.5	1	8.054281	1.002812	3.230414	0.460623	3.211767	1.966276	1.930628	0.630937
		5	200.0331	24.6746	88.64802	11.43347	71.79031	47.96229	47.57213	15.58843
		10	798.6732	99.89063	343.1072	47.94989	289.7206	186.8523	192.0431	63.89926
	0.8	1	8.287053	1.002375	5.790988	1.190124	3.23958	1.888944	1.232627	0.500644
		5	202.7911	24.66536	146.9629	29.12038	67.62575	47.69107	31.55474	12.56423
		10	818.1584	101.7289	601.5855	120.9016	293.5068	207.6913	126.2489	48.76478

Table 4.7: Estimated MSE values with $n = 100$, ρ=0.95 and $p = 5$

k	d	σ	OLS	RE	Liu	MLiu	JSE	KL	D	MRT
0.3	0.2	1	0.700046	0.597282	0.499186	0.412041	0.478847	0.523192	0.666804	0.573216
		5	17.31978	14.60626	12.58918	10.4355	6.395081	12.99682	16.77158	14.4096
		10	70.69768	58.29128	49.5358	42.1508	23.39362	52.29275	67.43793	58.57557
	0.5	1	0.684263	0.585567	0.534978	0.363824	0.477843	0.523633	0.652761	0.571462
		5	17.02508	15.0608	14.15845	9.386967	6.235395	12.8493	15.79692	14.19356
		10	67.92422	58.29184	56.52427	35.92359	24.65344	52.97197	64.59615	55.14421
	0.8	1	0.677485	0.598775	0.628635	0.315368	0.476682	0.501566	0.60142	0.55499
		5	17.54436	14.9115	16.31182	7.921313	6.558185	13.31831	15.2702	13.59937
		10	67.85145	58.50005	62.85318	31.71841	22.97436	51.84048	61.25293	53.05549
0.7	0.2	1	0.719608	0.509696	0.504949	0.415764	0.474731	0.364943	0.627369	0.484659
		5	17.3848	13.00673	12.64669	10.23579	6.373263	9.475111	16.33078	11.9345
		10	69.9319	51.12635	49.54573	41.81308	24.66282	36.23258	63.19949	48.53508
	0.5	1	0.693379	0.515638	0.570633	0.357494	0.487522	0.360533	0.575943	0.455784
		5	16.84341	12.82463	14.03451	8.868946	6.633509	8.861871	14.53926	11.05975
		10	68.67275	49.66949	56.47841	35.72763	24.88351	36.52531	57.47112	45.19244
	0.8	1	0.701735	0.510588	0.642786	0.316977	0.476305	0.372447	0.543461	0.410197
		5	17.37591	12.97673	15.91266	7.697426	6.296603	9.022297	13.27242	10.58104
		10	70.06555	50.62008	63.56288	31.63916	24.84275	36.87747	53.48749	39.94176
0.9	0.2	1	0.689147	0.472243	0.497323	0.41862	0.467984	0.308932	0.630728	0.434429
		5	17.11334	11.78168	12.54751	10.17676	6.598378	7.509189	14.98987	11.12511
		10	67.16036	46.78626	49.81799	41.94541	25.89015	30.96788	63.48564	45.73111
	0.5	1	0.684591	0.474079	0.53155	0.369609	0.47886	0.317349	0.56522	0.405679
		5	16.97811	11.68458	14.0109	9.074128	6.332651	7.689249	14.14543	10.32677
		10	67.01242	47.1507	55.95899	36.85756	25.66138	30.98032	56.97264	42.71501
	0.8	1	0.698607	0.485264	0.644074	0.317886	0.474104	0.302593	0.511899	0.371916
		5	17.25554	11.98541	15.57179	7.688414	6.328523	7.867748	12.77305	9.282673
		10	69.21875	46.05166	63.87413	31.61019	24.58357	31.73967	49.3765	38.08984

Table 4.8: Estimated MSE values with $n = 100$, ρ=0.95 and $p = 10$

k	d	σ	OLS	RE	Liu	MLiu	JSE	KL	D	MRT
0.3	0.2	1	1.647764	1.313763	1.069362	0.86198	0.842948	1.095843	1.538839	1.294281
		5	40.71	33.17957	26.30692	21.65499	13.55766	27.10629	39.62363	32.60837
		10	163.775	131.6603	105.7489	85.49798	159.6873	107.8976	154.1303	127.9928
	0.5	1	1.645468	1.350547	1.270969	0.73776	0.909437	1.082388	1.428379	1.211917
		5	39.50126	32.65191	31.31112	18.60043	13.29146	26.6243	36.29626	30.53751
		10	164.355	134.3591	124.924	73.56941	53.51524	107.5601	141.567	124.1502
	0.8	1	1.606608	1.308429	1.449291	0.633496	0.901763	1.089603	1.370979	1.175673
		5	40.86388	33.29127	37.05248	15.69909	13.64745	26.87651	35.50154	28.85724
		10	162.8487	131.6424	149.624	63.43105	52.3578	106.348	139.3918	118.7928
0.7	0.2	1	1.659695	1.080654	1.091251	0.86682	0.872424	0.716977	1.467913	1.039821
		5	41.74185	27.10053	26.91722	21.67839	13.24694	17.7788	37.45911	25.32314
		10	163.5662	108.2721	109.3077	86.32453	54.71174	73.17498	147.9726	103.6526
	0.5	1	1.626237	1.084177	1.24799	0.723993	0.875953	0.715541	1.302314	0.963899
		5	41.91594	27.20584	30.88999	18.35651	14.04073	18.19453	32.64567	23.42062
		10	162.0318	107.4106	125.1381	75.32203	53.53499	71.75687	131.3859	94.27214
	0.8	1	1.629941	1.070804	1.466796	0.626071	0.858485	0.711922	1.158164	0.880721
		5	40.0957	26.9383	36.7274	15.72446	13.52693	17.92247	28.25888	21.46445
		10	165.5159	111.242	149.3801	62.94494	52.63885	72.81521	114.9822	88.3346
0.9	0.2	1	1.663912	1.0067	1.073095	0.865602	0.846246	0.616002	1.416783	0.930729
		5	41.73189	25.29061	26.79739	21.46664	13.56245	15.48796	36.19314	23.47476
		10	163.1713	98.58982	106.9026	87.40375	51.57016	61.16043	137.8905	91.80948
	0.5	1	1.661782	0.989589	1.305327	0.74466	0.882593	0.61302	1.244106	0.854408
		5	41.67532	24.79394	31.84781	18.54993	13.66351	15.43741	30.82159	21.38242
		10	165.3744	101.6978	126.7813	71.69086	54.09368	62.22285	123.5861	85.04713
	0.8	1	1.666038	1.004342	1.497129	0.632349	0.892651	0.614048	1.068354	0.782716
		5	40.71283	25.54994	36.42413	15.95893	13.1287	15.34762	27.34089	19.96034
		10	165.1488	99.90803	146.7554	63.64983	52.2601	61.10938	107.1069	78.78281

Table 4.9: Estimated MSE values with $n = 100$, ρ=0.95 and $p = 15$

k	d	σ	OLS	RE	Liu	MLiu	JSE	KL	D	MRT
0.3	0.2	1	2.572112	2.032771	1.648923	1.276486	1.230908	1.596462	2.479989	1.964617
		5	67.42791	51.55236	41.13507	32.77824	20.99291	39.98169	62.99593	50.87378
		10	268.3297	202.0006	167.5809	130.0623	83.33916	159.8791	250.6883	200.2173
	0.5	1	2.643167	2.080777	1.979766	1.084366	1.216022	1.61184	2.322418	1.88121
		5	65.61477	51.68254	49.08291	27.65406	21.32872	39.9083	57.01546	46.78511
		10	265.8191	205.4113	201.2372	110.3256	82.09979	161.1382	235.9736	190.0205
	0.8	1	2.678161	2.081701	2.412455	0.9744	1.203015	1.603734	2.22391	1.826706
		5	67.4041	51.56105	59.83919	24.52412	21.41705	40.49527	53.25286	45.40392
		10	265.1311	206.7917	237.6746	97.0334	82.61019	159.8336	215.7192	181.1915
0.7	0.2	1	2.647089	1.650768	1.632388	1.328115	1.199673	1.071931	2.335296	1.54972
		5	68.2147	42.15528	40.70108	32.73141	21.52908	27.33427	58.92133	38.93095
		10	276.0539	161.8433	163.863	129.8865	85.20297	108.2032	234.9057	155.0318
	0.5	1	2.725553	1.647557	1.942303	1.11564	1.224581	1.09773	1.981452	1.446693
		5	66.17953	41.27135	48.24111	28.03793	21.16432	27.17617	50.37812	35.49785
		10	270.7805	167.0215	197.6894	108.7586	89.14918	108.4086	199.3251	141.9582
	0.8	1	2.747023	1.638523	2.406902	0.985177	1.222264	1.099068	1.771937	1.344456
		5	67.08282	41.5794	61.07357	24.86553	20.72457	27.10702	43.89465	33.37821
		10	268.5696	165.2787	241.8575	97.8039	81.81127	107.7733	176.9178	133.3491
0.9	0.2	1	2.722368	1.528707	1.638716	1.299102	1.224445	0.98093	2.269103	1.428495
		5	67.93411	38.34031	40.68461	33.02256	21.97023	24.25643	57.1763	34.78406
		10	269.221	151.4117	163.4738	129.4122	81.69785	95.08848	218.8238	140.8135
	0.5	1	2.661082	1.4946	1.993348	1.106123	1.253891	0.965911	1.871593	1.295419
		5	65.87686	37.59458	49.75211	27.59507	21.03756	24.62475	46.33346	32.46492
		10	267.016	152.8228	202.7524	111.5302	83.89605	96.34691	189.0426	130.2938
	0.8	1	2.687762	1.50224	2.460457	0.973756	1.22307	0.983395	1.641619	1.192827
		5	66.77469	37.50647	59.44664	24.3506	21.13343	24.00454	41.25485	30.10909
		10	272.2274	150.3865	236.0741	97.09121	80.65775	97.11156	164.3249	121.3612

estimate values of the MSE of all estimators; for a large number of parameters, the MSE value increases for fixed $n = 100$ and $\rho = 0.95$.
For a better understanding of the simulation result, we have ordered the estimator in the smallest MSE values in Tables 4.10 to 4.18. According to the tables, the OLS estimator gives the highest MSE values as expected. MLiu gives the least MSE for lower $k = 0.3$ with lower σ when Liu, KL gives a little higher value for fixed $n = 30, \rho = 0.90, p = 3$ in table 4.10 but with the increase of the value of σ, JSE estimator gives smaller MSE values than others. And for large values of $k = 0.9$, KL gives the least MSE values, also MLiu, JSE, and MRT give small MSE values for large values of k.
Again for the following table 4.11 for $n = 50, \rho = 0.90$, and $p = 3$, we can see that for the small value of k (0.3) MLiu gives the smallest value of MSE but for bigger σ JSE always gives the smallest one.
These results are almost the same in all of the tables except one (Table 4.14). In this table, we can see that for $k = 0.3$ MLiu gives the smallest one but with the increase of k (k = 0.7 and 0.9) KL gives the smallest MSE values and for k = 0.7 and d = 0.8 MLiu also gives the least MSE values.

5 Real-life Example

In this section, we consider the data set on Portland cement originally coming from Woods et al. (1932)[35], and this data set has been widely analyzed by Kaciranlar et al.(1999)[15] and Kibria (2003)[18] among others. The data set is given as follows:

$$X = \begin{pmatrix} 7 & 26 & 6 & 60 \\ 1 & 29 & 15 & 52 \\ 11 & 56 & 8 & 20 \\ 11 & 31 & 8 & 47 \\ 7 & 52 & 6 & 33 \\ 11 & 55 & 9 & 22 \\ 3 & 71 & 17 & 6 \\ 1 & 31 & 22 & 44 \\ 2 & 54 & 18 & 22 \\ 21 & 47 & 4 & 26 \\ 1 & 40 & 23 & 34 \\ 11 & 66 & 9 & 12 \\ 10 & 68 & 8 & 12 \end{pmatrix}, y = \begin{pmatrix} 78.5 \\ 74.3 \\ 104.3 \\ 87.6 \\ 95.9 \\ 109.2 \\ 102.7 \\ 72.5 \\ 93.1 \\ 115.9 \\ 83.8 \\ 113.3 \\ 109.4 \end{pmatrix}$$

We will consider the following regression model:

$$y_i = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \beta_4 X_4 + \varepsilon_i \tag{5.1}$$

where y_i is the heat evolved after 180 days of curing measured in calories per gram of cement, X_1 represnts tricalcium aluminate, X_2 represnts tricalcium silicate, X_3

Table 4.10: For $n = 30, \rho = 0.90, p = 3$

k	d	σ	1	2	3	4	5	6	7	8
0.3	0.2	1	MLiu	Liu	KL	JSE	MRT	RE	D	OLS
		5	JSE	MLiu	Liu	KL	MRT	RE	D	OLS
		10	JSE	MLiu	Liu	KL	MRT	RE	D	OLS
	0.5	1	MLiu	KL	MRT	JSE	Liu	RE	D	OLS
		5	MLiu	JSE	KL	MRT	Liu	RE	D	OLS
		10	JSE	MLiu	KL	MRT	Liu	RE	D	OLS
	0.8	1	MLiu	KL	MRT	JSE	RE	D	Liu	OLS
		5	MLiu	JSE	KL	MRT	RE	D	Liu	OLS
		10	MLiu	JSE	KL	MRT	RE	Liu	D	OLS
0.7	0.2	1	KL	MLiu	MRT	Liu	JSE	RE	D	OLS
		5	KL	MLiu	JSE	MRT	Liu	RE	D	OLS
		10	JSE	KL	MLiu	MRT	Liu	RE	D	OLS
	0.5	1	KL	MLiu	MRT	JSE	RE	D	Liu	OLS
		5	KL	MLiu	JSE	MRT	RE	Liu	D	OLS
		10	KL	MLiu	JSE	MRT	RE	Liu	D	OLS
	0.8	1	MLiu	KL	MRT	D	JSE	RE	Liu	OLS
		5	MLiu	KL	JSE	MRT	D	RE	Liu	OLS
		10	MLiu	JSE	KL	MRT	D	RE	Liu	OLS
0.9	0.2	1	KL	MLiu	MRT	Liu	RE	JSE	D	OLS
		5	KL	JSE	MLiu	MRT	Liu	RE	D	OLS
		10	KL	JSE	MLiu	MRT	Liu	RE	D	OLS
	0.5	1	KL	MLiu	MRT	RE	D	JSE	Liu	OLS
		5	KL	MLiu	JSE	MRT	RE	D	Liu	OLS
		10	KL	MLiu	JSE	MRT	D	RE	Liu	OLS
	0.8	1	KL	MLiu	MRT	D	JSE	RE	Liu	OLS
		5	KL	MLiu	JSE	MRT	D	RE	Liu	OLS
		10	KL	MLiu	JSE	MRT	D	RE	Liu	OLS

Table 4.11: For $n = 50, \rho = 0.90, p = 3$

k	d	σ	1	2	3	4	5	6	7	8
0.3	0.2	1	MLiu	Liu	KL	JSE	MRT	RE	D	OLS
		5	JSE	MLiu	Liu	KL	MRT	RE	D	OLS
		10	JSE	MLiu	Liu	KL	MRT	RE	D	OLS
	0.5	1	MLiu	KL	JSE	MRT	Liu	RE	D	OLS
		5	JSE	MLiu	KL	MRT	Liu	RE	D	OLS
		10	JSE	MLiu	KL	MRT	Liu	RE	D	OLS
	0.8	1	MLiu	KL	MRT	JSE	RE	D	Liu	OLS
		5	JSE	MLiu	KL	MRT	RE	D	Liu	OLS
		10	JSE	MLiu	KL	MRT	RE	Liu	D	OLS
0.7	0.2	1	KL	MLiu	MRT	Liu	JSE	RE	D	OLS
		5	JSE	KL	MLiu	MRT	Liu	RE	D	OLS
		10	JSE	KL	MLiu	MRT	Liu	RE	D	OLS
	0.5	1	MLiu	KL	MRT	RE	JSE	Liu	D	OLS
		5	JSE	MLiu	KL	MRT	Liu	RE	D	OLS
		10	JSE	MLiu	KL	MRT	RE	Liu	D	OLS
	0.8	1	MLiu	KL	MRT	JSE	RE	D	Liu	OLS
		5	JSE	MLiu	KL	MRT	D	RE	Liu	OLS
		10	JSE	MLiu	KL	MRT	D	RE	Liu	OLS
0.9	0.2	1	KL	MLiu	MRT	Liu	RE	JSE	D	OLS
		5	JSE	KL	MLiu	MRT	Liu	RE	D	OLS
		10	JSE	KL	MLiu	MRT	Liu	RE	D	OLS
	0.5	1	KL	MLiu	MRT	RE	JSE	Liu	D	OLS
		5	JSE	KL	MLiu	MRT	RE	D	Liu	OLS
		10	JSE	KL	MLiu	MRT	Liu	RE	D	OLS
	0.8	1	KL	MLiu	MRT	D	RE	JSE	Liu	OLS
		5	JSE	KL	MLiu	MRT	D	RE	Liu	OLS
		10	JSE	KL	MLiu	MRT	D	RE	Liu	OLS

Table 4.12: For $n = 100, \rho = 0.90, p = 3$

k	d	σ	1	2	3	4	5	6	7	8
0.3	0.2	1	MLiu	KL	Liu	JSE	RE	MRT	D	OLS
		5	JSE	MLiu	Liu	KL	MRT	D	RE	OLS
		10	JSE	MLiu	Liu	KL	MRT	RE	D	OLS
	0.5	1	MLiu	KL	JSE	RE	Liu	MRT	D	OLS
		5	JSE	MLiu	MRT	KL	RE	Liu	D	OLS
		10	JSE	MLiu	MRT	KL	Liu	D	RE	OLS
	0.8	1	MLiu	KL	JSE	MRT	RE	D	Liu	OLS
		5	JSE	MLiu	KL	Liu	MRT	RE	D	OLS
		10	JSE	MLiu	MRT	KL	D	RE	Liu	OLS
0.7	0.2	1	KL	MLiu	MRT	JSE	Liu	RE	D	OLS
		5	JSE	KL	MLiu	MRT	Liu	RE	D	OLS
		10	JSE	MLiu	KL	MRT	Liu	RE	D	OLS
	0.5	1	KL	MLiu	MRT	D	RE	JSE	Liu	OLS
		5	JSE	MLiu	KL	MRT	RE	D	Liu	OLS
		10	JSE	KL	MLiu	MRT	RE	D	Liu	OLS
	0.8	1	MLiu	KL	MRT	JSE	D	RE	Liu	OLS
		5	JSE	MLiu	KL	MRT	RE	D	Liu	OLS
		10	JSE	MLiu	KL	MRT	RE	D	Liu	OLS
0.9	0.2	1	KL	MLiu	RE	MRT	Liu	JSE	D	OLS
		5	JSE	KL	MLiu	MRT	RE	Liu	D	OLS
		10	JSE	KL	MLiu	MRT	Liu	RE	D	OLS
	0.5	1	KL	MLiu	MRT	JSE	RE	Liu	D	OLS
		5	JSE	KL	MLiu	MRT	RE	Liu	D	OLS
		10	JSE	KL	M	MRT	RE	Liu	D	OLS
	0.8	1	MLiu	KL	MRT	RE	D	JSE	Liu	OLS
		5	JSE	KL	MLiu	MRT	D	RE	Liu	OLS
		10	JSE	KL	MLiu	MRT	D	RE	Liu	OLS

Table 4.13: For $n = 50, \rho = 0.90, p = 5$

k	d	σ	1	2	3	4	5	6	7	8
0.3	0.2	1	MLiu	JSE	Liu	KL	MRT	RE	D	OLS
		5	JSE	MLiu	Liu	KL	MRT	RE	D	OLS
		10	JSE	MLiu	Liu	KL	MRT	RE	D	OLS
	0.5	1	MLiu	JSE	KL	Liu	MRT	RE	D	OLS
		5	JSE	MLiu	KL	Liu	MRT	RE	D	OLS
		10	JSE	MLiu	KL	Liu	MRT	RE	D	OLS
	0.8	1	MLiu	JSE	KL	MRT	R	D	Liu	OLS
		5	JSE	MLiu	KL	MRT	D	RE	Liu	OLS
		10	JSE	MLiu	KL	MRT	RE	D	Liu	OLS
0.7	0.2	1	KL	MLiu	JSE	Liu	MRT	RE	D	OLS
		5	JSE	KL	MLiu	MRT	Liu	RE	D	OLS
		10	JSE	KL	MLiu	MRT	Liu	RE	D	OLS
	0.5	1	KL	MLiu	MRT	JSE	RE	D	Liu	OLS
		5	JSE	MLiu	KL	MRT	RE	Liu	D	OLS
		10	JSE	KL	MLiu	MRT	RE	Liu	D	OLS
	0.8	1	MLiu	KL	MRT	JSE	RE	D	Liu	OLS
		5	JSE	MLiu	KL	MRT	RE	D	Liu	OLS
		10	JSE	MLiu	KL	MRT	RE	D	Liu	OLS
0.9	0.2	1	KL	MLiu	MRT	JSE	RE	Liu	D	OLS
		5	JSE	KL	MLiu	MRT	RE	Liu	D	OLS
		10	JSE	KL	MLiu	MRT	RE	Liu	D	OLS
	0.5	1	KL	MLiu	MRT	RE	JSE	D	Liu	OLS
		5	JSE	KL	MLiu	MRT	RE	D	Liu	OLS
		10	JSE	KL	MLiu	MRT	RE	D	Liu	OLS
	0.8	1	KL	MLiu	MRT	RE	JSE	D	Liu	OLS
		5	JSE	KL	MLiu	MRT	RE	D	Liu	OLS
		10	JSE	MLiu	KL	MRT	RE	D	Liu	OLS

Table 4.14: For $n = 50, \rho = 0.95, p = 5$

k	d	σ	1	2	3	4	5	6	7	8
0.3	0.2	1	MLiu	Liu	KL	JSE	MRT	RE	D	OLS
		5	JSE	MLiu	Liu	KL	MRT	RE	D	OLS
		10	JSE	MLiu	KL	Liu	MRT	RE	D	OLS
	0.5	1	MLiu	KL	JSE	MRT	Liu	RE	D	OLS
		5	MLiu	JSE	KL	MRT	Liu	RE	D	OLS
		10	MLiu	JSE	KL	MRT	Liu	RE	D	OLS
	0.8	1	MLiu	KL	JSE	MRT	RE	D	Liu	OLS
		5	MLiu	JSE	KL	MRT	RE	D	Liu	OLS
		10	MLiu	JSE	KL	MRT	RE	D	Liu	OLS
0.7	0.2	1	KL	MLiu	MRT	RE	Liu	JSE	D	OLS
		5	KL	JSE	MLiu	MRT	RE	Liu	D	OLS
		10	KL	JSE	MLiu	MRT	RE	Liu	D	OLS
	0.5	1	KL	MLiu	MRT	JSE	RE	Liu	D	OLS
		5	KL	MLiu	JSE	MRT	RE	Liu	D	OLS
		10	KL	MLiu	JSE	MRT	RE	Liu	D	OLS
	0.8	1	MLiu	KL	MRT	RE	JSE	D	Liu	OLS
		5	MLiu	KL	JSE	MRT	RE	D	Liu	OLS
		10	MLiu	KL	JSE	MRT	RE	D	Liu	OLS
0.9	0.2	1	KL	MLiu	MRT	RE	Liu	JSE	D	OLS
		5	KL	JSE	MLiu	MRT	RE	Liu	D	OLS
		10	KL	JSE	MLiu	MRT	RE	Liu	D	OLS
	0.5	1	KL	MLiu	MRT	RE	JSE	D	Liu	OLS
		5	KL	MLiu	JSE	MRT	RE	D	Liu	OLS
		10	KL	MLiu	JSE	MRT	RE	D	Liu	OLS
	0.8	1	KL	MLiu	MRT	RE	D	JSE	Liu	OLS
		5	KL	MLiu	MRT	JSE	RE	D	Liu	OLS
		10	KL	MLiu	MRT	JSE	RE	D	Liu	OLS

Table 4.15: For $n = 50, \rho = 0.99, p = 5$

k	d	σ	1	2	3	4	5	6	7	8
0.3	0.2	1	MLiu	KL	Liu	MRT	RE	JSE	D	OLS
		5	MLiu	KL	Liu	MRT	RE	JSE	D	OLS
		10	MLiu	KL	Liu	MRT	RE	JSE	D	OLS
	0.5	1	MLiu	KL	MRT	RE	Liu	JSE	D	OLS
		5	MLiu	KL	MRT	RE	JSE	Liu	D	OLS
		10	MLiu	KL	MRT	RE	JSE	Liu	D	OLS
	0.8	1	KL	MLiu	MRT	RE	D	JSE	Liu	OLS
		5	KL	MLiu	MRT	RE	JSE	D	Liu	OLS
		10	KL	MLiu	MRT	RE	D	JSE	Liu	OLS
0.7	0.2	1	MLiu	MRT	RE	KL	Liu	JSE	D	OLS
		5	MLiu	MRT	RE	KL	Liu	JSE	D	OLS
		10	MLiu	MRT	RE	KL	Liu	JSE	D	OLS
	0.5	1	MLiu	MRT	RE	KL	D	Liu	JSE	OLS
		5	MLiu	MRT	RE	KL	D	JSE	Liu	OLS
		10	MLiu	MRT	RE	KL	D	JSE	Liu	OLS
	0.8	1	MRT	MLiu	RE	KL	D	JSE	Liu	OLS
		5	MRT	MLiu	RE	KL	D	JSE	Liu	OLS
		10	MRT	MLiu	RE	KL	D	JSE	Liu	OLS
0.9	0.2	1	MLiu	MRT	RE	KL	Liu	JSE	D	OLS
		5	MLiu	MRT	RE	Liu	KL	JSE	D	OLS
		10	MLiu	MRT	RE	Liu	KL	JSE	D	OLS
	0.5	1	MLiu	MRT	RE	KL	D	JSE	Liu	OLS
		5	MLiu	MRT	RE	D	KL	JSE	Liu	OLS
		10	MLiu	MRT	RE	KL	D	JSE	Liu	OLS
	0.8	1	MRT	RE	MLiu	D	KL	JSE	Liu	OLS
		5	MRT	RE	MLiu	D	KL	JSE	Liu	OLS
		10	MRT	RE	MLiu	D	KL	JSE	Liu	OLS

Table 4.16: For $n = 100, \rho = 0.95, p = 5$

k	d	σ	1	2	3	4	5	6	7	8
0.3	0.2	1	MLiu	JSE	Liu	KL	MRT	RE	D	OLS
		5	JSE	MLiu	Liu	KL	MRT	RE	D	OLS
		10	JSE	MLiu	Liu	KL	RE	MRT	D	OLS
	0.5	1	MLiu	JSE	KL	Liu	MRT	RE	D	OLS
		5	JSE	MLiu	KL	Liu	MRT	RE	D	OLS
		10	JSE	MLiu	KL	MRT	Liu	RE	D	OLS
	0.8	1	MLiu	JSE	KL	MRT	RE	D	Liu	OLS
		5	JSE	MLiu	KL	MRT	RE	D	Liu	OLS
		10	JSE	MLiu	KL	MRT	RE	D	Liu	OLS
0.7	0.2	1	KL	MLiu	JSE	MRT	Liu	RE	D	OLS
		5	JSE	KL	MLiu	MRT	Liu	RE	D	OLS
		10	JSE	KL	MLiu	MRT	Liu	RE	D	OLS
	0.5	1	MLiu	KL	MRT	JSE	RE	Liu	D	OLS
		5	JSE	KL	MLiu	MRT	RE	Liu	D	OLS
		10	JSE	MLiu	KL	MRT	RE	Liu	D	OLS
	0.8	1	MLiu	KL	MRT	JSE	RE	D	Liu	OLS
		5	JSE	MLiu	KL	MRT	RE	D	Liu	OLS
		10	JSE	MLiu	KL	MRT	RE	D	Liu	OLS
0.9	0.2	1	KL	MLiu	MRT	RE	JSE	Liu	D	OLS
		5	JSE	KL	MLiu	MRT	RE	Liu	D	OLS
		10	JSE	KL	MLiu	MRT	RE	Liu	D	OLS
	0.5	1	KL	MLiu	MRT	RE	JSE	Liu	D	OLS
		5	JSE	KL	MLiu	MRT	RE	Liu	D	OLS
		10	JSE	KL	MLiu	MRT	RE	Liu	D	OLS
	0.8	1	KL	MLiu	MRT	JSE	RE	D	Liu	OLS
		5	JSE	MLiu	KL	MRT	RE	D	Liu	OLS
		10	JSE	MLiu	KL	MRT	RE	D	Liu	OLS

Table 4.17: For $n = 100, \rho = 0.95, p = 10$

k	d	σ	1	2	3	4	5	6	7	8
0.3	0.2	1	JSE	MLiu	Liu	KL	MRT	RE	D	OLS
		5	JSE	MLiu	Liu	KL	MRT	RE	D	OLS
		10	JSE	MLiu	Liu	KL	MRT	RE	D	OLS
	0.5	1	MLiu	JSE	KL	MRT	Liu	RE	D	OLS
		5	JSE	MLiu	KL	MRT	Liu	RE	D	OLS
		10	JSE	MLiu	KL	MRT	Liu	RE	D	OLS
	0.8	1	MLiu	JSE	KL	MRT	RE	D	Liu	OLS
		5	JSE	MLiu	KL	MRT	RE	D	Liu	OLS
		10	JSE	MLiu	KL	MRT	RE	D	Liu	OLS
0.7	0.2	1	KL	MLiu	JSE	MRT	RE	D	Liu	OLS
		5	JSE	KL	MLiu	MRT	RE	Liu	D	OLS
		10	JSE	KL	MLiu	MRT	RE	Liu	D	OLS
	0.5	1	MLiu	KL	JSE	MRT	RE	D	Liu	OLS
		5	JSE	MLiu	KL	MRT	RE	D	Liu	OLS
		10	JSE	MLiu	KL	MRT	RE	D	Liu	OLS
	0.8	1	KL	MLiu	JSE	MRT	RE	Liu	D	OLS
		5	JSE	KL	MLiu	MRT	RE	Liu	D	OLS
		10	JSE	KL	MLiu	MRT	RE	Liu	D	OLS
0.9	0.2	1	KL	MLiu	MLiu	JSE	RE	D	Liu	OLS
		5	JSE	KL	MLiu	MRT	RE	D	Liu	OLS
		10	JSE	KL	MLiu	MRT	RE	D	Liu	OLS
	0.5	1	KL	MLiu	MRT	JSE	RE	D	Liu	OLS
		5	JSE	KL	MLiu	MRT	RE	D	Liu	OLS
		10	JSE	KL	MLiu	MRT	RE	D	Liu	OLS
	0.8	1	KL	MLiu	MRT	JSE	RE	D	Liu	OLS
		5	JSE	KL	MLiu	MRT	RE	D	Liu	OLS
		10	JSE	KL	MLiu	MRT	RE	D	Liu	OLS

Table 4.18: For $n = 100, \rho = 0.95, p = 15$

k	d	σ	1	2	3	4	5	6	7	8
0.3	0.2	1	JSE	MLiu	KL	Liu	MRT	RE	D	OLS
		5	JSE	MLiu	KL	Liu	MRT	RE	D	OLS
		10	JSE	MLiu	KL	Liu	MRT	RE	D	OLS
	0.5	1	MLiu	JSE	KL	MRT	Liu	RE	D	OLS
		5	JSE	MLiu	KL	MRT	Liu	RE	D	OLS
		10	JSE	MLiu	KL	MRT	Liu	RE	D	OLS
	0.8	1	MLiu	JSE	KL	MRT	RE	D	Liu	OLS
		5	JSE	MLiu	KL	MRT	RE	D	Liu	OLS
		10	JSE	MLiu	KL	MRT	RE	D	Liu	OLS
0.7	0.2	1	KL	JSE	MLiu	MRT	Liu	RE	D	OLS
		5	JSE	KL	MLiu	MRT	Liu	RE	D	OLS
		10	JSE	KL	MLiu	MRT	RE	Liu	D	OLS
	0.5	1	KL	MLiu	JSE	MRT	RE	Liu	D	OLS
		5	JSE	KL	MLiu	MRT	RE	Liu	D	OLS
		10	JSE	KL	MLiu	MRT	RE	Liu	D	OLS
	0.8	1	MLiu	KL	JSE	MRT	RE	D	Liu	OLS
		5	JSE	MLiu	KL	MRT	RE	D	Liu	OLS
		10	JSE	MLiu	KL	MRT	RE	D	Liu	OLS
0.9	0.2	1	KL	JSE	MLiu	MRT	RE	Liu	D	OLS
		5	JSE	KL	MLiu	MRT	RE	Liu	D	OLS
		10	JSE	KL	MLiu	MRT	RE	Liu	D	OLS
	0.5	1	KL	MLiu	JSE	MRT	RE	D	Liu	OLS
		5	JSE	KL	MLiu	MRT	RE	D	Liu	OLS
		10	JSE	KL	MLiu	MRT	RE	D	Liu	OLS
	0.8	1	MLiu	KL	MLiu	JSE	RE	D	Liu	OLS
		5	JSE	KL	MLiu	MRT	RE	D	Liu	OLS
		10	JSE	MLiu	KL	MRT	RE	D	Liu	OLS

represents tetracalcium aluminoferrite, and X_4 represents β-dicalcium silicate. We can fit a linear regression to y and (X_1, X_2, X_3, X_4) and the coefficient vector is $(\beta_0, \beta_1, \beta_2, \beta_3, \beta_4)$. Then, the design matrix is $X = (1, X_1, X_2, X_3, X_4)$, where 1 is a 13×1 vector where all elements are 1. The eigenvalue of $X^{'}X$ is $\lambda_1 = 44676.21, \lambda_2 = 5965.42, \lambda_3 = 809.95, \lambda_4 = 105.42, \lambda_5 = 0.00123$ and the condition number is approximately $3.662e + 007$. The eigenvalues and condition number indicate the severe multicollinearity among the regressors. The ordinary least squares estimator of β and σ^2,

$$\hat{\beta} = (62.4054, 1.5511, 0.5102, 0.1019, -0.1441)^{'}$$

with MSE($\hat{\beta}$)= 4912.09 and $\hat{\sigma}^2 = 5.983$.

Table 5.1: Regression Coefficients and MSE for $\hat{k}_{AM}$=0.7948, $\hat{k}_{KL}(min) = 0.001218094$ and d=0.95

coef	OLS	RE	Liu	MLiu	JSE	KL	D	MRT
ϑ_0	62.40537	0.142583	59.29125	-59.0453	27.60752	0.045478	0.147605	0.095992
ϑ_1	1.551103	2.180653	1.58245	2.773665	0.686193	2.19255	2.181144	2.17089
ϑ_2	0.510168	1.154382	0.542421	1.76804	0.225693	1.152864	1.154205	1.157227
ϑ_3	0.101909	0.749152	0.134181	1.360517	0.045084	0.758011	0.749489	0.74229
ϑ_4	-0.14406	0.486529	-0.1125	1.086798	-0.06373	0.485864	0.486396	0.488544
MSE	4912.127	3878.358	4433.769	4411.087	2173.077	3890.473	3877.735	3884.138

Table 5.2: Regression Coefficients and MSE for $\hat{k}_{GM}$=28.9913,$\hat{k}_{KL}(HMN) = 0.004747493$ and d=0.95

coef	OLS	RE	Liu	MLiu	JSE	KL	D	MRT
ϑ_0	62.40537	0.045543	59.29125	-59.0453	-36.8466	0.045478	0.147605	0.095992
ϑ_1	1.551103	1.873706	1.58245	2.773665	2.5719523	2.19255	2.181144	2.17089
ϑ_2	0.510168	1.225298	0.542421	1.76804	1.5330997	1.152864	1.154205	1.157227
ϑ_3	0.101909	0.535757	0.134181	1.360517	0.045084	1.1461043	0.749489	0.74229
ϑ_4	-0.14406	0.532899	-0.1125	1.086798	-0.06373	0.8585387	0.486396	0.488544
MSE	4912.127	3890.053	4411.087	4411.087	2173.077	11574.63	3890.024	3890.458

The regression analysis with fixed d and different values of k are provided in Tables 5.1 and 5.2. From Tables 5.1 and 5.2, we observed that JSE performed the best, which supported the simulation study to some extent..

6 Some Concluding Remarks

In this paper, we consider some one and two parameter estimators, namely, OLS, ridge estimator (RE), Liu estimator, Modified Liu estimator (MLiu), James- Stein estimator (JSE), KL estimator, D estimator by Dorugade (2014)[6] and Modified Ridge Type estimator (MRT) for estimating the regression parameters in the linear regression model. The bias, covariance and MSE expressions of the estimators are given. These estimators depend on the biasing parameters k and d and their corresponding estimators are outlined. We theoretically compared the performance of the estimators in terms of smaller mean squared error (MSE) criteria. A Monte Carlo simulation study has been conducted to compare the estimators numerically. Finally, for illustration purposes, a real-life data is analyzed. The performance of the estimators depends on the sample size, number of regressors and amount of correlation among regressors. From both simulation study and application, it appears that the JSE estimator performed better in most of the simulation conditions followed by RE, Liu, MLiu, KL, D and MRT which are supported by Amin et al. (2020)[3] and Akram et al. (2021)[2]. Therefore, we recommend the practitioners to use JSE estimator among other estimators.

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