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A COMPREHENSIVE REVIEW OF THE HERMITE-HADAMARD INEQUALITY PERTAINING TO FRACTIONAL DIFFERENTIAL OPERATORS

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Abstract. A review on Hermite-Hadamard type inequalities connected with a different classes of convexities and fractional differential operators is presented. In the various classes of convexities it includes, classical convex functions, quasi-convex functions, p-convex functions, strongly-m-convex functions, strongly-(θ, m)-convex functions, (s, m)-convex functions, $(\theta, h-m)$-convex functions, strongly $(\theta, h-m)$-convex functions, (h, m)-convex functions of the second type, m-convex functions, h-convex functions, (h, m)-convex functions, relative-convex functions, exponentially $(\theta, h-m)$-convex functions, harmonically h-convex functions and geometric-arithmetically s-convex functions. In the fractional differential operators it includes, Caputo fractional derivative, k-Caputo fractional derivative and Hilfer fractional derivative.

1 Introduction

The term convex function is a family of important functions that is widely acknowledged in the field of mathematical analysis. This family represents significant parts of the theory of inequality. Convex functions have also been extensively used and applied in a variety of study areas, including physics, financial operations, optimization, engineering etc. The notion of modified convexity and the theory of inequality are frequently employed in optimization. Because of their prominence and effectiveness, the Hermite-Hadamard (H-H) integral inequalities with convex functions are a prominent research subject for many mathematicians.

The subject fractional calculus addressed the research of asserted fractional derivatives and integrations over complex domains and their utilization. Fractional calculus has gained considerable popularity over the past ten years. Numerous investigators are researchers are intrigued by this topic due to its numerous applications in various fields, for example, designing, material science, fluid mechanics,

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probability theory, image processing, biomathematics and viscoelasticity etc. Recently, it has been observed and reported that a number of mathematicians have been employing their notations and methods to investigate various definitions that might be applicable to fractional-order integrals and derivatives.

Due to its numerous implementations in physics and mathematics, mathematical inequalities have major implications in both the study of mathematics and other branches of mathematics, for example see the papers [1]-[4]. A convex function is one of the most important functions used to investigate different intriguing inequalities, which is stated that:

A real-valued function Υ is called convex, if

$$\Upsilon(t\mathtt{v}_1 + (1-t)\mathtt{v}_2) \leq t\Upsilon(\mathtt{v}_1) + (1-t)\Upsilon(\mathtt{v}_2)$$

holds true $\forall\ \mathtt{v}_1, \mathtt{v}_2 \in I$ and $t \in [0,1]$.

The analysis of convex functions, and in specific, the H-H inequality, has gained the attention and interest of many scholars in recent years, which is stated that:

$$\Upsilon\left(\frac{\mathtt{v}_1+\mathtt{v}_2}{2}\right) \leq \frac{1}{\mathtt{v}_2-\mathtt{v}_1}\int_{\mathtt{v}_1}^{\mathtt{v}_2}\Upsilon(x)dx \leq \frac{\Upsilon(\mathtt{v}_1)+\Upsilon(\mathtt{v}_2)}{2}. \tag{1.1}$$

The above inequality (1.1) was studied first time by Hermite [5] and examined by Hadamard [6] in 1893.

In [7], Dragomir investigated the H-H result for differentiable function, which is given as:

Theorem 1. *Assume that a real-valued function Υ is differentiable on I°, $\mathtt{v}_1, \mathtt{v}_2 \in I^\circ$ with $\mathtt{v}_1 < \mathtt{v}_2$. If $|\Upsilon'|$ is convex on $[\mathtt{v}_1, \mathtt{v}_2]$, then*

$$\left|\frac{\Upsilon(\mathtt{v}_1)+\Upsilon(\mathtt{v}_2)}{2} - \frac{1}{\mathtt{v}_2-\mathtt{v}_1}\int_{\mathtt{v}_1}^{\mathtt{v}_2}\Upsilon(x)dx\right| \leq \frac{(\mathtt{v}_2-\mathtt{v}_1)}{8}(|\Upsilon'(\mathtt{v}_1)| + |\Upsilon'(\mathtt{v}_2)|). \tag{1.2}$$

Very recently in [8] a comprehensive and up-to-date review on H-H-type inequalities for different kinds of convexities and different kinds of fractional integral operators is presented. The present paper compliments the paper [8] presenting a comprehensive and up-to-date review of H-H-type inequalities pertaining to fractional differential operators (FDO). Thus, paper [8] and this paper are comprehensive and up-to-date reviews on H-H-type inequalities for both integral and differential fractional operators. We believe that the present review will motivate and provide a platform for the researchers working on H-H-type inequalities to learn about the available work on the topic before developing new results.

This review paper is constructed in the following manner. In Section 2, we present H-H type inequalities via Caputo fractional derivative (CFD), in Section 3 we collect H-H type inequalities via k-CFD, while H-H type inequalities via Hadamard fractional derivative (HFD) are included in Section 4. A variety of classes of convexities are associated with H-H type inequalities, like classical convex functions, quasi-convex functions, p-convex functions, strongly-m-convex functions, strongly-(θ, m)-convex functions, (s, m)-convex functi-ons, $(\theta, h-m)$-convex functions, strongly $(\theta, h-m)$-convex functions, (h, m)-convex functions of the second type, m-convex functions, h-convex functions, (h, m)-convex functions, relative -convex functions, exponentially $(\theta, h-m)$-convex functions, harmonically h-convex functions and geometric-arithmetically s-convex functions.

Note that our aim here is a more comprehensive and complete review and the incorporation of as many results as appropriate is considered to demonstrate the development and progress on the subject. For this regard, any proofs (which are very lengthy) are excluded, and the reader is directed to the related article as a result.

2 Hermite-Hadamard Type Inequalities via Caputo Fractional Derivative

Let us start with the definition of the left-sided and right-sided CFD.

Definition 2 ([9]). *Let $\alpha > 0$ and $\alpha \notin \{1, 2, 3, \ldots\}$, $n = [\alpha] + 1$, $\Upsilon \in AC^m[\mathtt{v}_1, \mathtt{v}_2]$. The CFD for right and left-sided of order α are stated that:*

$$({}^C D^{\alpha}_{\mathtt{v}_2-}\Upsilon)(x) = \frac{1}{\Gamma(n-\alpha)} \int_{\mathtt{v}_1}^{x} \frac{\Upsilon^{(n)}(t)}{(x-t)^{\alpha-n+1}} dt, \quad x > \mathtt{v}_1$$

and

$$({}^C D^{\alpha}_{\mathtt{v}_1+}\Upsilon)(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_{x}^{b} \frac{\Upsilon^{(n)}(t)}{(t-x)^{\alpha-n+1}} dt, \quad x < \mathtt{v}_2.$$

In the following we give H-H type inequalities for n-times differentiable convex functions via CFD.

Theorem 3 ([10]). *Let $\Upsilon : [\mathtt{v}_1, v_2] \to \mathbb{R}$, $0 \le \mathtt{v}_1 < \mathtt{v}_2$ be such that $\Upsilon \in C^m[\mathtt{v}_1, \mathtt{v}_2]$. Also suppose $\Upsilon^{(n)}$ be convex and positive mapping on $[\mathtt{v}_1, \mathtt{v}_2]$. Then fractional inequality pertaining to CFD is given as:*

$$\begin{aligned}\Upsilon^{(n)}\Big(\frac{\mathtt{v}_1+\mathtt{v}_2}{2}\Big) &\le \frac{\Gamma(n-\alpha+1)}{2(\mathtt{v}_2-\mathtt{v}_1)^{n-\alpha}}\Big[{}^C D^{\alpha}_{\mathtt{v}_1+}\Upsilon^{(n)}(\mathtt{v}_2) + (-1)^n {}^C D^{\alpha}_{\mathtt{v}_2-}\Upsilon^{(n)}(\mathtt{v}_1)\Big] \\ &\le \frac{\Upsilon^{(n)}(\mathtt{v}_1)+\Upsilon^{(n)}(\mathtt{v}_2)}{2}.\end{aligned}$$

Theorem 4 ([10]). *Let $\Upsilon : [\mathtt{v}_1, \mathtt{v}_2] \to \mathbb{R}$, $0 \le \mathtt{v}_1 < \mathtt{v}_2$ be such that $\Upsilon \in C^{n+1}[\mathtt{v}_1, \mathtt{v}_2]$. Also let $|\Upsilon^{(n+1)}|$ is convex on $[\mathtt{v}_1, \mathtt{v}_2]$. Then fractional inequality pertaining to CFD is given as:*

$$\left|\frac{\Upsilon^{(n)}(\mathtt{v}_1)+\Upsilon^{(n)}(\mathtt{v}_2)}{2} - \frac{\Gamma(n-\alpha+1}{2(\mathtt{v}_2-\mathtt{v}_1)^{n-\alpha}}\Big[{}^C D^{\alpha}_{\mathtt{v}_1+}\Upsilon(\mathtt{v}_2) + (-1)^n {}^C D^{\alpha}_{\mathtt{v}_2-}\Upsilon(\mathtt{v}_1)\Big]\right|$$

$$\leq \frac{\mathtt{v}_2-\mathtt{v}_1}{2(n-\alpha+1)}\Big(1-\frac{1}{2^{n-\alpha}}\Big)\Big[|\Upsilon^{(n+1)}(\mathtt{v}_1)|+|\Upsilon^{(n+1)}(\mathtt{v}_2)|\Big].$$

Theorem 5 ([11]). *Let $\Upsilon:[\mathtt{v}_1,\mathtt{v}_2]\to\mathbb{R}$ be a positive mapping with $0\leq\mathtt{v}_1<\mathtt{v}_2$ and $\Upsilon\in C^n[\mathtt{v}_1,\mathtt{v}_2]$. If $\Upsilon^{(n)}$ is a convex on $[\mathtt{v}_1,\mathtt{v}_2]$, then fractional inequality pertaining to CFD is given as:*

$$\begin{aligned}
&\Upsilon^{(n)}\Big(\frac{\mathtt{v}_1+\mathtt{v}_2}{2}\Big)\\
\leq\ &\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(\mathtt{v}_2-\mathtt{v}_1)^{n-\alpha}}\Big[{}^{C}D^{\alpha}_{\left(\frac{\mathtt{v}_1+\mathtt{v}_2}{2}\right)+}\Upsilon(\mathtt{v}_2)+(-1)^n\ {}^{C}D^{\alpha}_{\left(\frac{\mathtt{v}_1+\mathtt{v}_2}{2}\right)-}\Upsilon(\mathtt{v}_1)\Big]\\
\leq\ &\frac{\Upsilon^{(n)}(\mathtt{v}_1)+\Upsilon^{(n)}(\mathtt{v}_2)}{2}.
\end{aligned}$$

Theorem 6 ([11]). *Let $\Upsilon:[\mathtt{v}_1,\mathtt{v}_2]\to\mathbb{R}$ be a differential mapping on $(\mathtt{v}_1,\mathtt{v}_2)$ with $\mathtt{v}_1<\mathtt{v}_2$ and $\Upsilon\in C^{n+1}[\mathtt{v}_1,\mathtt{v}_2]$. If $|\Upsilon^{(n+1)}|^q$ is convex on $[\mathtt{v}_1,\mathtt{v}_2]$ for $q\geq 1$, then fractional inequalities pertaining to CFD are given as:*

$$\begin{aligned}
&\Bigg|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(\mathtt{v}_2-\mathtt{v}_1)^{n-\alpha}}\Big[{}^{C}D^{\alpha}_{\left(\frac{\mathtt{v}_1+\mathtt{v}_2}{2}\right)+}\Upsilon(\mathtt{v}_2)+(-1)^n\ {}^{C}D^{\alpha}_{\left(\frac{\mathtt{v}_1+\mathtt{v}_2}{2}\right)-}\Upsilon(\mathtt{v}_1)\Big]\\
&-\Upsilon^{(n)}\Big(\frac{\mathtt{v}_1+\mathtt{v}_2}{2}\Big)\Bigg|\\
\leq\ &\frac{\mathtt{v}_2-\mathtt{v}_1}{4(n-\alpha+1)}\Big(\frac{1}{2(n-\alpha+2)}\Big)^{\frac{1}{q}}\Big[\Big[(n-\alpha+1)|\Upsilon^{(n+1)}(\mathtt{v}_1)|^q\\
&+(n-\alpha+3)|\Upsilon^{(n+1)}(\mathtt{v}_2)|^q\Big]^{\frac{1}{q}}+\Big[(n-\alpha+3)|\Upsilon^{(n+1)}(\mathtt{v}_1)|^q\\
&+(n-\alpha+1)|\Upsilon^{(n+1)}(\mathtt{v}_2)|^q\Big]^{\frac{1}{q}}\Big].
\end{aligned}$$

In the next theorems we give Hermite-Jensen-Mercer type inequalities via CFD.

Theorem 7 ([12]). *Suppose that $\Upsilon:[\mathtt{v}_1,\mathtt{v}_2]\to\mathbb{R}$ is a positive function, with $0\leq\mathtt{v}_1<\mathtt{v}_2$ and $\Upsilon\in C^n[\mathtt{v}_1,\mathtt{v}_2]$. If $\Upsilon^{(n)}$ is a convex function on $[\mathtt{v}_1,\mathtt{v}_2]$, then fractional inequalities pertaining to CFD are given as:*

$$\begin{aligned}
&\Upsilon^{(n)}\Big(\mathtt{v}_1+\mathtt{v}_2-\frac{x+y}{2}\Big)\\
\leq\ &\Upsilon^{(n)}(\mathtt{v}_1)+\Upsilon^{(n)}(\mathtt{v}_2)-\frac{\Gamma(n-\alpha+1)}{2(y-x)^{n-\alpha}}\Big[({}^{C}D^{\alpha}_{x+}\Upsilon)(y)+(-1)^n({}^{C}D^{\alpha}_{y-}\Upsilon)(x)\Big]\\
\leq\ &\Upsilon^{(n)}(\mathtt{v}_1)+\Upsilon^{(n)}(\mathtt{v}_2)-\Upsilon^{(n)}\Big(\frac{x+y}{2}\Big),
\end{aligned}$$

for all $x,y\in[\mathtt{v}_1,\mathtt{v}_2]$ and $\alpha>0$.

Theorem 8 ([12]). *Assume that Υ is as in Theorem 7. If $\Upsilon^{(n)}$ is a convex on $[\mathtt{v}_1, \mathtt{v}_2]$, then fractional inequalities pertaining to CFD are given as:*

$$\begin{aligned}\Upsilon^{(n)}\Big(\mathtt{v}_1+\mathtt{v}_2-\frac{x+y}{2}\Big) &\le \frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(y-x)^{n-\alpha}}\Big[\big({}^{C}D^{\alpha}_{(\mathtt{v}_1+\mathtt{v}_2-\frac{x+y}{2})+}\Upsilon\big)(\mathtt{v}_1+\mathtt{v}_2-x)\\ &\quad +(-1)^n\big({}^{C}D^{\alpha}_{(\mathtt{v}_1+\mathtt{v}_2-\frac{x+y}{2})-)}\Upsilon\big)(\mathtt{v}_1+\mathtt{v}_2-y)\Big]\\ &\le \Upsilon^{(n)}(\mathtt{v}_1)+\Upsilon^{(n)}(\mathtt{v}_2)-\frac{^{(n)}(x)+\Upsilon^{(n)}(y)}{2},\end{aligned}$$

for all $x, y \in [\mathtt{v}_1, \mathtt{v}_2]$ and $\alpha > 0$.

Theorem 9 ([12]). *Let $\Upsilon : [\mathtt{v}_1, \mathtt{v}_2] \to \mathbb{R}$ be a differentiable mapping on $(\mathtt{v}_1, \mathtt{v}_2)$ with $\mathtt{v}_1 < \mathtt{v}_2$ and $\Upsilon \in C^{n+1}[\mathtt{v}_1, \mathtt{v}_2]$. If $|\Upsilon^{n+1}|$ is a convex on $[\mathtt{v}_1, \mathtt{v}_2]$, then inequality pertaining to CFD is given as:*

$$\begin{aligned}&\Bigg|\Upsilon^{(n)}\Big(\mathtt{v}_1+\mathtt{v}_2-\frac{x+y}{2}\Big)-\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(y-x)^{n-\alpha}}\Big[\big({}^{C}D^{\alpha}_{(\mathtt{v}_1+\mathtt{v}_2-\frac{x+y}{2})+}\Upsilon\big)(\mathtt{v}_1+\mathtt{v}_2-x)\\ &+(-1)^n\big({}^{C}D^{\alpha}_{(\mathtt{v}_1+\mathtt{v}_2-\frac{x+y}{2})-}\Upsilon\big)(\mathtt{v}_1+\mathtt{v}_2-y)\Big]\Bigg|\\ &\le \frac{y-x}{2(n-\alpha+1)}\Big[|\Upsilon^{n+1}(\mathtt{v}_1)|+|\Upsilon^{n+1}(\mathtt{v}_2)|-\frac{|\Upsilon^{n+1}(x)|+|\Upsilon^{n+1}(y)|}{2}\Big],\end{aligned}$$

for all $x, y \in [\mathtt{v}_1, \mathtt{v}_2]$ and $\alpha > 0$.

H-H type fractional inequalities are given in the next theorems for CFD by utilizing the property of quasi-convex functions.

Definition 10 ([13]). *A real-valued function Υ is quasi-convex, if*

$$\Upsilon(tx+(1-t)y) \le \max\{\Upsilon(x), \Upsilon(y)\},$$

$\forall\ x, y \in I$ *and* $t \in [0, 1]$.

Theorem 11 ([14]). *Let $\Upsilon : [\mathtt{v}_1, \mathtt{v}_2] \to \mathbb{R}$ be a function such that $\Upsilon \in AC^{n+1}[\mathtt{v}_1, \mathtt{v}_2]$. (Here $AC[\mathtt{v}_1, \mathtt{v}_2]$ is the space of absolutely continuous functions on $[\mathtt{v}_1, \mathtt{v}_2]$ and $AC^m[\mathtt{v}_1, \mathtt{v}_2]$ is the space of all functions $\Upsilon \in C^m[\mathtt{v}_1, \mathtt{v}_2]$ with $\Upsilon^{(m-1)} \in AC[\mathtt{v}_1, \mathtt{v}_2]$). Also let $|\Upsilon^{(n+1)}|$ be quasi-convex function on $[\mathtt{v}_1, \mathtt{v}_2]$. Then fractional inequality pertaining to CFD is stated as:*

$$\begin{aligned}&\left|\frac{\Upsilon^{(n)}(\mathtt{v}_1)+\Upsilon^{(n)}(\mathtt{v}_2)}{2}-\frac{\Gamma(n-\alpha+1)}{2(\mathtt{v}_2-\mathtt{v}_1)^{n-\alpha}}\Big[{}^{C}D^{\alpha}_{\mathtt{v}_1+}\Upsilon)(\mathtt{v}_2)+(-1)^{n\,C}D^{\alpha}_{\mathtt{v}_2-}\Upsilon)(\mathtt{v}_1)\Big]\right|\\ \le\ &\frac{\mathtt{v}_2-\mathtt{v}_1}{n-\alpha+1}\Big(1-\frac{1}{2^{n-\alpha}}\Big)\max\{|\Upsilon^{(n+1)}(\mathtt{v}_1)|, |\Upsilon^{(n+1)}(\mathtt{v}_2)|\}.\end{aligned}$$

Theorem 12 ([14]). *Assume that* Υ *is as in Theorem 11. Then*

$$\left|\frac{\Upsilon^{(n)}(\mathrm{v}_1)+\Upsilon^{(n)}(\mathrm{v}_2)}{2}-\frac{\Gamma(n-\alpha+1)}{2(\mathrm{v}_2-\mathrm{v}_1)^{n-\alpha}}\left[{}^{C}D^{\alpha}_{\mathrm{v}_1+}\Upsilon)(\mathrm{v}_2)+(-1)^{n C}D^{\alpha}_{\mathrm{v}_2-}\Upsilon)(\mathrm{v}_1)\right]\right|$$
$$\leq \frac{\mathrm{v}_2-\mathrm{v}_1}{(n-\alpha+1)^{\frac{1}{p}}}\max\{|\Upsilon^{(n+1)}(\mathrm{v}_1)|,|\Upsilon^{(n+1)}(\mathrm{v}_2)|\},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 13 ([14]). *Let* Υ *be as in Theorem 11. Then:*

$$\left|\frac{\Upsilon^{(n)}(\mathrm{v}_1)+\Upsilon^{(n)}(\mathrm{v}_2)}{2}-\frac{\Gamma(n-\alpha+1)}{2(\mathrm{v}_2-\mathrm{v}_1)^{n-\alpha}}\left[{}^{C}D^{\alpha}_{\mathrm{v}_1+}\Upsilon)(\mathrm{v}_2)+(-1)^{n C}D^{\alpha}_{\mathrm{v}_2-}\Upsilon)(\mathrm{v}_1)\right]\right|$$
$$\leq \frac{\mathrm{v}_2-\mathrm{v}_1}{n-\alpha+1}\left(1-\frac{1}{2^{n-\alpha}}\right)\max\{|\Upsilon^{(n+1)}(\mathrm{v}_1)|^q,|\Upsilon^{(n+1)}(\mathrm{v}_2)|^q\}.$$

Now we present H-H type integral inequalities involving p-convexity via CFD.

Definition 14 ([15]). *A function* $\Upsilon : I\subset(0,\infty)\to\mathbb{R}$ *is said to be p-convex, if*

$$\Upsilon\left([tx^p+(1-t)y^p]^{\frac{1}{p}}\right)\leq t\Upsilon(x)+(1-t)\Upsilon(y),$$

$\forall x,y\in I$ *and* $t\in[0,1]$.

Theorem 15 ([16]). *Let* $\Upsilon : [\mathrm{v}_1,\mathrm{v}_2]\subset(0,\infty)\to\mathbb{R}$ *be a function such that* $\Upsilon\in C^m[\mathrm{v}_1,\mathrm{v}_2]$. *Also let* Υ^m *is positive p-convex. Then:*

(i) for $p>0$ *we have:*

$$\Upsilon\left(\left[\frac{\mathrm{v}_1{}^p+\mathrm{v}_2{}^p}{2}\right]^{\frac{1}{p}}\right)$$
$$\leq \frac{\Gamma(m-\alpha+1)}{2(\mathrm{v}_2{}^p-\mathrm{v}_1{}^p)^{m-\alpha}}\left[({}^{C}D_{\mathrm{v}_1{}^p+}\Upsilon)(\mu(\mathrm{v}_2{}^p))+(-1)^m({}^{C}D_{\mathrm{v}_2{}^p-}\Upsilon)(\mu(\mathrm{v}_1{}^p))\right]$$
$$\leq \frac{\Upsilon^m(\mathrm{v}_1)+\Upsilon^m(\mathrm{v}_2)}{2},$$

where $\mu(s)=s^{\frac{1}{p}}$, $\forall\ s\in[\mathrm{v}_1{}^p,\mathrm{v}_2{}^p]$.

(ii) for $p<0$ *we have:*

$$\Upsilon\left(\left[\frac{\mathrm{v}_1{}^p+\mathrm{v}_2{}^p}{2}\right]^{\frac{1}{p}}\right)$$
$$\leq \frac{\Gamma(m-\alpha+1)}{2(\mathrm{v}_1{}^p-\mathrm{v}_2{}^p)^{m-\alpha}}\left[(-1)^m({}^{C}D_{\mathrm{v}_1{}^p+}\Upsilon)(\mu(\mathrm{v}_2{}^p))+({}^{C}D_{\mathrm{v}_2{}^p-}\Upsilon)(\mu(\mathrm{v}_1{}^p))\right]$$

$$\leq \quad \frac{\Upsilon^m(\mathtt{v}_1)+\Upsilon^m(\mathtt{v}_2)}{2},$$

where $\mu(s)=s^{\frac{1}{p}},\ \forall\ s\in[\mathtt{v}_2{}^p,\mathtt{v}_1{}^p]$.

Theorem 16 ([16]). *Let* $\Upsilon:[\mathtt{v}_1,\mathtt{v}_2]\subset(0,\infty)\to\mathbb{R}$ *be a function such that* $\Upsilon\in C^m[\mathtt{v}_1,\mathtt{v}_2]$, *with* $\mathtt{v}_1<\mathtt{v}_2$. *Also let* $|\Upsilon^{m+1}|^q$ *is p-convex. Then:*

(i) for $p>1$ *we have:*

$$\Bigg|\frac{\Upsilon^m(\mathtt{v}_1)+\Upsilon^m(\mathtt{v}_2)}{2}-\frac{\Gamma(m-\alpha+1)}{2(\mathtt{v}_2{}^p-\mathtt{v}_1{}^p)^{m-\alpha}}\Big[({}^CD_{\mathtt{v}_1{}^p+}\Upsilon)(\mu(\mathtt{v}_2{}^p))$$
$$+(-1)^m({}^CD_{\mathtt{v}_2{}^p-}\Upsilon)(\mu(\mathtt{v}_1{}^p))\Big]$$
$$\leq\frac{\mathtt{v}_2{}^p-\mathtt{v}_1{}^p}{2p}\Big[\frac{\mathtt{v}_2{}^{1-p}}{2}\,{}_2F_1\Big(1-\frac{1}{p},1;2;1-\frac{\mathtt{v}_1{}^p}{\mathtt{v}_2{}^p}\Big)\Big]^{1-\frac{1}{q}}\Big(1-\frac{1}{2^{m-\alpha}}\Big)^{\frac{1}{q}}$$
$$\times\Big(\frac{|\Upsilon^{m+1}(\mathtt{v}_1)|^q+|\Upsilon^{m+1}(\mathtt{v}_2)|^q}{m-\alpha+1}\Big)^{\frac{1}{q}},$$

where $q\geq1$.

(ii) for $p<1$ *we have:*

$$\Bigg|\frac{\Upsilon^m(\mathtt{v}_1)+\Upsilon^m(\mathtt{v}_2)}{2}-\frac{\Gamma(m-\alpha+1)}{2(\mathtt{v}_1{}^p-\mathtt{v}_2{}^p)^{m-\alpha}}\Big[(-1)^m({}^CD_{\mathtt{v}_1{}^p+}\Upsilon)(\mu(\mathtt{v}_2{}^p))$$
$$+({}^CD_{\mathtt{v}_2{}^p-}\Upsilon)(\mu(\mathtt{v}_1{}^p))\Big]$$
$$\leq\frac{\mathtt{v}_2{}^p-\mathtt{v}_1{}^p}{2p}\Big[\frac{\mathtt{v}_2{}^{p-1}}{2}\,{}_2F_1\Big(1-\frac{1}{p},1;2;1-\frac{\mathtt{v}_2{}^p}{\mathtt{v}_1{}^p}\Big)\Big]^{1-\frac{1}{q}}\Big(1-\frac{1}{2^{m-\alpha}}\Big)^{\frac{1}{q}}$$
$$\times\Big(\frac{|\Upsilon^{m+1}(\mathtt{v}_1)|^q+|\Upsilon^{m+1}(\mathtt{v}_2)|^q}{m-\alpha+1}\Big)^{\frac{1}{q}},$$

where${}_{(\cdot)}F_{(\cdot)}(a,b;c;z)$ *is the hypergeometric function.*

Theorem 17 ([16]). *Let* $\Upsilon:[\mathtt{v}_1,\mathtt{v}_2]\subset(0,\infty)\to\mathbb{R}$ *be a function such that* $\Upsilon\in C^m[\mathtt{v}_1,\mathtt{v}_2]$, *with* $\mathtt{v}_1<\mathtt{v}_2$. *Also let* $|\Upsilon^{m+1}|^q$ *be p-convex with* $q\geq1$, *then:*

(i) for $p>1$ *we have:*

$$\Bigg|\frac{\Upsilon^m(\mathtt{v}_1)+\Upsilon^m(\mathtt{v}_2)}{2}-\frac{\Gamma(m-\alpha+1)}{2(\mathtt{v}_2{}^p-\mathtt{v}_1{}^p)^{m-\alpha}}\Big[({}^CD_{\mathtt{v}_1{}^p+}\Upsilon)(\mu(\mathtt{v}_2{}^p))$$
$$+(-1)^m({}^CD_{\mathtt{v}_2{}^p-}\Upsilon)(\mu(\mathtt{v}_1{}^p))\Big]$$
$$\leq\frac{\mathtt{v}_2{}^p-\mathtt{v}_1{}^p}{2p}\Big(\frac{2}{m-\alpha+1}\Big)^{1-\frac{1}{q}}\Big[\frac{\mathtt{v}_2{}^{1-p}}{2}\,{}_2F_1\Big(1-\frac{1}{p},2;3;1-\frac{\mathtt{v}_1{}^p}{\mathtt{v}_2{}^p}\Big)|\Upsilon^{m+1}(\mathtt{v}_1)|^q$$
$$+\frac{\mathtt{v}_2{}^{1-p}}{2}\,{}_2F_1\Big(1-\frac{1}{p},1;3;1-\frac{\mathtt{v}_1{}^p}{\mathtt{v}_2{}^p}\Big)|\Upsilon^{m+1}(\mathtt{v}_2)|^q\Big]^{\frac{1}{q}}.$$

(ii) for $p<1$ we have:

$$\begin{aligned}&\Bigg|\frac{\Upsilon^m(\mathtt{v}_1)+\Upsilon^m(\mathtt{v}_2)}{2}-\frac{\Gamma(m-\alpha+1)}{2(\mathtt{v}_1{}^p-\mathtt{v}_2{}^p)^{m-\alpha}}\Big[(-1)^m({}^C D_{\mathtt{v}_1{}^p+}\Upsilon)(\mu(\mathtt{v}_2{}^p))\\&+({}^C D_{\mathtt{v}_2{}^p-}\Upsilon)(\mu(\mathtt{v}_1{}^p))\Big]\\ \le\ &\frac{\mathtt{v}_2{}^p-\mathtt{v}_1{}^p}{2p}\Big(\frac{2}{m-\alpha+1}\Big)^{1-\frac{1}{q}}\Big[\frac{\mathtt{v}_2{}^{p-1}}{2}\ {}_2F_1\Big(1-\frac{1}{p},1;3;1-\frac{\mathtt{v}_1{}^p}{\mathtt{v}_2{}^p}\Big)|\Upsilon^{m+1}(\mathtt{v}_1)|^q\\&+\frac{\mathtt{v}_2{}^{p-1}}{2}\ {}_2F_1\Big(1-\frac{1}{p},2;3;1-\frac{\mathtt{v}_2{}^p}{\mathtt{v}_1{}^p}\Big)|\Upsilon^{m+1}(\mathtt{v}_2)|^q\Big]^{\frac{1}{q}}.\end{aligned}$$

The next theorems concern H-H type inequalities for CFD and strongly convex functions.

Definition 18 ([17]). *A function $\Upsilon: I\to\mathbb{R}$ is called strongly convex with modulus C if it satisfies*

$$\Upsilon(tx+(1-t)y)\le t\Upsilon(x)+(1-t)\Upsilon(y)-Ct(1-t)|x-y|^2,$$

for all $x,y\in I$, $t\in[0,1$ and $C>0$.

Theorem 19 ([18]). *Let $\Upsilon:[\mathtt{v}_1,\mathtt{v}_2]\to\mathbb{R}$ be the positive function such that $\Upsilon\in C^n[\mathtt{v}_1,\mathtt{v}_2]$ and $0\le\mathtt{v}_1<\mathtt{v}_2$. If $\Upsilon^{(n)}$ is strongly convex function with modulus C, then fractional inequalities pertaining to CFD are given as:*

$$\begin{aligned}&\Upsilon^{(n)}\Big(\frac{\mathtt{v}_1+\mathtt{v}_2}{2}\Big)+\frac{C(\mathtt{v}_2-\mathtt{v}_1)^2\big[(\alpha-n+2)+(n-\alpha)^2\big]}{4(n-\alpha+1)(n-\beta+2)}\\ \le\ &\frac{\Gamma(n-\alpha+1)}{2(\mathtt{v}_2-\mathtt{v}_1)^{n-\alpha}}\Big[\Big({}^C D^{\alpha}_{\mathtt{v}_1+}\Upsilon\Big)(\mathtt{v}_2)+(-1)^n\Big({}^C D^{\alpha}_{\mathtt{v}_2-}\Upsilon\Big)(\mathtt{v}_1)\Big]\\ \le\ &\frac{\Upsilon^{(n)}(\mathtt{v}_1)+\Upsilon^{(n)}(\mathtt{v}_2)}{2}-\frac{C(n-\alpha)(\mathtt{v}_2-\mathtt{v}_1)^2}{(n-\alpha+1)(n-\alpha+2)},\end{aligned}$$

with $\alpha>0$.

Theorem 20 ([18]). *Let the statement of Theorem 19 is satisfied. Then*

$$\begin{aligned}&\Upsilon^{(n)}\Big(\frac{\mathtt{v}_1+\mathtt{v}_2}{2}\Big)+\frac{C(\mathtt{v}_2-\mathtt{v}_1)^2}{2(n-\alpha+1)(n-\alpha+2)}\\ \le\ &\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(\mathtt{v}_2-\mathtt{v}_1)^{n-\alpha}}\Big[\Big({}^C D^{\alpha}_{(\frac{\mathtt{v}_1+\mathtt{v}_2}{2})+}\Upsilon\Big)(\mathtt{v}_2)+(-1)^n\Big({}^C D^{\alpha}_{(\frac{\mathtt{v}_1+\mathtt{v}_2}{2})-}\Upsilon\Big)(\mathtt{v}_1)\Big]\\ \le\ &\frac{\Upsilon^{(n)}(\mathtt{v}_1)+\Upsilon^{(n)}(\mathtt{v}_2)}{2}-\frac{C(n-\alpha)(\mathtt{v}_2-\mathtt{v}_1)^2(n-\alpha+3)}{4(n-\alpha+1)(n-\alpha+2)},\end{aligned}$$

with $\alpha>0$.

Theorem 21 ([18]). *Let $\Upsilon : [\mathrm{v}_1, \mathrm{v}_2] \to \mathbb{R}$ be a function such that $\Upsilon \in C^{n+1}[\mathrm{v}_1, \mathrm{v}_2]$ and $0 \leq \mathrm{v}_1 < \mathrm{v}_2$. If $|\Upsilon^{(n+1)}|$ is a strongly convex on $[\mathrm{v}_1, \mathrm{v}_2]$, then fractional inequalities pertaining to CFD are given as:*

$$\begin{aligned}
&\Bigg| \frac{\Upsilon^{(n)}(\mathrm{v}_1) + \Upsilon^{(n)}(\mathrm{v}_2)}{2} - \frac{\Gamma(n-\alpha+1)}{2(\mathrm{v}_2-\mathrm{v}_1)^{n-\alpha}} \\
&\times \Big[\Big({}^{C}D^{\alpha}_{\mathrm{v}_1+}\Upsilon\Big)(\mathrm{v}_2) + (-1)^n \Big({}^{C}D^{\alpha}_{\mathrm{v}_2-}\Upsilon\Big)(\mathrm{v}_1) \Big] \Bigg| \\
\leq\ & \frac{\mathrm{v}_2-\mathrm{v}_1}{2(n-\alpha+1)} \Big(1 - \frac{1}{2^{n-\alpha}}\Big) \Big[\Upsilon^{(n+1)}(\mathrm{v}_1) + \Upsilon^{(n+1)}(\mathrm{v}_2) \Big] \\
& - \frac{C(\mathrm{v}_2-\mathrm{v}_1)^3}{(n-\alpha+2)(n-\alpha+3)} \Big(1 - \frac{n-\alpha+4}{2^{n-\alpha+2}}\Big),
\end{aligned}$$

with $\alpha > 0$.

Next, we give H-H results for strongly m-convexity with modulus $C \geq 0$ via CFD.

Definition 22 ([19]). *A real-valued function Υ is strongly m-convex with modulus $C \geq 0$ if*

$$\Upsilon(t\mathrm{v}_1 + (1-t)\mathrm{v}_2) \leq t\Upsilon(\mathrm{v}_1) + m(1-t)\Upsilon(\mathrm{v}_2) - Cmt(1-t)|\mathrm{v}_1 - \mathrm{v}_2|^2,$$

$\forall \mathrm{v}_1, \mathrm{v}_2 \in I$ and $t \in [0,1]$.

Theorem 23 ([19]). *Let $\Upsilon \in AC^n[\mathrm{v}_1, \mathrm{v}_2]$, $0 \leq \mathrm{v}_1 < m\mathrm{v}_2$ be a positive function. If $\Upsilon^{(n)}$ is a strongly m-convex with modulus $C \geq 0$, $m \in (0,1]$, then fractional inequality pertaining to CFD is given as:*

$$\begin{aligned}
& \Upsilon^{(n)}\Big(\frac{m\mathrm{v}_2 + \mathrm{v}_1}{2}\Big) + \frac{mC(n-\alpha)}{4(n-\alpha+2)} \Big\{ (\mathrm{v}_2-\mathrm{v}_1)^2 + \frac{2(\mathrm{v}_2-\mathrm{v}_1)((\mathrm{v}_1/m) - m\mathrm{v}_2)}{n-\alpha+1} \\
& + \frac{2((\mathrm{v}_1/m) - m\mathrm{v}_2)^2}{(n-\alpha)(n-\alpha+1)} \Big\} \\
\leq\ & \frac{\Gamma(n-\alpha+1)}{2(m\mathrm{v}_2-\mathrm{v}_1)^{n-\alpha}} \Big[m^{n-\alpha+1}(-1)^n \Big({}^{C}D^{\alpha}_{\mathrm{v}_2-}\Upsilon\Big)\Big(\frac{\mathrm{v}_1}{m}\Big) + \Big({}^{C}D^{\alpha}_{\mathrm{v}_1+}\Upsilon\Big)(m\mathrm{v}_2) \Big] \\
\leq\ & \frac{n-\alpha}{2(n-\alpha+1)} \Bigg\{ \frac{m^2\Upsilon^{(n)}(\mathrm{v}_1/m^2) + m\Upsilon^{(n)}(\mathrm{v}_2)}{n-\alpha} \\
& + \Big[m\Upsilon^{(n)}(\mathrm{v}_2) + \Upsilon^{(n)}(\mathrm{v}_1) \Big] - \frac{Cm\Big((\mathrm{v}_2-\mathrm{v}_1)^2 + (\mathrm{v}_2 - (\mathrm{v}_1/m^2))^2\Big)}{n-\alpha+2} \Bigg\},
\end{aligned}$$

with $\alpha > 0$.

Theorem 24 ([19]). *Let $\Upsilon \in AC^n[\mathtt{v}_1, \mathtt{v}_2]$, $0 \leq \mathtt{v}_1 < m\mathtt{v}_2$ be a positive function. If $\Upsilon^{(n)}$ is a strongly m-convex with $C \geq 0$, $m \in (0,1]$, then fractional inequality pertaining to CFD is given as:*

$$\begin{aligned}
&\Upsilon^{(n)}\left(\frac{m\mathtt{v}_2+\mathtt{v}_1}{2}\right)+\frac{mC(n-\alpha)}{8(n-\alpha+2)}\left\{\frac{(\mathtt{v}_2-\mathtt{v}_1)^2}{2}\right.\\
&+\frac{(\mathtt{v}_2-\mathtt{v}_1)((\mathtt{v}_1/m)-m\mathtt{v}_2)(n-\alpha+3)}{n-\alpha+1}\\
&\left.+\frac{((\mathtt{v}_1/m)-m\mathtt{v}_2)^2[(n-\alpha)^2+5n-5\alpha+8]}{2(n-\alpha)(n-\alpha+1)}\right\}\\
\leq\;&\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(m\mathtt{v}_2-\mathtt{v}_1)^{n-\alpha}}\left[m^{n-\alpha+1}(-1)^n\left({}^{C}D^{\alpha}_{\left(\frac{\mathtt{v}_1+\mathtt{v}_2 m}{2}\right)-}\Upsilon\right)\left(\frac{\mathtt{v}_1}{m}\right)\right.\\
&\left.+\left({}^{C}D^{\alpha}_{\left(\frac{\mathtt{v}_1+\mathtt{v}_2 m}{2}\right)+}\Upsilon\right)(m\mathtt{v}_2)\right]\\
\leq\;&\frac{n-\alpha}{4(n-\alpha+1)}\left\{\frac{(n-\alpha+2)[m\Upsilon^{(n)}(\mathtt{v}_2)+m^2\Upsilon^{(n)}(\mathtt{v}_1/m^2)]}{n-\alpha}\right.\\
&\left.+[m\Upsilon^{(n)}(\mathtt{v}_2)+\Upsilon^{(n)}(\mathtt{v}_1)]-\frac{Cm(n-\alpha+3)\left[(\mathtt{v}_2-\mathtt{v}_1)^2+m(\mathtt{v}_2-(\mathtt{v}_1/m^2))^2\right]}{2(n-\alpha+2)}\right\},
\end{aligned}$$

with $\alpha > 0$.

Theorem 25 ([19]). *Let $\Upsilon \in AC^{n+1}[\mathtt{v}_1, \mathtt{v}_2]$, $\mathtt{v}_1 < \mathtt{v}_2$ be a differentiable mapping on $(\mathtt{v}_1, \mathtt{v}_2)$. If $|\Upsilon^{(n+1)}|$ is a strongly m-convex function with $C \geq 0$ on $[a, m\mathtt{v}_2]$, $m \in (0,1]$, then fractional inequality pertaining to CFD is given as:*

$$\begin{aligned}
&\left|\frac{\Upsilon^{(n)}(\mathtt{v}_1)+\Upsilon^{(n)}(\mathtt{v}_2)}{2}-\frac{\Gamma(n-\alpha+1)}{2(\mathtt{v}_2-\mathtt{v}_1)^{n-\alpha}}\right.\\
&\left.\times\left[\left({}^{C}D^{\alpha}_{\mathtt{v}_1+}\Upsilon\right)(\mathtt{v}_2)+(-1)^n\left({}^{C}D^{\alpha}_{\mathtt{v}_2-}\Upsilon\right)(\mathtt{v}_1)\right]\right|\\
\leq\;&\frac{\mathtt{v}_2-\mathtt{v}_1}{2(n-\alpha+1)}\left(1-\frac{1}{2^{n-\alpha}}\right)\left[|\Upsilon^{(n+1)}(\mathtt{v}_1)|+m\left|\Upsilon^{(n+1)}\left(\frac{\mathtt{v}_2}{m}\right)\right|\right]\\
&-\frac{Cm(\mathtt{v}_2-\mathtt{v}_1)((\mathtt{v}_2/m)-\mathtt{v}_1)^2}{(n-\alpha+2)(n-\alpha+3)}\left(1-\frac{n-\alpha+4}{2^{n-\alpha+2}}\right),
\end{aligned}$$

with $\alpha > 0$.

Theorem 26 ([19]). *Let $\Upsilon \in AC^{n+1}[\mathtt{v}_1, \mathtt{v}_2]$, $\mathtt{v}_1 < \mathtt{v}_2$ be a differentiable mapping on $(\mathtt{v}_1, \mathtt{v}_2)$. If $|\Upsilon^{(n+1)}|^q$ is a strongly m-convex with $C \geq 0$ on $[\mathtt{v}_1, \mathtt{v}_2]$, $m \in (0,1]$, for $q > 1$, then fractional inequality pertaining to CFD is given as:*

$$\left|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(m\mathtt{v}_2-\mathtt{v}_1)^{n-\alpha}}\left[m^{n-\alpha+1}(-1)^n\left({}^{C}D^{\alpha}_{\left(\frac{\mathtt{v}_1+\mathtt{v}_2 m}{2}\right)-}\Upsilon\right)\left(\frac{\mathtt{v}_1}{m}\right)\right.\right.$$

$$+\left({}^{C}D^{\alpha}_{\left(\frac{\mathrm{v}_1+\mathrm{v}_2 m}{2}\right)+}\Upsilon\right)(m\mathrm{v}_2)\Big] - \frac{1}{2}\Big[\Upsilon^{(n)}\Big(\frac{m\mathrm{v}_2+\mathrm{v}_1}{2}\Big) + m\Upsilon^{(n)}\Big(\frac{m\mathrm{v}_2+\mathrm{v}_1}{2m}\Big)\Big]\Bigg|$$

$$\leq \frac{m\mathrm{v}_2-\mathrm{v}_1}{16}\Big(\frac{4}{np-\alpha p+1}\Big)^{\frac{1}{p}}\Big[\Big(\Big(|\Upsilon^{(n+1)}(\mathrm{v}_1)| + (3m)^{\frac{1}{q}}|\Upsilon^{(n+1)}(\mathrm{v}_2)|\Big)^q$$
$$-\frac{2Cm(\mathrm{v}_2-\mathrm{v}_1)^2}{3}\Big)^{\frac{1}{q}} + \Big(\Big((3m)^{\frac{1}{q}}\Big|\Upsilon^{(n+1)}\Big(\frac{\mathrm{v}_1}{m^2}\Big)\Big| + |\Upsilon^{(n+1)}(\mathrm{v}_2)|\Big)^q$$
$$-\frac{2Cm(\mathrm{v}_2-(\mathrm{v}_1/m^2))^2}{3}\Big)^{\frac{1}{q}}\Big].$$

Next we present H-H type inequalities for strongly (θ, m)-convex functions with modulus $C \geq 0$ via CFD.

Definition 27 ([20]). *A real-valued function Υ is strongly (θ, m)-convex with $C \geq 0$, if*

$$\Upsilon(tx + m(1-t)y) \leq t^\theta + m(1-t^\theta)\Upsilon(y) - Cmt^\theta(1-t^\theta)|y-x|^2,$$

holds true $\forall\ x, y \in [0, +\infty]$, $(\theta, m) \in [0,1]^2$, and $t \in [0,1]$.

Theorem 28 ([20]). *Let $\Upsilon : [\mathrm{v}_1, \mathrm{v}_2] \to \mathbb{R}$ be a positive function with $\Upsilon \in C^n[\mathrm{v}_1, \mathrm{v}_2]$, $(\theta, m) \in [0,1]^2$, and $0 \leq \mathrm{v}_1 < m\mathrm{v}_2$, $m \neq 0$. If $\Upsilon^{(n)}$ is a strongly (θ, m)-convex with modulus C, then fractional inequality pertaining to CFD is given as:*

$$\Upsilon^{(n)}\Big(\frac{m\mathrm{v}_2+\mathrm{v}_1}{2}\Big) + \frac{mC(n-\alpha)}{2^\alpha(n-\alpha+2)}\Big(1-\frac{1}{2^\theta}\Big)\Big\{(\mathrm{v}_2-\mathrm{v}_1)^2$$
$$+\frac{2(\mathrm{v}_2-\mathrm{v}_1)((\mathrm{v}_1/m)-m\mathrm{v}_2)}{n-\alpha+1} + \frac{2((\mathrm{v}_1/m)-m\mathrm{v}_2)^2}{(n-\alpha)(n-\alpha+1)}\Big\}$$
$$\leq \frac{\Gamma(n-\alpha+1)}{(m\mathrm{v}_2-\mathrm{v}_1)^{n-\alpha}}\Big[\Big(1-\frac{1}{2^\theta}\Big)m^{n-\alpha+1}(-1)^n({}^{C}D^{\alpha}_{\mathrm{v}_2-}\Upsilon)\Big(\frac{\mathrm{v}_1}{m}\Big) + \frac{1}{2^\theta}({}^{C}D^{\alpha}_{\mathrm{v}_1+}\Upsilon)(m\mathrm{v}_2)\Big]$$
$$\leq \frac{n-\alpha}{n-\alpha+\theta}\Big\{\Big(1-\frac{1}{2^\theta}\Big)\frac{m^2\Upsilon^{(n)}(\mathrm{v}_1/m^2) + m\Upsilon^{(n)}(\mathrm{v}_2)}{n-\alpha} + \frac{m\Upsilon^{(n)}(\mathrm{v}_2)+m\Upsilon^{(n)}(\mathrm{v}_1)}{2^\theta}$$
$$-\frac{Cma\Big(m(2^\theta-1)(\mathrm{v}_2-(\mathrm{v}_1/m^2))^2 + (\mathrm{v}_2-\mathrm{v}_1)^2\Big)}{2^\theta(n-\alpha+2\theta)}\Big\},$$

with $\alpha > 0$ and $n = [\alpha] + 1$.

The following results on H-H type inequalities are based on (s, m)-convex functions and CFD.

Definition 29 ([21]). *A real-valued function Υ is (s, m)-convex in the 2^{nd} sense, if*

$$\Upsilon(tx + m(1-t)y) \leq t^s\Upsilon(x) + m(1-t)^s\Upsilon(y)$$

holds true $\forall\ x, y \in [0, a]$, $s, m \in (0,1]$, and $t \in [0,1]$.

Theorem 30 ([22]). *Assume that a real-valued function Υ is n-times differentiable. If $\Upsilon^{(n)}$ is (s,m)-convex, then for $\alpha,\theta>1$, $x\in[\mathrm{v}_1,\mathrm{v}_2]$ with $n>\max\{\alpha,\theta\}$, the fractional inequality pertaining to CFD is given as:*

$$\begin{aligned}
&\Gamma(n-\alpha+1)({}^{C}D^{\alpha-1}_{\mathrm{v}_1+}\Upsilon)(x)+\Gamma(n-\theta+1)({}^{C}D^{\theta-1}_{\mathrm{v}_2-}\Upsilon)(x)\\
\leq\ & \frac{(x-\mathrm{v}_1)^{n-\alpha+1}\Upsilon^{(n)}(\mathrm{v}_1)+(-1)^n(\mathrm{v}_2-x)^{n-\theta+1}\Upsilon^{(n)}(\mathrm{v}_2)}{s+1}\\
&+m\Big(\frac{(x-\mathrm{v}_1)^{n-\alpha+1}+(-1)^n(\mathrm{v}_2-x)^{n-\theta+1}}{s+1}\Big)\Upsilon^{n}\Big(\frac{x}{m}\Big).
\end{aligned}$$

Theorem 31 ([22]). *Assume that a real-valued function Υ is n-times differentiable. If $\Upsilon^{(n)}$ is (s,m)-convex and integrable on $[\mathrm{v}_1,\mathrm{v}_2]$, Then fractional inequalities pertaining to CFD are given as:*

$$\begin{aligned}
\frac{2^s}{n-\alpha}\Upsilon^{(n)}\Big(\frac{\mathrm{v}_1+m\mathrm{v}_2}{2}\Big) \leq\ & \frac{\Gamma(n-\alpha)}{(m\mathrm{v}_2-\mathrm{v}_1)^{n-\alpha}}({}^{C}D^{\alpha}_{\mathrm{v}_1+}\Upsilon^{(n)})(m\mathrm{v}_2)\\
&+m\frac{\Gamma(n-\alpha)}{\Big(\mathrm{v}_2-\frac{\mathrm{v}_1}{m}\Big)^{n-\alpha}}(-1)^n({}^{C}D^{\alpha}_{\mathrm{v}_2-}\Upsilon^{(n)})\Big(\frac{\mathrm{v}_1}{m}\Big)\\
\leq\ & \frac{\Upsilon^{(n)}(\mathrm{v}_1)}{n-\alpha+s}+m\Upsilon^{(n)}\Big(\frac{\mathrm{v}_1}{m}\Big)\beta(s+1,n-\alpha)\\
&+\Upsilon^{(n)}(\mathrm{v}_2)\Big[m\beta(s+1,n-\alpha)+\frac{m^2}{n-\alpha+s}\Big].
\end{aligned}$$

where β represents Euler Beta function.

H-H inequality results via strongly $(\theta,h-m)$-convexity pertaining to CFD are presented in the following.

Definition 32 ([23]). *A real-valued function Υ is $(\theta,h-m)$-convex, if*

$$\Upsilon(tx+m(1-t)y)\leq h(t^\theta)\Upsilon(x)+mh(1-t^\theta)\Upsilon(y)$$

holds true $\forall\ x,y\in[0,b]$, where h be a non-negative real-valued function, $t\in[0,1]$, and $(\theta,m)\in[0,1]^2$

Definition 33 ([24]). *A real-valued function Υ is $(\theta,h-m)$-convex with modulus $C\geq 0$, if*

$$\Upsilon(tx+m(1-t)y)\leq h(t^\theta)\Upsilon(x)+mh(1-t^\theta)\Upsilon(y)-mCh(t^\theta)h(1-t^\theta)|y-x|^2$$

holds true $\forall\ x,y\in[0,b]$, where h be a non-negative real-valued function, $t\in[0,1]$ and $(\theta,m)\in[0,1]^2$.

Theorem 34 ([25]). *Suppose that $\Upsilon \in C^n[\mathtt{v}_1, \mathtt{v}_2]$ and $\Upsilon^{(n)}$ is a strongly $(\theta, h-m)$-convex. Then*

$$\begin{aligned}
&\Upsilon^{(n)}\Big(\frac{m\mathtt{v}_2+\mathtt{v}_1}{2}\Big)+\frac{mC(n-\alpha)}{n-\alpha+2}h\Big(\frac{1}{2^\theta}\Big)h\Big(1-\frac{1}{2^\theta}\Big)\Big\{(\mathtt{v}_2-\mathtt{v}_1)^2\\
&+\frac{2(\mathtt{v}_2-\mathtt{v}_1)\Big(\frac{\mathtt{v}_1}{m}-m\mathtt{v}_2\Big)}{n-\alpha+1}+\frac{2\Big(\frac{\mathtt{v}_1}{m}-m\mathtt{v}_2\Big)^2}{(n-\alpha)(n-\alpha+1)}\Big\}\\
\leq\;&\frac{\theta(n-\alpha+1)}{(m\mathtt{v}_2-\mathtt{v}_1)^{n-\alpha}}\Big[h\Big(1-\frac{1}{2^\theta}\Big)m^{n-\alpha+1}(-1)^n({}^CD^\alpha_{\mathtt{v}_2-}\Upsilon)\Big(\frac{\mathtt{v}_1}{m}\Big)\\
&+h\Big(\frac{1}{2^\theta}\Big)({}^CD^\alpha_{\mathtt{v}_1+}\Upsilon)(m\mathtt{v}_2)\Big],
\end{aligned}$$

where $m \in (0,1]$, $0 \leq \mathtt{v}_1 < m\mathtt{v}_2$ and $\alpha > 0$.

Theorem 35 ([25]). *Suppose that Υ is as in Theorem 34. Then we have:*

$$\begin{aligned}
&\Upsilon^{(n)}\Big(\frac{m\mathtt{v}_2+\mathtt{v}_1}{2}\Big)+\frac{mC(n-\alpha)}{2(n-\alpha+2)}h\Big(\frac{1}{2^\theta}\Big)h\Big(1-\frac{1}{2^\theta}\Big)\\
&\times\Big\{\frac{(\mathtt{v}_2-\mathtt{v}_1)^2}{2}+\frac{(\mathtt{v}_2-\mathtt{v}_1)\Big(\frac{\mathtt{v}_1}{m}-m\mathtt{v}_2\Big)(n-\alpha+3)}{n-\alpha+1}\\
&+\frac{\Big(\frac{\mathtt{v}_1}{m}-m\mathtt{v}_2\Big)^2[(n-\alpha)^2+5n-5\alpha+8]}{2(n-\alpha)(n-\alpha+1)}\Big\}\\
\leq\;&\frac{2^{n-\alpha}\theta(n-\alpha+1)}{(m\mathtt{v}_2-\mathtt{v}_1)^{n-\alpha}}\Big[h\Big(1-\frac{1}{2^\theta}\Big)m^{n-\alpha+1}({}^CD^\alpha_{\left(\frac{\mathtt{v}_1+\mathtt{v}_2m}{2m}\right)-}\Upsilon)\Big(\frac{\mathtt{v}_1}{m}\Big)\\
&+h\Big(\frac{1}{2^\theta}\Big)({}^CD^\alpha_{\left(\frac{\mathtt{v}_1+\mathtt{v}_2m}{2m}\right)+}\Upsilon)(m\mathtt{v}_2)\Big],
\end{aligned}$$

where $m \in (0,1]$, $0 \leq \mathtt{v}_1 < m\mathtt{v}_2$ and $\alpha > 0$.

Theorem 36 ([25]). *Suppose that $\Upsilon \in C^n[\mathtt{v}_1, \mathtt{v}_2]$ and $|\Upsilon^{(n+1)}|$ is a strongly $(\theta, h-m)$-convex and $h(x+y) \leq h(x)h(y)$. Then we have*

$$\begin{aligned}
&\left|\frac{\Upsilon^{(n)}(\mathtt{v}_1)+\Upsilon^{(n)}(\mathtt{v}_2)}{2}-\frac{\theta(n-\alpha+1)}{2(\mathtt{v}_2-\mathtt{v}_1)^{n-\alpha}}\Big[{}^CD^\alpha_{\mathtt{v}_1+}\Upsilon)(\mathtt{v}_2)+(-1)^n({}^CD^\alpha_{\mathtt{v}_2-}\Upsilon)(\mathtt{v}_1)\right|\\
\leq\;&\frac{\mathtt{v}_2-\mathtt{v}_1}{2}\Bigg\{\Big[\frac{(2^{np-\alpha p+1}-1)^{\frac{1}{p}}-1}{2^{n-\alpha+\frac{1}{p}}(np-\alpha p+1)^{\frac{1}{p}}}\Big]\Big[|\Upsilon^{(n+1)}(\mathtt{v}_1)|\Big(\Big(\int_0^{1/2}(h(u^\theta))^q du\Big)^{\frac{1}{q}}\\
&+\Big(\int_{1/2}^1(h(u^\theta))^q du\Big)^{\frac{1}{q}}\Big)+m\Big|\Upsilon^{(n+1)}\Big(\frac{\mathtt{v}_1}{m}\Big)\Big|\Big(\Big(\int_0^{1/2}(h(1-u^\theta))^q du\Big)^{\frac{1}{q}}\\
&+\Big(\int_{1/2}^1(h(1-u^\theta))^q du\Big)^{\frac{1}{q}}\Big)\Big]-\frac{Cmh(1)\Big(\frac{\mathtt{v}_2}{m}-\mathtt{v}_1\Big)^2(2^{n-\alpha}-1)}{2^{n-\alpha}(n-\alpha+1)}\Bigg\}
\end{aligned}$$

where $m \in (0,1]$, $0 \le \mathtt{v}_1 < m\mathtt{v}_2$, $\alpha > 0$ *and* $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 37 ([25]). *Suppose that* $\Upsilon \in C^{n+1}[\mathtt{v}_1, \mathtt{v}_2]$ *and* $|\Upsilon^{(n+1)}|^q$, $q > 1$ *is a strongly convex function. Then we have*

$$\Bigg| \frac{2^{n-\alpha-1}\theta(n-\alpha+1)}{(m\mathtt{v}_2 - \mathtt{v}_1)^{n-\alpha}} \Big[{}^{C}D^{\alpha}_{\left(\frac{\mathtt{v}_1+\mathtt{v}_2 m}{2}\right)+}\Upsilon)(\mathtt{v}_2)$$
$$+ m^{n-\alpha+1}(-1)^n ({}^{C}D^{\alpha}_{\left(\frac{\mathtt{v}_1+\mathtt{v}_2 m}{2}\right)-}\Upsilon)\left(\frac{\mathtt{v}_1}{m}\right)\Big]$$
$$- \frac{1}{2}\left[\Upsilon^{(n)}\left(\frac{\mathtt{v}_1 + \mathtt{v}_2 m}{2}\right) + m\Upsilon^{(n)}\left(\frac{\mathtt{v}_1 + \mathtt{v}_2 m}{2m}\right)\right]\Bigg|$$
$$\le \frac{m\mathtt{v}_2 - \mathtt{v}_1}{4(n-\alpha+1)^{\frac{1}{p}}} \Bigg[\Bigg(|\Upsilon^{(n+1)}(\mathtt{v}_1)|^q \int_0^1 h\left(\frac{u}{2}\right)^{\theta} u^{n-\alpha} du$$
$$+ m|\Upsilon^{(n+1)}(\mathtt{v}_2)|^q \int_0^1 h\left(1 - \left(\frac{u}{2}\right)^{\theta}\right) u^{n-\alpha} du$$
$$- Cm(\mathtt{v}_2 - \mathtt{v}_1)^2 \int_0^1 h\left(\frac{u}{2}\right)^{\theta} h\left(1 - \left(\frac{u}{2}\right)^{\theta}\right) u^{n-\alpha} du \Bigg)^{\frac{1}{q}}$$
$$+ \Bigg(m\left|\Upsilon^{(n+1)}\left(\frac{\mathtt{v}_1}{m^2}\right)\right|^q \int_0^1 h\left(1 - \left(\frac{u}{2}\right)^{\theta}\right) u^{n-\alpha} du + |\Upsilon^{(n+1)}(\mathtt{v}_2)|^q \int_0^1 h\left(\frac{u}{2}\right)^{\theta} u^{n-\alpha} du$$
$$- Cm\left(\mathtt{v}_2 - \frac{\mathtt{v}_1}{m^2}\right)^2 \int_0^1 h\left(\frac{u}{2}\right)^{\theta} h\left(1 - \left(\frac{u}{2}\right)^{\theta}\right) u^{n-\alpha} du \Bigg)^{\frac{1}{q}} \Bigg],$$

where $m \in (0,1]$, $0 \le \mathtt{v}_1 < m\mathtt{v}_2$ *and* $\alpha > 0$.

3 Hermite-Hadamard type Inequalities via k-Caputo Fractional Derivatives

Definition 38 ([26]). *The k-CFD for right and left-sided of order* α *are stated as:*

$$({}^{C}D^{\alpha,k}_{\mathtt{v}_2-}\Upsilon)(x) = \frac{1}{k\Gamma_k\left(n - \frac{\alpha}{k}\right)} \int_{\mathtt{v}_1}^{x} \frac{\Upsilon^{(n)}(t)}{(x-t)^{\frac{\alpha}{k}-n+1}} dt, \quad x > a$$

and

$$({}^{C}D^{\alpha}_{\mathtt{v}_1+}\Upsilon)(x) = \frac{(-1)^n}{k\Gamma_k\left(n - \frac{\alpha}{k}\right)} \int_{x}^{b} \frac{\Upsilon^{(n)}(t)}{(t-x)^{\frac{\alpha}{k}-n+1}} dt, \quad x < b,$$

where $\alpha > 0$ *and* $n = [\alpha] + 1$, $\Upsilon \in AC^m[\mathtt{v}_1, \mathtt{v}_2]$ *and* $\Gamma_k(\alpha)$ *represents the k-Gamma function stated as* $\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1} e^{\frac{-t^k}{k}} dt$.

H-H inequality results for n-th derivatives pertaining to k-CFD are presented in the following.

Theorem 39 ([26]). *Let $\Upsilon : [\mathrm{v}_1, \mathrm{v}_2] \to \mathbb{R}$ be a positive function such that $\Upsilon \in C^n[\mathrm{v}_1, \mathrm{v}_2]$, $\mathrm{v}_1 < \mathrm{v}_2$. If $\Upsilon^{(n)}$ is a convex function on $[\mathrm{v}_1, \mathrm{v}_2]$, then fractional inequalities for k-CFD are given as:*

$$\begin{aligned}
&\Upsilon^{(n)}\Big(\frac{\mathrm{v}_1+\mathrm{v}_2}{2}\Big)\\
\leq\ & \frac{2^{n-\frac{\alpha}{k}-1}k\Gamma_k\Big(n-\frac{\alpha}{k}+k\Big)}{(\mathrm{v}_2-\mathrm{v}_1)^{n-\frac{\alpha}{k}}}\Big[\Big({}^{C}D^{\alpha,k}_{\left(\frac{\mathrm{v}_1+\mathrm{v}_2}{2}\right)_+}\Upsilon\Big)(\mathrm{v}_2)+(-1)^n\Big({}^{C}D^{\alpha,k}_{\left(\frac{\mathrm{v}_1+\mathrm{v}_2}{2}\right)_-}\Upsilon\Big)(\mathrm{v}_1)\Big]\\
\leq\ & \frac{\Upsilon^{(n)}(\mathrm{v}_1)+\Upsilon^{(n)}(\mathrm{v}_2)}{2}.
\end{aligned}$$

Theorem 40 ([26]). *Let $\Upsilon : [\mathrm{v}_1, \mathrm{v}_2] \to \mathbb{R}$ be a function such that $\Upsilon \in C^n[\mathrm{v}_1, \mathrm{v}_2]$, $\mathrm{v}_1 < \mathrm{v}_2$. If $|\Upsilon^{(n+1)}|^q$ is convex on $[\mathrm{v}_1, \mathrm{v}_2]$ for $q \geq 1$, then fractional inequality for k-CFD is given as:*

$$\begin{aligned}
&\Bigg|\frac{2^{n-\frac{\alpha}{k}-1}k\Gamma_k\Big(n-\frac{\alpha}{k}+k\Big)}{(\mathrm{v}_2-\mathrm{v}_1)^{n-\frac{\alpha}{k}}}\Big[\Big({}^{C}D^{\alpha,k}_{\left(\frac{\mathrm{v}_1+\mathrm{v}_2}{2}\right)_+}\Upsilon\Big)(\mathrm{v}_2)+(-1)^n\Big({}^{C}D^{\alpha,k}_{\left(\frac{\mathrm{v}_1+\mathrm{v}_2}{2}\right)_-}\Upsilon\Big)(\mathrm{v}_1)\Big]\\
&-\Upsilon^{(n)}\Big(\frac{\mathrm{v}_1+\mathrm{v}_2}{2}\Big)\Bigg|\\
\leq\ & \frac{\mathrm{v}_2-\mathrm{v}_1}{4\Big(n-\frac{\alpha}{k}+1\Big)}\Bigg(\frac{1}{2\Big(n-\frac{\alpha}{k}+2\Big)}\Bigg)^{\frac{1}{q}}\Bigg[\Big(n-\frac{\alpha}{k}+1\Big)|\Upsilon^{(n+1)}(\mathrm{v}_1)|^q\\
&+\Big(n-\frac{\alpha}{k}+3\Big)|\Upsilon^{(n+1)}(\mathrm{v}_2)|^q\Bigg]^{\frac{1}{q}}+\Bigg[\Big(n-\frac{\alpha}{k}+3\Big)|\Upsilon^{(n+1)}(\mathrm{v}_1)|^q\\
&+\Big(n-\frac{\alpha}{k}+1\Big)|\Upsilon^{(n+1)}(\mathrm{v}_2)|^q\Bigg]^{\frac{1}{q}}.
\end{aligned}$$

Theorem 41 ([26]). *Let Υ satisfies the assumptions of Theorem 40. Then fractional inequality for k-CFD is given as:*

$$\begin{aligned}
&\Bigg|\frac{2^{n-\frac{\alpha}{k}-1}k\Gamma_k\Big(n-\frac{\alpha}{k}+k\Big)}{(\mathrm{v}_2-\mathrm{v}_1)^{n-\frac{\alpha}{k}}}\Big[\Big({}^{C}D^{\alpha,k}_{\left(\frac{\mathrm{v}_1+\mathrm{v}_2}{2}\right)_+}\Upsilon\Big)(\mathrm{v}_2)+(-1)^n\Big({}^{C}D^{\alpha,k}_{\left(\frac{\mathrm{v}_1+\mathrm{v}_2}{2}\right)_-}\Upsilon\Big)(\mathrm{v}_1)\Big]\\
&-\Upsilon^{(n)}\Big(\frac{\mathrm{v}_1+\mathrm{v}_2}{2}\Big)\Bigg|\\
\leq\ & \frac{\mathrm{v}_2-\mathrm{v}_1}{4}\Big(\frac{1}{np-\frac{\alpha}{k}p+1}\Big)^{\frac{1}{p}}\Bigg[\Bigg(\frac{|\Upsilon^{(n+1)}(\mathrm{v}_1)|^q+3|\Upsilon^{(n+1)}(\mathrm{v}_2)|^q}{4}\Bigg)^{\frac{1}{q}}
\end{aligned}$$

$$+\Big(\frac{3|\Upsilon^{(n+1)}(\mathtt{v}_1)|^q+|\Upsilon^{(n+1)}(\mathtt{v}_2)|^q}{4}\Big)^{\frac{1}{q}}\Big],$$

with $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 42 ([27])**.** *Let* $\Upsilon:[\mathtt{v}_1,\mathtt{v}_2]\to\mathbb{R}$, $0\le\mathtt{v}_1<\mathtt{v}_2$ *be the function such that* $\Upsilon\in AC^m[\mathtt{v}_1,\mathtt{v}_2]$. *If* $\Upsilon^{(n)}$ *is convex on* $[\mathtt{v}_1,\mathtt{v}_2]$, *then fractional inequalities for k-CFD are given as:*

$$\begin{aligned}\Upsilon^{(n)}\Big(\frac{\mathtt{v}_1+\mathtt{v}_2}{2}\Big) &\le \frac{k\Gamma_k\Big(n-\frac{\alpha}{k}+k\Big)}{2(\mathtt{v}_2-\mathtt{v}_1)^{n-\frac{\alpha}{k}}}\Big[{}^{C}D^{\alpha,k}_{\mathtt{v}_1+}\Upsilon^{(n)}(\mathtt{v}_2)+(-1)^n\,{}^{C}D^{\alpha,k}_{\mathtt{v}_2-}\Upsilon^{(n)}(\mathtt{v}_1)\Big]\\ &\le \frac{\Upsilon^{(n)}(\mathtt{v}_1)+\Upsilon^{(n)}(\mathtt{v}_2)}{2}.\end{aligned}$$

Theorem 43 ([27])**.** *Let* $\Upsilon:[\mathtt{v}_1,\mathtt{v}_2]\to\mathbb{R}$, $0\le\mathtt{v}_1<\mathtt{v}_2$ *be the function such that* $\Upsilon\in AC^{n+1}[\mathtt{v}_1,\mathtt{v}_2]$. *If* $|\Upsilon^{(n+1)}|$ *is convex on* $[\mathtt{v}_1,\mathtt{v}_2]$, *then fractional inequality for k-CFD is given as:*

$$\begin{aligned}&\Bigg|\frac{\Upsilon^{(n)}(\mathtt{v}_1)+\Upsilon^{(n)}(\mathtt{v}_2)}{2}-\frac{k\Gamma_k\Big(n-\frac{\alpha}{k}+k\Big)}{2(\mathtt{v}_2-\mathtt{v}_1)^{n-\frac{\alpha}{k}}}\\ &\times\Big[{}^{C}D^{\alpha,k}_{\mathtt{v}_1+}\Upsilon^{(n)}(\mathtt{v}_2)+(-1)^n\,{}^{C}D^{\alpha,k}_{\mathtt{v}_2-}\Upsilon^{(n)}(\mathtt{v}_1)\Big]\Bigg|\\ \le\;& \frac{\mathtt{v}_2-\mathtt{v}_1}{2(n-\frac{\alpha}{k}+1)}\Big(1-\frac{1}{2^{n-\frac{\alpha}{k}}}\Big)\Big[|\Upsilon^{(n+1)}(\mathtt{v}_1)|+|\Upsilon^{(n+1)}(\mathtt{v}_2)|\Big].\end{aligned}$$

Theorem 44 ([27])**.** *Let* $\Upsilon:[\mathtt{v}_1,\mathtt{v}_2]\to\mathbb{R}$ *be a function such that* $\Upsilon\in AC^n[\mathtt{v}_1,\mathtt{v}_2]$, $\mathtt{v}_1<\mathtt{v}_2$. *Also let* $\Upsilon^{(n)}$ *be convex on* $[\mathtt{v}_1,\mathtt{v}_2]$ *and* $g:[\mathtt{v}_1,\mathtt{v}_2]\to\mathbb{R}$ *be such that* $g\in AC^n[\mathtt{v}_1,\mathtt{v}_2]$. *If* $g^{(n)}$ *is integrable, nonnegative and symmetric to* $\frac{\mathtt{v}_1+\mathtt{v}_2}{2}$, *then fractional inequality for k-CFD is given as:*

$$\begin{aligned}&\Upsilon^{(n)}\Big(\frac{\mathtt{v}_1+\mathtt{v}_2}{2}\Big)\Big[({}^{C}D^{\alpha,k}_{\mathtt{v}_1+}g)(\mathtt{v}_2)+(-1)^n({}^{C}D^{\alpha,k}_{\mathtt{v}_2-}g)(\mathtt{v}_1)\Big]\\ \le\;& ({}^{C}D^{\alpha,k}_{\mathtt{v}_1+}\Upsilon\star g)(\mathtt{v}_2)+(-1)^n{}^{C}D^{\alpha,k}_{\mathtt{v}_2-}\Upsilon\star g)(\mathtt{v}_1)\\ \le\;& \frac{\Upsilon^{(n)}(\mathtt{v}_1)+\Upsilon^{(n)}(\mathtt{v}_2)}{2}\Big[({}^{C}D^{\alpha,k}_{\mathtt{v}_1+}g)(\mathtt{v}_2)+(-1)^n({}^{C}D^{\alpha,k}_{\mathtt{v}_2-}g)(\mathtt{v}_1)\Big],\end{aligned}$$

where the convolution $\Upsilon\star g$ *of the functions* Υ *and* g *for Caputo k-fractional derivatives are defined as follows*

$$({}^{C}D^{\alpha,k}_{\mathtt{v}_1+}\Upsilon\star g)(x)=\frac{1}{\Gamma(n-\alpha)}\int_{\mathtt{v}_1}^{x}\frac{\Upsilon^{(n)}(t)g^{(n)}(t)}{(x-t)^{\alpha-n+1}}dt,\;\; x>a,$$

$$({}^{C}D^{\alpha,k}_{\mathtt{v}_2-}\Upsilon\star g)(x)=\frac{(-1)^n}{\Gamma(n-\alpha)}\int_{x}^{b}\frac{\Upsilon^{(n)}(t)g^{(n)}(t)}{(t-x)^{\alpha-n+1}}dt,\;\; x<b.$$

Theorem 45 ([27]). *Let $\Upsilon : [\mathrm{v}_1, \mathrm{v}_2] \to \mathbb{R}$ be a function such that $\Upsilon \in AC^{n+1}[\mathrm{v}_1, \mathrm{v}_2]$, $\mathrm{v}_1 < \mathrm{v}_2$. If $|\Upsilon^{(n+1)}|$ is convex function on $[\mathrm{v}_1, \mathrm{v}_2]$ and $g : [\mathrm{v}_1, \mathrm{v}_2] \to \mathbb{R}$ is such that $g \in AC^n[\mathrm{v}_1, \mathrm{v}_2]$, $g^{(n)}$ is symmetric to $\frac{\mathrm{v}_1+\mathrm{v}_2}{2}$, then fractional inequality for k-CFD is given as:*

$$\Bigg| \frac{\Upsilon^{(n)}(\mathrm{v}_1) + \Upsilon^{(n)}(\mathrm{v}_2)}{2} \Big[({}^C D^{\alpha,k}_{\mathrm{v}_1+} g)(\mathrm{v}_2) + (-1)^n ({}^C D^{\alpha,k}_{\mathrm{v}_2-} g)(\mathrm{v}_1) \Big]$$
$$- \Big({}^C D^{\alpha,k}_{\mathrm{v}_1+} \Upsilon \star g)(\mathrm{v}_2) + (-1)^n \; {}^C D^{\alpha,k}_{\mathrm{v}_2-} \Upsilon \star g)(\mathrm{v}_1) \Big] \Bigg|$$
$$\leq \frac{(\mathrm{v}_2 - \mathrm{v}_1)^{n-\frac{\alpha}{k}+1} \|g\|_\infty}{\Big(n - \frac{\alpha}{k} + 1\Big) \Gamma_k \Big(n - \frac{\alpha}{k} + k\Big)} \Big(1 - \frac{1}{2^{n-\frac{\alpha}{k}}}\Big) [|\Upsilon^{(n+1)}(\mathrm{v}_1)| + |\Upsilon^{(n+1)}(\mathrm{v}_2)|],$$

where $\|g\|_\infty = \sup_{t \in [\mathrm{v}_1, \mathrm{v}_2]} |g(t)|$.

Theorem 46 ([27]). *Let $\Upsilon : [\mathrm{v}_1, \mathrm{v}_2] \to \mathbb{R}$ be a function such that $\Upsilon \in AC^{n+1}[\mathrm{v}_1, \mathrm{v}_2]$, $\mathrm{v}_1 < \mathrm{v}_2$. Also let $|\Upsilon^{(n+1)}|^q$, $q \geq 1$ be convex function on $[\mathrm{v}_1, \mathrm{v}_2]$ and $g : [\mathrm{v}_1, \mathrm{v}_2] \to \mathbb{R}$ be such that $g \in AC^b[\mathrm{v}_1, \mathrm{v}_2]$. If $g^{(n)}$ is symmetric to $\frac{\mathrm{v}_1+\mathrm{v}_2}{2}$, then fractional inequality for k-CFD is given as:*

$$\Bigg| \frac{\Upsilon^{(n)}(\mathrm{v}_1) + \Upsilon^{(n)}(\mathrm{v}_2)}{2} \Big[({}^C D^{\alpha,k}_{\mathrm{v}_1+} g)(\mathrm{v}_2) + (-1)^n ({}^C D^{\alpha,k}_{\mathrm{v}_2-} g)(\mathrm{v}_1) \Big]$$
$$- \Big({}^C D^{\alpha,k}_{\mathrm{v}_1+} \Upsilon \star g)(\mathrm{v}_2) + (-1)^n \; {}^C D^{\alpha,k}_{\mathrm{v}_2-} \Upsilon \star g)(\mathrm{v}_1) \Big] \Bigg|$$
$$\leq \frac{2(\mathrm{v}_2 - \mathrm{v}_1)^{n-\frac{\alpha}{k}+1} \|g\|_\infty}{\Big(n - \frac{\alpha}{k} + 1\Big) k \Gamma_k \Big(n - \frac{\alpha}{k} + k\Big) (\mathrm{v}_2 - \mathrm{v}_1)^{\frac{1}{q}}} \Big(1 - \frac{1}{2^{n-\frac{\alpha}{k}}}\Big)$$
$$\times \Big(\frac{|\Upsilon^{(n+1)}(\mathrm{v}_1)| + |\Upsilon^{(n+1)}(\mathrm{v}_2)||}{2} \Big)^{\frac{1}{q}}.$$

In the following we give H-H-Mercer type inequalities for k-CFD.

Theorem 47 ([28]). *Suppose that $\Upsilon : [\mathrm{v}_1, \mathrm{v}_2] \to \mathbb{R}$ is a positive function with $0 \leq \mathrm{v}_1 < \mathrm{v}_2$ and $\Upsilon \in C^n[\mathrm{v}_1, \mathrm{v}_2]$. If $\Upsilon^{(n)}$ is a convex function on $[\mathrm{v}_1, \mathrm{v}_2]$ then fractional inequalities for k-CFD are given as:*

$$\Upsilon^{(n)}\Big(\mathrm{v}_1 + \mathrm{v}_2 - \frac{x+y}{2}\Big)$$
$$\leq \Upsilon^{(n)}(\mathrm{v}_1) + \Upsilon^{(n)}(\mathrm{v}_2) - \frac{\Gamma_k\Big(n - \frac{\alpha}{k} + k\Big)}{2(\mathrm{v}_2 - \mathrm{v}_1)^{n-\frac{\alpha}{k}}} \Big[{}^C D^{\alpha,k}_{\mathrm{v}_1+} \Upsilon(y) + (-1)^{n C} D^{\alpha,k}_{\mathrm{v}_2-} \Upsilon(x) \Big]$$

$$\leq \quad \Upsilon^{(n)}(\mathtt{v}_1) + \Upsilon^{(n)}(\mathtt{v}_2) - \Upsilon^{(n)}\Big(\frac{x+y}{2}\Big),$$

for all $x, y \in [\mathtt{v}_1, \mathtt{v}_2]$ *and* $\alpha > 0,\ k \geq 1$.

Theorem 48 ([28]). *Suppose that the statement of Theorem 47 are satisfied. Then fractional inequalities for k-CFD are given as:*

$$\begin{aligned} &\Upsilon^{(n)}\Big(\mathtt{v}_1 + \mathtt{v}_2 - \frac{x+y}{2}\Big) \\ \leq\ & \frac{2^{n-\frac{\alpha}{k}-1}\Gamma_k\big(n-\frac{\alpha}{k}+k\big)}{(y-x)^{n-\frac{\alpha}{k}}}\Big[{}^{C}D^{\alpha,k}_{\left(\mathtt{v}_1+\mathtt{v}_2-\frac{x+y}{2}\right)+}\Upsilon(\mathtt{v}_1+\mathtt{v}_2-x) \\ & +(-1)^{n\,C}D^{\alpha,k}_{\left(\mathtt{v}_1+\mathtt{v}_2-\frac{x+y}{2}\right)-}\Upsilon(\mathtt{v}_1+\mathtt{v}_2-y)\Big] \\ \leq\ & \Upsilon^{(n)}(\mathtt{v}_1) + \Upsilon^{(n)}(\mathtt{v}_2) - \Upsilon^{(n)}\Big(\frac{x+y}{2}\Big), \end{aligned}$$

for all $x, y \in [\mathtt{v}_1, \mathtt{v}_2]$ *and* $\alpha > 0,\ k \geq 1$.

Theorem 49 ([28]). *Suppose that if* $\Upsilon : [\mathtt{v}_1, \mathtt{v}_2] \to \mathbb{R}$ *is a differentiable function on* $(\mathtt{v}_1, \mathtt{v}_2)$ *with* $0 \leq \mathtt{v}_1 < \mathtt{v}_2$ *and* $\Upsilon \in C^{n+1}[\mathtt{v}_1, \mathtt{v}_2]$. *If* $|\Upsilon^{(n+1)}|$ *is a convex function on* $[\mathtt{v}_1, \mathtt{v}_2]$ *then fractional inequalities for k-CFD are given as:*

$$\begin{aligned} &\Bigg|\frac{\Upsilon^{(n)}(\mathtt{v}_1+\mathtt{v}_2-x)+\Upsilon^{(n)}(\mathtt{v}_1+\mathtt{v}_2-y)}{2} - \frac{\Gamma_k\Big(n-\frac{\alpha}{k}+k\Big)}{2(\mathtt{v}_2-\mathtt{v}_1)^{n-\frac{\alpha}{k}}} \\ &\times\Big[{}^{C}D^{\alpha,k}_{(\mathtt{v}_1+\mathtt{v}_2-y)+}\Upsilon(\mathtt{v}_1+\mathtt{v}_2-x) + (-1)^n\ {}^{C}D^{\alpha,k}_{(\mathtt{v}_1+\mathtt{v}_2-x)-}\Upsilon(\mathtt{v}_1+\mathtt{v}_2-y)\Big]\Bigg| \\ \leq\ & \frac{y-x}{n-\frac{\alpha}{k}+k}\Big(1-\frac{1}{2^{n-\frac{\alpha}{k}}}\Big)\Big[\Upsilon^{(n+1)}(\mathtt{v}_1)+\Upsilon^{(n+1)}(\mathtt{v}_2) - \frac{\Upsilon^{(n+1)}(x)+\Upsilon^{(n+1)}(y)}{2}\Big], \end{aligned}$$

for all $x, y \in [\mathtt{v}_1, \mathtt{v}_2]$ *and* $\alpha > 0,\ k \geq 1$.

Theorem 50 ([28]). *Assume that* Υ *is as in Theorem 49. If* $|\Upsilon^{(n+1)}|^q$ *is convex on* $[\mathtt{v}_1, \mathtt{v}_2]$, $q > 1$, *then fractional inequalities for k-CFD are given as:*

$$\begin{aligned} &\Bigg|\Upsilon^{(n)}\Big(\mathtt{v}_1+\mathtt{v}_2-\frac{x+y}{2}\Big) - \frac{2^{n-\frac{\alpha}{k}-1}\Gamma_k\big(n-\frac{\alpha}{k}+k\big)}{(y-x)^{n-\frac{\alpha}{k}}}\Big[{}^{C}D^{\alpha,k}_{\left(\mathtt{v}_1+\mathtt{v}_2-\frac{x+y}{2}\right)+}\Upsilon(\mathtt{v}_1+\mathtt{v}_2-x) \\ &+(-1)^n\ {}^{C}D^{\alpha,k}_{\left(\mathtt{v}_1+\mathtt{v}_2-\frac{x+y}{2}\right)-}\Upsilon(\mathtt{v}_1+\mathtt{v}_2-y)\Big]\Bigg| \\ \leq\ & \frac{y-x}{4}\Big(\frac{1}{np-\frac{\alpha}{k}p+1}\Big)^{\frac{1}{p}}\Big[\Big(|\Upsilon^{(n+1)}(\mathtt{v}_1)|^q + |\Upsilon^{(n+1)}(\mathtt{v}_2)|^q \end{aligned}$$

$$-\Big(\frac{|\Upsilon^{(n+1)}(x)|^q+3|\Upsilon^{(n+1)}(y)|^q}{4}\Big)\Big)^{\frac{1}{q}}+\Big(|\Upsilon^{(n+1)}(\mathtt{v}_1)|^q+|\Upsilon^{(n+1)}(\mathtt{v}_2)|^q$$
$$-\Big(\frac{3|\Upsilon^{(n+1)}(x)|^q+|\Upsilon^{(n+1)}(y)|^q}{4}\Big)\Big)^{\frac{1}{q}}\Big],$$

for all $x,y\in[\mathtt{v}_1,\mathtt{v}_2]$ *and* $\alpha>0,\ k\geq 1$.

H-H inequalities for (h,m)-convex modified functions of the second type can be presented, using the k-CFD, given in the next theorems.

Definition 51 ([29]). *Let* $h:[0,1]\to\mathbb{R}$ *nonnegative function,* $h\neq 0$. *A function* $\Upsilon:[0,+\infty)\to[0,+\infty)$ *is called* (h,m)*-convex modified of the second type, if*

$$\Upsilon(tx+m(1-t)y)\leq h^s(t)\Upsilon(x)+m(1-h(t))^s\Upsilon(y)$$

holds for all $x,y\in[0,+\infty)$ *and* $t\in[0,1]$, *where* $m\in[0,1]$, $s\in[-1,1]$.

Theorem 52 ([29]). *Let* Υ *be a positive function such that* $\Upsilon\in C^n[\mathtt{v}_1,\mathtt{v}_2]$ $\mathtt{v}_1<\mathtt{v}_2$. *If* $\Upsilon^{(n)}$ *is a modified* (h,m)*-convex of* 2^{nd} *type with* $0<\mathtt{v}_1<m\mathtt{v}_2<+\infty$ *and* $m\in(0,1]$ *then fractional inequality is given as:*

$$\begin{aligned}
&\Upsilon^{(n)}\Big(\frac{\mathtt{v}_1+\mathtt{v}_2}{2}\Big)\\
\leq\ &\frac{k\Gamma_k(n-\alpha)(r+1)^{n-\alpha}}{2(\mathtt{v}_2-\mathtt{v}_1)^{n-\alpha}}{}^CD^{\alpha,k}_{\left(\frac{\mathtt{v}_1+r\mathtt{v}_2}{r+1}\right)+}\Upsilon(\mathtt{v}_2)\\
&+(-1)^n\frac{k\Gamma_k(n-\alpha)(r+1)^{n-\alpha}}{2(\mathtt{v}_2-\mathtt{v}_1)^{n-\alpha}}{}^CD^{\alpha,k}_{\left(\frac{r\mathtt{v}_1+\mathtt{v}_2}{r+1}\right)-}\Upsilon(\mathtt{v}_1)\\
\leq\ &\Big(\frac{n-\alpha}{2}\Big)\Big\{\Big[h^s\Big(\frac{1}{2}\Big)\Upsilon^{(n)}(\mathtt{v}_1)+\Big(1-h\Big(\frac{1}{2}\Big)^s\Upsilon^{(n)}(\mathtt{v}_2)\Big]\int_0^1 t^{n-\alpha-1}h^s\Big(\frac{t}{r+1}\Big)dt\\
&+m\Big[h^s\Big(\frac{1}{2}\Big)\Upsilon^{(n)}\Big(\frac{\mathtt{v}_2}{\mathtt{v}_1}\Big)\\
&+\Big(1-h\Big(\frac{1}{2}\Big)^s\Upsilon^{(n)}\Big(\frac{\mathtt{v}_2}{m}\Big)\Big]\int_0^1 t^{n-\alpha-1}\Big(1-h\Big(\frac{r+1-t}{r+1}\Big)\Big)^s dt\Big\},
\end{aligned}$$

where $r\in[0,1]$.

Theorem 53 ([29]). *Assume that* Υ *is as in Theorem 52. If* $|\Upsilon^{(n+1)}|$ *is a modified* (h,m)*-convex of* 2^{nd} *type on* $\Big[\mathtt{v}_1,\frac{\mathtt{v}_2}{m}\Big]$, *then fractional inequality is given as:*

$$\begin{aligned}
&\Bigg|-\Big(\Upsilon^{(n)}\Big(\frac{\mathtt{v}_1+r\mathtt{v}_2}{r+1}\Big)+\Upsilon^{(n)}\Big(\frac{r\mathtt{v}_1+\mathtt{v}_2}{r+1}\Big)\Big)\\
&+\frac{(r+1)^{n-\frac{\alpha}{k}}k\Gamma_k(n-\frac{\alpha}{k}+1)}{(\mathtt{v}_2-\mathtt{v}_1)^{n-\frac{\alpha}{k}}}\Big({}^CD^{\alpha,k}_{\left(\frac{r\mathtt{v}_1+\mathtt{v}_2}{r+1}\right)-}\Upsilon(\mathtt{v}_1)+(-1)^n{}^CD^{\alpha,k}_{\left(\frac{r\mathtt{v}_1+\mathtt{v}_2}{r+1}\right)+}\Upsilon(\mathtt{v}_2)\Big)\Bigg|
\end{aligned}$$

$$\leq \frac{\mathrm{v}_2-\mathrm{v}_1}{r+1}\Big\{\Big(|\Upsilon^{(n+1)}(\mathrm{v}_1)|+|\Upsilon^{(n+1)}(\mathrm{v}_2)|\Big)\int_0^1 t^{n-\frac{\alpha}{k}}h^s\Big(\frac{t}{r+1}\Big)dt$$
$$+m\Big(\Big|\Upsilon^{(n+1)}\Big(\frac{\mathrm{v}_1}{m}\Big)\Big|+\Big|\Upsilon^{(n+1)}\Big(\frac{\mathrm{v}_2}{m}\Big)\Big|\Big)\int_0^1 t^{n-\frac{\alpha}{k}}\Big(1-h\Big(1-\frac{t}{r+1}\Big)\Big)^s dt,$$

where $r \in [0,1]$.

The above result can be improved, if we consider $|\Upsilon^{(n+1)}|^q$.

Theorem 54 ([29]). *Assume that* Υ *is as in Theorem 52. If* $|\Upsilon^{(n+1)}|^q$ *is a modified* (h,m)*-convex of* 2^{nd} *type on* $\Big[\mathrm{v}_1,\frac{\mathrm{v}_2}{m}\Big]$, *then fractional inequality is given as:*

$$\Big|-\Big(\Upsilon^{(n)}\Big(\frac{\mathrm{v}_1+r\mathrm{v}_2}{r+1}\Big)+\Upsilon^{(n)}\Big(\frac{r\mathrm{v}_1+\mathrm{v}_2}{r+1}\Big)\Big)$$
$$+\frac{(r+1)^{n-\frac{\alpha}{k}}k\Gamma_k\big(n-\frac{\alpha}{k}+1\big)}{(\mathrm{v}_2-\mathrm{v}_1)^{n-\frac{\alpha}{k}}}\Big({}^{C}D^{\alpha,k}_{\left(\frac{r\mathrm{v}_1+\mathrm{v}_2}{r+1}\right)-}\Upsilon(\mathrm{v}_1)+(-1)^n\,{}^{C}D^{\alpha,k}_{\left(\frac{r\mathrm{v}_1+\mathrm{v}_2}{r+1}\right)+}\Upsilon(\mathrm{v}_2)\Big)\Big|$$
$$\leq \frac{\mathrm{v}_2-\mathrm{v}_1}{r+1}\Big(\frac{1}{p(n-\frac{\alpha}{k})+1}\Big)^{\frac{1}{p}}\Big\{\Big(|\Upsilon^{(n+1)}(\mathrm{v}_1)|^q\int_0^1 h^s\Big(\frac{t}{r+1}\Big)dt$$
$$+m\Big|\Upsilon^{(n+1)}\Big(\frac{\mathrm{v}_2}{m}\Big)\Big|^q\int_0^1\Big(1-h\Big(1-\frac{t}{r+1}\Big)\Big)^s dt\Big)^{\frac{1}{q}}$$
$$+\Big(|\Upsilon^{(n+1)}(\mathrm{v}_2)|^q\int_0^1 h^s\Big(\frac{t}{r+1}\Big)dt$$
$$+m\Big|\Upsilon^{(n+1)}\Big(\frac{\mathrm{v}_1}{m}\Big)\Big|^q\int_0^1\Big(1-h\Big(1-\frac{t}{r+1}\Big)\Big)^s dt\Big)^{\frac{1}{q}}\Big\},$$

where $r \in [0,1]$.

Next, H-H results involving m-convexity pertaining to k-CFD.

Definition 55 ([30]). *A real-valued function* Υ *is* m*-convex, if*

$$\Upsilon(tx+m(1-t)y)\leq t\Upsilon(x)+m(1-t)\Upsilon(y),$$

$\forall\, x,y \in [0,b]$, $m \in [0,1]$, *and* $t \in [0,1]$.

Theorem 56 ([31]). *Let* $\Upsilon : [0,\infty) \to \mathbb{R}$ *be a positive function such that* $\Upsilon \in C^n[0,\infty)$. *If* $\Upsilon^{(n)}$ *is* m*-convex on* $[0,\infty)$ *and* $0 \leq \mathrm{v}_1 < m\mathrm{v}_2$, *then fractional inequalities for* k*-CFD are given as:*

$$\Upsilon^{(n)}\Big(\frac{\mathrm{v}_1+m\mathrm{v}_2}{2}\Big)$$
$$\leq \frac{k\Gamma_k\Big(n-\frac{\alpha}{k}+k\Big)}{2(m\mathrm{v}_2-\mathrm{v}_1)^{n-\frac{\alpha}{k}}}\Big[({}^{C}D^{\alpha,k}_{\mathrm{v}_1+}\Upsilon)(m\mathrm{v}_2)+(-1)^n m^{\alpha-\frac{\alpha}{k}+1}({}^{C}D^{\alpha,k}_{\mathrm{v}_2-}\Upsilon)\Big(\frac{\mathrm{v}_1}{m}\Big)\Big]$$

$$\leq \frac{n-\frac{\alpha}{k}}{2\left(n-\frac{\alpha}{k}+1\right)}\left[\Upsilon^{(n)}(\mathtt{v}_1)-m^2\Upsilon^{(n)}\left(\frac{\mathtt{v}_1}{m^2}\right)\right] + \frac{m}{2}\left[\Upsilon^{(n)}(\mathtt{v}_2)+m\Upsilon^{(n)}\left(\frac{\mathtt{v}_1}{m^2}\right)\right].$$

Theorem 57 ([31]). *Assume that Υ is as in Theorem 56. If $|\Upsilon^{n+1}|$ is m-convex on $[\mathtt{v}_1,\mathtt{v}_2]$, then fractional inequality for k-CFD is given as:*

$$\Bigg|\frac{\Upsilon^{(n)}(\mathtt{v}_1)+\Upsilon^{(n)}(m\mathtt{v}_2)}{2} - \frac{k\Gamma_k\left(n-\frac{\alpha}{k}+k\right)}{2(m\mathtt{v}_2-\mathtt{v}_1)^{n-\frac{\alpha}{k}}}\Big[({}^C D^{\alpha,k}_{\mathtt{v}_1+}\Upsilon)(m\mathtt{v}_2) + (-1)^n m^{\alpha-\frac{\alpha}{k}+1}({}^C D^{\alpha,k}_{m\mathtt{v}_2-}\Upsilon)(\mathtt{v}_1)\Big]$$

$$\leq \frac{m\mathtt{v}_2-\mathtt{v}_1}{2\left(n-\frac{\alpha}{k}+1\right)}\left(1-\frac{1}{2^{n-\frac{\alpha}{k}}}\right)[\Upsilon^{(n+1)}(\mathtt{v}_1)+m\Upsilon^{(n+1)}(\mathtt{v}_2)].$$

Theorem 58 ([31]). *Assume that Υ is as in Theorem 56. If $\Upsilon^{(n)}$ is m-convex on $[\mathtt{v}_1,\mathtt{v}_2]$. Then fractional inequalities for k-CFD are given as:*

$$\Upsilon^{(n)}\left(\frac{\mathtt{v}_1+m\mathtt{v}_2}{2}\right)$$

$$\leq \frac{2^{n-\frac{\alpha}{k}-1}k\Gamma_k\left(n-\frac{\alpha}{k}+k\right)}{(m\mathtt{v}_2-\mathtt{v}_1)^{n-\frac{\alpha}{k}}}\Big[({}^C D^{\alpha,k}_{\left(\frac{\mathtt{v}_1+m\mathtt{v}_2}{2}\right)+}\Upsilon)(m\mathtt{v}_2) + (-1)^n m^{\alpha-\frac{\alpha}{k}+1}({}^C D^{\alpha,k}_{\left(\frac{\mathtt{v}_1+m\mathtt{v}_2}{2m}\right)-}\Upsilon)\left(\frac{\mathtt{v}_1}{m}\right)\Big]$$

$$\leq \frac{n-\frac{\alpha}{k}}{4\left(n-\frac{\alpha}{k}+1\right)}\left[\Upsilon^{(n)}(\mathtt{v}_1)-m^2\Upsilon^{(n)}\left(\frac{\mathtt{v}_1}{m^2}\right)\right]+\frac{m}{2}\left[\Upsilon^{(n)}(\mathtt{v}_2)+m\Upsilon^{(n)}\left(\frac{\mathtt{v}_1}{m^2}\right)\right].$$

Theorem 59 ([31]). *Assume that Υ is as in Theorem 56. If $|\Upsilon^{(n+1)}|^q$ is m-convex on $[\mathtt{v}_1,\mathtt{v}_2]$ for $q>1$, then fractional inequality for k-CFD is given as:*

$$\Bigg|\frac{2^{n-\frac{\alpha}{k}-1}k\Gamma_k\left(n-\frac{\alpha}{k}+k\right)}{(m\mathtt{v}_2-\mathtt{v}_1)^{n-\frac{\alpha}{k}}}\Big[({}^C D^{\alpha,k}_{\left(\frac{\mathtt{v}_1+m\mathtt{v}_2}{2}\right)+}\Upsilon)(m\mathtt{v}_2) + (-1)^n m^{\alpha-\frac{\alpha}{k}+1}({}^C D^{\alpha,k}_{\left(\frac{\mathtt{v}_1+m\mathtt{v}_2}{2m}\right)-}\Upsilon)\left(\frac{\mathtt{v}_1}{m}\right)\Big] - \frac{1}{2}\left[\Upsilon^{(n)}\left(\frac{\mathtt{v}_1+m\mathtt{v}_2}{2}\right)+m\Upsilon^{(n)}\left(\frac{\mathtt{v}_1+m\mathtt{v}_2}{2m}\right)\right]\Bigg|$$

$$\leq \frac{m\mathtt{v}_2-\mathtt{v}_1}{4}\left(\frac{1}{np-\frac{\alpha p}{k}+1}\right)^{\frac{1}{p}}\left[\left(\frac{|\Upsilon^{(n+1)}(\mathtt{v}_1)|^q+3m|\Upsilon^{(n+1)}(\mathtt{v}_2)|^q}{4}\right)^{\frac{1}{q}}\right.$$

$$+\Big(\frac{3m|\Upsilon^{(n+1)}\big(\frac{v_1}{m^2}\big)|^q + |\Upsilon^{(n+1)}(v_2)|^q}{4}\Big)^{\frac{1}{q}},$$

where $\dfrac{1}{p}+\dfrac{1}{q}=1.$

The results regarding H-H type inequalities involving harmonic h-convexity and harmonically symmetric functions for k-CFD are given in the following.

Definition 60 ([32]). *A real-valued function* Υ *is harmonic* h*-convex, if*

$$\Upsilon\Big(\frac{xy}{tx+(1-t)y}\Big) \le h(t)\Upsilon(y) + h(1-t)\Upsilon(x),$$

$\forall\ x,y\in[v_1,v_2]$ *and* $t\in[0,1]$, *where* h *is a real-valued positive function.*

Definition 61 ([33]). *A real-valued function* Υ *is harmonic symmetric with respect to* $\dfrac{2v_1v_2}{v_1+v_2}$ *if*

$$\Upsilon(x) = \Upsilon\left(\frac{1}{\frac{1}{v_1}+\frac{1}{v_2}-\frac{1}{x}}\right),$$

for all $x\in[v_1,v_2]$.

Theorem 62 ([34]). *Let function* $\Upsilon:[v_1,v_2]\to\mathbb{R}$ *be differentiable such that* $\Upsilon^{(n)}$ *is harmonic* h*-convex and* $\Upsilon^{(n)}\in L_1[v_1,v_2]$. *If* $g:[v_1,v_2]\to\mathbb{R}$ *is a function such that* $g^{(n)}$ *is non-negative, integrable and harmonic symmetric with respect to* $\dfrac{2v_1v_2}{v_1+v_2}$, *then fractional inequalities for* k*-CFD are given as:*

$$\begin{aligned}
&\frac{(-1)^n}{h\left(\frac{1}{2}\right)}\Upsilon^{(n)}\Big(\frac{2v_1v_2}{v_1+v_2}\Big)\,{}^{C}D^{\alpha,k}_{\frac{1}{v_1}-}(g\circ r)\Big(\frac{1}{v_2}\Big)\\
\le\ & \Big[{}^{C}D^{\alpha,k}_{\frac{1}{v_2}+}((\Upsilon\star g)\circ r)\Big(\frac{1}{v_1}\Big) + (-1)^{n}\,{}^{C}D^{\alpha,k}_{\frac{1}{v_1}-}((\Upsilon\star g)\circ r)\Big(\frac{1}{v_2}\Big)\Big]\\
\le\ & [\Upsilon^{(n)}(v_1)+\Upsilon^{(n)}(v_2)]\int_{\frac{1}{v_2}}^{\frac{1}{v_1}}\frac{\big(g^{(n)}\circ r\big)(x)}{\Big(x-\frac{1}{v_2}\Big)^{\frac{\alpha}{k}-n+1}}\overline{h}(x)dx,
\end{aligned}$$

where $r(x)=\frac{1}{x}$ *and* $\overline{h}(x)=h\Big(\frac{v_1v_2}{v_2-v_1}\Big(x-\frac{1}{v_2}\Big)\Big)+h\Big(\frac{v_1v_2}{v_2-v_1}\Big(\frac{1}{v_1}-x\Big)\Big)$ *for all* $x\in\Big[\frac{1}{v_2},\frac{1}{v_1}\Big]$.

Theorem 63 ([34]). *Let* $\Upsilon: I\subset(0,\infty)\to\mathbb{R}$ *be a function such that* $\Upsilon\in L_1[v_1,v_2]$, *where* $v_1,v_2\in I$ *with* $v_1<v_2$. *If* $\Upsilon^{(n)}$ *is harmonically* h*-convex function on* $[v_1,v_2]$, *then fractional inequalities for* k*-CFD are given as:*

$$\frac{1}{h\left(\frac{1}{2}\right)\left(n-\frac{\alpha}{k}\right)}\Big(\frac{v_2-v_1}{v_1v_2}\Big)^{n-\frac{\alpha}{k}}\Upsilon^{(n)}\Big(\frac{2v_1v_2}{v_1+v_2}\Big)$$

$$\leq \quad k\Gamma_k\Big(n-\frac{\alpha}{k}\Big)\Big[{}^CD^{\alpha,k}_{\frac{1}{\mathrm{v}_2}+}(\Upsilon\circ r)\Big(\frac{1}{\mathrm{v}_1}\Big)+(-1)^n\,{}^CD^{\alpha,k}_{\frac{1}{\mathrm{v}_1}-}(\Upsilon\circ r)\Big(\frac{1}{\mathrm{v}_2}\Big)\Big]$$
$$\leq \quad [\Upsilon^{(n)}(\mathrm{v}_1)+\Upsilon^{(n)}(\mathrm{v}_2)]\int_{\frac{1}{\mathrm{v}_2}}^{\frac{1}{\mathrm{v}_1}}\frac{1}{\Big(x-\frac{1}{\mathrm{v}_2}\Big)^{\frac{\alpha}{k}-n+1}}\overline{h}(x)dx,$$

where $r(x)=\dfrac{1}{x}$ *and* $\overline{h}(x)=h\Big(\frac{\mathrm{v}_1\mathrm{v}_2}{\mathrm{v}_2-\mathrm{v}_1}\Big(x-\frac{1}{\mathrm{v}_2}\Big)\Big)+h\Big(\frac{\mathrm{v}_1\mathrm{v}_2}{\mathrm{v}_2-\mathrm{v}_1}\Big(\frac{1}{\mathrm{v}_1}-x\Big)\Big)$ *for all* $x\in\Big[\frac{1}{\mathrm{v}_2},\frac{1}{\mathrm{v}_1}\Big]$.

Here we give the H-H type inequalities involving (h,m)-convexity pertaining to k-CFD.

Definition 64 ([35]). *A real-valued function* Υ *is called* (h,m)*-convex, if*

$$\Upsilon(tx+m(1-t)y)\leq h(t)\Upsilon(x)+mh(1-t)\Upsilon(y).$$

$\forall\, x,y\in[0,b]$, *where* $m\in[0,1]$, $t\in(0,1)$, *and* $h:J\to\mathbb{R}$ *is a nonnegative function.*

Theorem 65 ([36]). *Let* $\Upsilon:[0,\infty)\to\mathbb{R}$ *be a function such that* $\Upsilon\in C^n[\mathrm{v}_1,\mathrm{v}_2]$. *Also, let* $\Upsilon^{(n)}$ *be* (h,m)*-convex with* $m\in(0,1]$. *Then fractional inequality for* k*-CFD is given as:*

$$\Upsilon^{(n)}\Big(\frac{m\mathrm{v}_2+\mathrm{v}_1}{2}\Big)$$
$$\leq \quad h\Big(\frac{1}{2}\Big)\frac{k\Gamma_k\Big(n-\frac{\alpha}{k}+k\Big)}{(m\mathrm{v}_2-\mathrm{v}_1)^{n-\frac{\alpha}{k}}}\Big[m^{\alpha+1}(-1)^n({}^CD^{\alpha,k}_{\mathrm{v}_2-}\Upsilon)\Big(\frac{\mathrm{v}_1}{m}\Big)+({}^CD^{\alpha,k}_{\mathrm{v}_1+}\Upsilon)(m\mathrm{v}_2)\Big]$$
$$\leq \quad \Big(n-\frac{\alpha}{k}\Big)h\Big(\frac{1}{2}\Big)\Big\{\Big[m^2\Upsilon^{(n)}\Big(\frac{\mathrm{v}_1}{m^2}\Big)+m\Upsilon^{(n)}(\mathrm{v}_2)\int_0^1 t^{n-\frac{\alpha}{k}-1}h(1-t)dt$$
$$+[m\Upsilon^{(n)}(\mathrm{v}_2)+\Upsilon^{(n)}(\mathrm{v}_1)]\int_0^1 t^{n-\frac{\alpha}{k}-1}h(t)dt\Big]\Big\}.$$

Theorem 66 ([36]). *Assume that* Υ *is as in Theorem 65. Then we have*

$$\Upsilon^{(n)}\Big(\frac{\mathrm{v}_1+\mathrm{v}_2m}{2}\Big)$$
$$\leq \quad 2^{n-\frac{\alpha}{k}}h\Big(\frac{1}{2}\Big)\frac{k\Gamma_k\Big(n-\frac{\alpha}{k}+k\Big)}{(m\mathrm{v}_2-\mathrm{v}_1)^{n-\frac{\alpha}{k}}}\Big[({}^CD^{\alpha,k}_{\big(\frac{\mathrm{v}_1+\mathrm{v}_2m}{2}\big)+}\Upsilon)(m\mathrm{v}_2)$$
$$+m^{\alpha+1}(-1)^n({}^CD^{\alpha,k}_{\big(\frac{\mathrm{v}_1+\mathrm{v}_2m}{2}\big)-}\Upsilon)\Big(\frac{\mathrm{v}_1}{m}\Big)\Big]$$
$$\leq \quad \Big(n-\frac{\alpha}{k}\Big)h\Big(\frac{1}{2}\Big)\Big\{\Big[m^2\Upsilon^{(n)}\Big(\frac{\mathrm{v}_1}{m^2}\Big)\int_0^1 t^{n-\frac{\alpha}{k}-1}h\Big(\frac{2-t}{2}\Big)dt$$

$$+m\Upsilon^{(n)}(\mathrm{v}_2)\int_0^1 t^{n-\frac{\alpha}{k}-1}h\Big(\frac{2-t}{2}\Big)dt+[m\Upsilon^{(n)}(\mathrm{v}_2)$$
$$+\Upsilon^{(n)}(\mathrm{v}_1)]\int_0^1 t^{n-\frac{\alpha}{k}-1}h\Big(\frac{t}{2}\Big)dt\Big]\Big\}.$$

Theorem 67 ([36])**.** *Assume that Υ is as in Theorem 65. If $|\Upsilon^{(n+1)}|$ is an (h,m)-convex, then fractional inequality for k-CFD is given as:*

$$\begin{aligned}&\left|\frac{\Upsilon(m\mathrm{v}_2)+\Upsilon(\mathrm{v}_1)}{2}-\frac{k\Gamma_k\Big(n-\frac{\alpha}{k}+k\Big)}{2(m\mathrm{v}_2-\mathrm{v}_1)^{n-\frac{\alpha}{k}}}\Big[({}^CD^{\alpha,k}_{b+}\Upsilon)(m\mathrm{v}_2)+({}^CD^{\alpha,k}_{b+}\Upsilon)(\mathrm{v}_1)\Big]\right|\\ \leq\ &\frac{(m\mathrm{v}_2-\mathrm{v}_1)[|\Upsilon'(\mathrm{v}_1)|+m|\Upsilon'(\mathrm{v}_2)|]}{2}\Big[\Big[\frac{2^{n-\frac{\alpha}{k}p+1}-1}{2^{n-\frac{\alpha}{k}p+1}\Big(np-\frac{\alpha}{k}p+1\Big)}\Big]^{\frac{1}{p}}\\ &-\Big[\frac{1}{2^{n-\frac{\alpha}{k}p+1}\Big(np-\frac{\alpha}{k}p+1\Big)}\Big]^{\frac{1}{p}}\Big]\Big(\Big[\int_0^{1/2}(h(t))^qdt\Big]^{\frac{1}{q}}+\Big[\int_{1/2}^1(h(t))^qdt\Big]^{\frac{1}{q}}\Big),\end{aligned}$$

where $\dfrac{1}{p}+\dfrac{1}{q}=1$, $q>1$, $m\in(0,1]$ *and* $h^q\in[0,1]$.

Theorem 68 ([36])**.** *Suppose that Υ is as in Theorem 65. If $|\Upsilon^{(n+2)}|$ is an (h,m)-convex, then fractional inequality for k-CFD is given as:*

$$\begin{aligned}&\left|\frac{\Upsilon^{(n)}(m\mathrm{v}_2)+\Upsilon^{(n)}(\mathrm{v}_1)}{2}-\frac{k\Gamma_k\Big(n-\frac{\alpha}{k}+k\Big)}{2(m\mathrm{v}_2-\mathrm{v}_1)^{n-\frac{\alpha}{k}}}\Big[({}^CD^{\alpha,k}_{b+}\Upsilon)(m\mathrm{v}_2)+({}^CD^{\alpha,k}_{b+}\Upsilon)(\mathrm{v}_1)\Big]\right|\\ \leq\ &\frac{(m\mathrm{v}_2-\mathrm{v}_1)^2}{2\Big(n-\frac{\alpha}{k}+1\Big)}\Big(1-\frac{2}{p(\alpha+1)+1}\Big)^{\frac{1}{p}}\Big(\int_0^1(h(t))^qdt\Big)^{\frac{1}{q}}\\ &\times\Big[|\Upsilon^{(n+2)}(\mathrm{v}_1)|+m[|\Upsilon^{(n+2)}(\mathrm{v}_2)|\Big],\end{aligned}$$

where $\dfrac{1}{p}+\dfrac{1}{q}=1$, $q>1$, $m\in(0,1]$ *and* $h^q\in[0,1]$.

We give in the next theorems certain H-H type inequalities involving relative convexity pertaining to k-CFD.

Definition 69 ([37])**.** *A real-valued set T_g is relative convex with respect to real-valued function g, if*

$$(1-t)u+tg(v)\in T_g,$$

where $u,v\in\mathbb{R}$ such that $u,g(v)\in T_g$, $t\in[0,1]$.

Definition 70 ([38]). *A real-valued function Υ is relative convex, if there exists an arbitrary real-valued function g such that*

$$\Upsilon((1-t)u+tg(v)) \leq (1-t)\Upsilon(u)+t\Upsilon(g(v)),$$

where $u, v \in \mathbb{R}$ such that $u, g(v) \in T_g$, $t \in [0,1]$.

Theorem 71 ([39]). *Let $\Upsilon : T_g \to \mathbb{R}$ be a positive function such that $\Upsilon \in C^n[\mathrm{v}_1, g(\mathrm{v}_2)]$, $\mathrm{v}_1 < g(\mathrm{v}_2)$. If $\Upsilon^{(n)}$ is relative convex on T_g, then fractional inequalities for k-CFD are given as:*

$$\begin{aligned}
&\Upsilon^{(n)}\Big(\frac{\mathrm{v}_1+g(\mathrm{v}_2)}{2}\Big)\\
\leq\ & \frac{k\Gamma_k\Big(n-\frac{\alpha}{k}+1\Big)}{2(g(\mathrm{v}_2)-\mathrm{v}_1)^{n-\frac{\alpha}{k}}}\Big[({}^{C}D^{\alpha,k}_{\mathrm{v}_1+}\Upsilon)(g(\mathrm{v}_2))+(-1)^n({}^{C}D^{\alpha,k}_{g(\mathrm{v}_2)-}\Upsilon)(\mathrm{v}_1)\Big]\\
\leq\ & \frac{\Upsilon^{(n)}(\mathrm{v}_1)+\Upsilon^{(n)}(g(\mathrm{v}_2))}{2}.
\end{aligned}$$

Theorem 72 ([39]). *Assume that Υ is as in Theorem 71. If $|\Upsilon^{(n+1)}|$ is relative convex on T_g, then fractional inequality for k-CFD is given as:*

$$\begin{aligned}
&\Bigg|\frac{\Upsilon^{(n)}(\mathrm{v}_1)+\Upsilon^{(n)}(g(\mathrm{v}_2))}{2}-\frac{k\Gamma_k\Big(n-\frac{\alpha}{k}+1\Big)}{2(g(\mathrm{v}_2)-\mathrm{v}_1)^{n-\frac{\alpha}{k}}}\Big[({}^{C}D^{\alpha,k}_{\mathrm{v}_1+}\Upsilon)(g(\mathrm{v}_2))\\
&+(-1)^n({}^{C}D^{\alpha,k}_{g(\mathrm{v}_2)-}\Upsilon)(\mathrm{v}_1)\Big]\Bigg|\\
\leq\ & \frac{g(\mathrm{v}_2)-\mathrm{v}_1}{2\Big(n-\frac{\alpha}{k}+1\Big)}\Big(1-\frac{1}{2^{n-\frac{\alpha}{k}}}\Big)\Big[|\Upsilon^{(n+1)}(\mathrm{v}_1)|+|\Upsilon^{(n+1)}(g(\mathrm{v}_2))|\Big].
\end{aligned}$$

Theorem 73 ([39]). *Assume that Υ is as in Theorem 71. If $\Upsilon^{(n)}$ is relative convex on T_g, then fractional inequalities for k-CFD are given as:*

$$\begin{aligned}
&\Upsilon^{(n)}\Big(\frac{g(\mathrm{v}_1)+g(\mathrm{v}_2)}{2}\Big)\\
\leq\ & \frac{k\Gamma_k\Big(n-\frac{\alpha}{k}+1\Big)}{2(g(\mathrm{v}_2)-g(\mathrm{v}_1))^{n-\frac{\alpha}{k}}}\Big[({}^{C}D^{\alpha,k}_{\mathrm{v}_1+}\Upsilon)(g(\mathrm{v}_2))+(-1)^n({}^{C}D^{\alpha,k}_{g(\mathrm{v}_2)-}\Upsilon)(\mathrm{v}_1)\Big]\\
\leq\ & \frac{\Upsilon^{(n)}(g(\mathrm{v}_1))+\Upsilon^{(n)}(g(\mathrm{v}_2))}{2}.
\end{aligned}$$

Theorem 74 ([39]). *Assume that Υ is as in Theorem 71. If $|\Upsilon^{(n+1)}|$ is relative convex on T_g, then fractional inequality for k-CFD is given as:*

$$\Bigg|\frac{\Upsilon^{(n)}(g(\mathrm{v}_1))+\Upsilon^{(n)}(g(\mathrm{v}_2))}{2}-\frac{k\Gamma_k\Big(n-\frac{\alpha}{k}+1\Big)}{2(g(\mathrm{v}_2)-g(\mathrm{v}_1))^{n-\frac{\alpha}{k}}}\Big[({}^{C}D^{\alpha,k}_{\mathrm{v}_1+}\Upsilon)(g(\mathrm{v}_2))$$

$$+(-1)^n({}^C D^{\alpha,k}_{g(\mathrm{v}_2)-}\Upsilon)(g(\mathrm{v}_1))\Big]\Bigg|$$
$$\leq \frac{g(\mathrm{v}_2)-g(\mathrm{v}_1)}{2\left(n-\frac{\alpha}{k}+1\right)}\left(1-\frac{1}{2^{n-\frac{\alpha}{k}}}\right)\Big[|\Upsilon^{(n+1)}(g(\mathrm{v}_1))|+|\Upsilon^{(n+1)}(g(\mathrm{v}_2))|\Big].$$

Next, we give the H-H inequalities involving exponentially $(\theta, h-m)$-convexity pertaining to k-CFD.

Definition 75 ([40]). *A real-valued function Υ is exponentially $(\theta, h-m)$-convex, if*

$$\Upsilon(tx+m(1-t)y)\leq h(t^\theta)\frac{\Upsilon(x)}{e^{\eta x}}+mh(1-t^\theta)\frac{\Upsilon(y)}{e^{\eta y}},$$

where $t\in(0,1)$, $\eta\in\mathbb{R}$, $(\theta,m)\in(0,1]^2$ and $h:J\to\mathbb{R}$ be a non-negative function.

Theorem 76 ([41]). *Let $\alpha>0$, $k\geq 1$ and $[\mathrm{v}_1,\mathrm{v}_2]\subset[0,+\infty)$, $\Upsilon:[0,+\infty)\to\mathbb{R}$ be a function such that $\Upsilon\in AC^n[\mathrm{v}_1,\mathrm{v}_2]$, where $\mathrm{v}_1<m\mathrm{v}_2$. Also, let $\Upsilon^{(n)}$ be an exponentially $(\theta,h-m)$-convex. Then fractional inequalities for k-CFD are given as:*

$$\frac{1}{g(\eta)}\Upsilon^{(n)}\left(\frac{m\mathrm{v}_2+\mathrm{v}_1}{2}\right)\leq\frac{k\Gamma_k\left(n-\frac{\alpha}{k}+k\right)}{(m\mathrm{v}_2-\mathrm{v}_1)^{n-\frac{\alpha}{k}}}$$
$$\times\left(h\left(1-\frac{1}{2^\theta}\right)m^{n-\frac{\alpha}{k}+1}(-1)^n({}^C D^{\alpha,k}_{\mathrm{v}_2-}\Upsilon)\left(\frac{\mathrm{v}_1}{m}\right)+h\left(\frac{1}{2^\theta}\right)({}^C D^{\alpha,k}_{\mathrm{v}_1+}\Upsilon)(m\mathrm{v}_2)\right)$$
$$\leq\frac{kn-\alpha}{k}\left\{\left(h\left(1-\frac{1}{2^\theta}\right)m^2\frac{\Upsilon^{(n)}\left(\frac{\mathrm{v}_1}{m^2}\right)}{e^{\frac{\eta\mathrm{v}_1}{m^2}}}+h\left(\frac{1}{2^\theta}\right)m\frac{\Upsilon^{(n)}(\mathrm{v}_2)}{e^{\eta\mathrm{v}_2}}\right)\int_0^1 t^{n-\frac{\alpha}{k}-1}h(1-t^\theta)dt\right.$$
$$\left.+\left(h\left(1-\frac{1}{2^\theta}\right)m\frac{\Upsilon^{(n)}(\mathrm{v}_2)}{e^{\eta\mathrm{v}_2}}+h\left(\frac{1}{2^\theta}\right)\frac{\Upsilon^{(n)}(\mathrm{v}_1)}{e^{\eta\mathrm{v}_1}}\right)\int_0^1 t^{n-\frac{\alpha}{k}-1}h(t^\theta)dt\right\},$$

where $(\theta,m)\in(0,1]^2$, $\eta\in\mathbb{R}$, $g(\eta)=\frac{1}{e^{\eta\mathrm{v}_2}}$ for $\eta<0$ and $g(\eta)=\frac{1}{e^{\frac{\eta\mathrm{v}_1}{m}}}$ for $\eta\geq 0$.

Theorem 77 ([41]). *Assume that Υ is defined with the conditions in the Theorem 76. If function $\Upsilon^{(n)}$ is an exponentially $(\theta,h-m)$-convex, then we have:*

$$\frac{1}{g(\eta)}\Upsilon^{(n)}\left(\frac{m\mathrm{v}_2+\mathrm{v}_1}{2}\right)\leq\frac{2^{n-\frac{\alpha}{k}}k\Gamma_k\left(n-\frac{\alpha}{k}+k\right)}{(m\mathrm{v}_2-\mathrm{v}_1)^{n-\frac{\alpha}{k}}}$$
$$\times\left(h\left(1-\frac{1}{2^\theta}\right)m^{n-\frac{\alpha}{k}+1}(-1)^n({}^C D^{\alpha,k}_{\left(\frac{\mathrm{v}_1+\mathrm{v}_2 m}{2m}\right)-}\Upsilon)\left(\frac{\mathrm{v}_1}{m}\right)\right.$$
$$\left.+h\left(\frac{1}{2^\theta}\right)({}^C D^{\alpha,k}_{\left(\frac{\mathrm{v}_1+\mathrm{v}_2 m}{2}\right)+}\Upsilon)(m\mathrm{v}_2)\right)$$

$$\leq \frac{kn-\alpha}{k}\Bigg\{\Bigg(h\Big(1-\frac{1}{2^\theta}\Big)m^2\frac{\Upsilon^{(n)}\Big(\frac{\mathrm{v}_1}{m^2}\Big)}{e^{\frac{\eta \mathrm{v}_1}{m^2}}}$$
$$+h\Big(\frac{1}{2^\theta}\Big)m\frac{\Upsilon^{(n)}(\mathrm{v}_2)}{e^{\eta \mathrm{v}_2}}\Bigg)\int_0^1 t^{n-\frac{\alpha}{k}-1}h\Big(1-\Big(\frac{t}{2}\Big)^\theta\Big)dt$$
$$+\Bigg(h\Big(1-\frac{1}{2^\theta}\Big)m\frac{\Upsilon^{(n)}(\mathrm{v}_2)}{e^{\eta \mathrm{v}_2}}+h\Big(\frac{1}{2^\theta}\Big)\frac{\Upsilon^{(n)}(\mathrm{v}_1)}{e^{\eta \mathrm{v}_1}}\Bigg)\int_0^1 t^{n-\frac{\alpha}{k}-1}h\Big(\Big(\frac{t}{2}\Big)^\theta\Big)dt\Bigg\},$$

where $(\theta,m)\in(0,1]^2$, $\eta\in\mathbb{R}$, $g(\eta)=\frac{1}{e^{\eta \mathrm{v}_2}}$ *for* $\eta<0$ *and* $g(\eta)=\frac{1}{e^{\frac{\eta \mathrm{v}_1}{m}}}$ *for* $\eta\geq 0$.

Theorem 78 ([41])**.** *Assume that* Υ *is defined with the conditions in the Theorem 76. If function* $|\Upsilon^{(n+1)}|$ *is an exponentially* $(\theta,h\!-\!m)$*-convex, then fractional inequality for k-CFD is given as:*

$$\Bigg|\frac{\Upsilon^{(n)}(m\mathrm{v}_2)+\Upsilon^{(n)}(\mathrm{v}_1)}{2}-\frac{k\Gamma_k\Big(n-\frac{\alpha}{k}+k\Big)}{(m\mathrm{v}_2-\mathrm{v}_1)^{n-\frac{\alpha}{k}}}$$
$$\times\Big[({}^C D^{\alpha,k}_{b+}\Upsilon)(m\mathrm{v}_2)+({}^C D^{\alpha,k}_{m\mathrm{v}_2-}\Upsilon)(\mathrm{v}_1)\Big]\Bigg|$$
$$\leq \frac{m\mathrm{v}_2-\mathrm{v}_1}{2}\left(\frac{(2^{np-\frac{\alpha}{k}p+1}-1)^{\frac{1}{p}}}{\Big(2^{np-\frac{\alpha}{k}p+1}\Big(np-\frac{\alpha}{k}p+1\Big)\Big)^{\frac{1}{p}}-1}\right)$$
$$\times\Bigg(\frac{|\Upsilon^{(n+1)}(\mathrm{v}_1)}{e^{\eta \mathrm{v}_1}}\Bigg(\Big(\int_0^{1/2}(h(t^\theta))^q dt\Big)^{\frac{1}{q}}+\Big(\int_{1/2}^1(h(t^\theta))^q dt\Big)^{\frac{1}{q}}\Bigg)$$
$$+m\frac{|\Upsilon^{(n+1)}(\mathrm{v}_2)}{e^{\eta \mathrm{v}_2}}\Bigg(\Big(\int_0^{1/2}(h(1-t^\theta))^q dt\Big)^{\frac{1}{q}}+\Big(\int_{1/2}^1(h(1-t^\theta))^q dt\Big)^{\frac{1}{q}}\Bigg)\Bigg),$$

where $\frac{1}{p}+\frac{1}{q}=1$, $(\theta,m)\in(0,1]^2$ *and* $\eta\in\mathbb{R}$.

4 Hermite-Hadamard Type Inequalities via Hilfer Fractional Derivative

Let $AC[\mathrm{v}_1,\mathrm{v}_2]$ be the space of absolutely continuous functions on $[\mathrm{v}_1,\mathrm{v}_2]$ and $AC^m[\mathrm{v}_1,\mathrm{v}_2]$ be the space of all functions $\Upsilon\in C^m[\mathrm{v}_1,\mathrm{v}_2]$ with $\Upsilon^{(m-1)}\in AC[\mathrm{v}_1,\mathrm{v}_2]$. We denote $K_\gamma(t)=\dfrac{t^{\gamma-1}}{\Gamma(\gamma)}$.

Definition 79 ([42])**.** *Let* $\Upsilon\in L_1[a.b]$, $\Upsilon\star K_{(1-\beta)(n-\gamma)}\in AC^n[\mathrm{v}_1,\mathrm{v}_2]$, $n-1<\gamma<n$, $0\leq\beta\leq 1$, $n\in\mathbb{N}$. *Then the derivative*

$$(D^{\gamma,\beta}_{\mathrm{v}_1+}\Upsilon)(t)=\Big(I^{\beta(n-\gamma)}_{\mathrm{v}_1+}\frac{d^n}{dt^n}\Big(I^{(1-\beta)(n-\gamma)}_{\mathrm{v}_1+}\Upsilon(t)\Big)\Big),$$

is called Hilfer fractional derivative (HFD).

Definition 80 ([43]). *Let $\Upsilon : I \subset \mathbb{R}^+ \to \mathbb{R}^+$ and $s \in (0,1]$. A function f is geometric-arithmetically s-convex function on I if for every $x, y \in I$ and $t \in [0,1]$, we have*

$$\Upsilon(x^t y^{1-t}) \leq t^s \Upsilon(x) + (1-t)^s \Upsilon(y).$$

Theorem 81 ([44]). *Let $\Upsilon \in L_1[\mathtt{v}_1, \mathtt{v}_2]$, $\Upsilon \star K_{(1-\beta)(n-\gamma)} \in AC^n[\mathtt{v}_1, \mathtt{v}_2]$, $n \in \mathbb{N}$ and $D^{\gamma+\beta(n-\gamma)}_{(\mathtt{v}_1,\mathtt{v}_2)}\Upsilon : [\mathtt{v}_1, \mathtt{v}_2] \to \mathbb{R}$ be a positive function with $0 \leq \mathtt{v}_1 < \mathtt{v}_2$, $n-1 < \gamma < n$, $0 \leq \beta \leq 1$ and $D^{\gamma+\beta(n-\gamma)}_{(\mathtt{v}_1,\mathtt{v}_2)}\Upsilon \in L_1[\mathtt{v}_1, \mathtt{v}_2]$. If function $D^{\gamma+\beta(n-\gamma)}_{(\mathtt{v}_1,\mathtt{v}_2)}\Upsilon$ is convex on $[\mathtt{v}_1, \mathtt{v}_2]$ and $F(x) = \Upsilon(x) + \Upsilon(\mathtt{v}_1 + \mathtt{v}_2 - x)$, then fractional inequality is given as:*

$$\begin{aligned} D^{\gamma+\beta(n-\gamma)}_{(\mathtt{v}_1,\mathtt{v}_2)}\Upsilon\Big(\frac{\mathtt{v}_1+\mathtt{v}_2}{2}\Big) &\leq \frac{\Gamma(\beta(n-\gamma)+1)}{(\mathtt{v}_2-\mathtt{v}_1)^{\beta(n-\gamma)}}\Big[D^{\gamma,\beta}_{\mathtt{v}_1+}F(\mathtt{v}_2) + D^{\gamma,\beta}_{\mathtt{v}_2-}F(\mathtt{v}_1)\Big] \\ &\leq D^{\gamma+\beta(n-\gamma)}_{\mathtt{v}_1+}F(\mathtt{v}_2) + D^{\gamma+\beta(n-\gamma)}_{\mathtt{v}_2-}F(\mathtt{v}_1). \end{aligned}$$

Theorem 82 ([44]). *Assume that $D^{\gamma+\beta(n-\gamma)+2}_{(\mathtt{v}_1,\mathtt{v}_2)}\Upsilon$ is defined with the conditions in the Theorem 81. If function $D^{\gamma+\beta(n-\gamma)+1}_{(\mathtt{v}_1,\mathtt{v}_2)}\Upsilon$ is convex on $[\mathtt{v}_1, \mathtt{v}_2]$, then fractional inequality is given as:*

$$\begin{aligned} &\Bigg|\frac{D^{\gamma+\beta(n-\gamma)}_{\mathtt{v}_1+}\Upsilon(\mathtt{v}_2) + D^{\gamma+\beta(n-\gamma)}_{\mathtt{v}_2-}\Upsilon(\mathtt{v}_1)}{2} \\ &-\frac{\Gamma(\beta(n-\gamma)+1)}{2(\mathtt{v}_2-\mathtt{v}_1)^{\beta(n-\gamma)}}\Big[D^{\gamma,\beta}_{\mathtt{v}_1+}F(\mathtt{v}_2) + D^{\gamma,\beta}_{\mathtt{v}_2-}F(\mathtt{v}_1)\Big]\Bigg| \\ \leq\ & \frac{\mathtt{v}_2-\mathtt{v}_1}{2(\beta(n-\gamma)+1)}\Big(1-\frac{1}{2^{\beta(n-\gamma)}}\Big)\Big(|D^{\gamma+\beta(n-\gamma)+1}_{(\mathtt{v}_1,\mathtt{v}_2)}\Upsilon(\mathtt{v}_2)| + |D^{\gamma+\beta(n-\gamma)+1}_{(\mathtt{v}_1,\mathtt{v}_2)}\Upsilon(\mathtt{v}_1)|\Big). \end{aligned}$$

Theorem 83 ([44]). *Assume that $D^{\gamma+\beta(n-\gamma)+2}_{(\mathtt{v}_1,\mathtt{v}_2)}\Upsilon$ is defined with the conditions in the Theorem 81. If $|D^{\gamma+\beta(n-\gamma)+2}_{(\mathtt{v}_1,\mathtt{v}_2)}\Upsilon|$ is measurable, decreasing and geometric-arithmetically s-convex on $[\mathtt{v}_1, \mathtt{v}_2]$, then fractional inequality is given as:*

$$\begin{aligned} &\Bigg|\frac{D^{\gamma+\beta(n-\gamma)}_{\mathtt{v}_1+}\Upsilon(\mathtt{v}_2) + D^{\gamma+\beta(n-\gamma)}_{\mathtt{v}_2-}\Upsilon(\mathtt{v}_1)}{2} \\ &-\frac{\Gamma(\beta(n-\gamma)+1)}{2(\mathtt{v}_2-\mathtt{v}_1)^{\beta(n-\gamma)}}\Big[D^{\gamma,\beta}_{\mathtt{v}_1+}F(\mathtt{v}_2) + D^{\gamma,\beta}_{\mathtt{v}_2-}F(\mathtt{v}_1)\Big]\Bigg| \\ \leq\ & \frac{(\mathtt{v}_2-\mathtt{v}_1)^2\Big(|D^{\gamma+\beta(n-\gamma)+2}_{(\mathtt{v}_1,\mathtt{v}_2)}\Upsilon(\mathtt{v}_2)| + |D^{\gamma+\beta(n-\gamma)+2}_{(\mathtt{v}_1,\mathtt{v}_2)}\Upsilon(\mathtt{v}_1)|\Big)}{2(\beta(n-\gamma)+1)} \\ &\times\Big(\frac{1}{s+1} - \frac{1}{\beta(n-\gamma)+s+2}\Big). \end{aligned}$$

Theorem 84 ([44]). *Assume that* $D^{\gamma+\beta(n-\gamma)+2}_{(\mathtt{v}_1,\mathtt{v}_2)}\Upsilon$ *is defined with the conditions in the Theorem 81. If function* $|D^{\gamma+\beta(n-\gamma)+2}_{(\mathtt{v}_1,\mathtt{v}_2)}\Upsilon|^q$ *is measurable, decreasing and geometric-arithmetically s-convex on* $[0,\mathtt{v}_2]$*, then fractional inequality is given as:*

$$\begin{aligned}&\left|\frac{\Gamma(\beta(n-\gamma)+1)}{2(\mathtt{v}_2-\mathtt{v}_1)^{\beta(n-\gamma)}}\Big[D^{\gamma,\beta}_{\mathtt{v}_1+}\Upsilon(\mathtt{v}_2)+D^{\gamma,\beta}_{\mathtt{v}_2-}\Upsilon(\mathtt{v}_1)\Big]-D^{\gamma+\beta(n-\gamma)}_{(\mathtt{v}_1,\mathtt{v}_2)}\Upsilon\Big(\frac{\mathtt{v}_1+\mathtt{v}_2}{2}\Big)\right|\\ \leq\ &\frac{(\mathtt{v}_2-\mathtt{v}_1)^2}{2(\beta(n-\gamma)+1)}\left(\frac{|D^{\gamma+\beta(n-\gamma)+2}_{(\mathtt{v}_1,\mathtt{v}_2)}\Upsilon(\mathtt{v}_2)|^q+|D^{\gamma+\beta(n-\gamma)+2}_{(\mathtt{v}_1,\mathtt{v}_2)}\Upsilon(\mathtt{v}_1)|^q}{s+1}\right)^{\frac{1}{q}}\\ &\times\left(\frac{(\beta(n-\gamma)+1)2^{-p-1}+(\beta(n-\gamma)+0.5)^{p+1}-(\beta(n-\gamma))^{p+1}}{p+1}\right)^{\frac{1}{q}},\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

5 Hermite-Hadamard Type Inequalities via ψ-Caputo Fractional Derivative

Now we give H-H Inequalities for ψ-CFD.

Definition 85 ([45]). *Let* $\psi'(t)\neq 0$ *and* $\alpha>0$, $n\in\mathbb{N}$. *The* ψ*-CFDs of a function* f *with respect to another function* ψ *of order* α*, are defined by*

$$ {}^{C}D^{\alpha;\psi}_{\mathtt{v}_1+}\Upsilon(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n\int_a^t\psi'(s)(\psi(t)-\psi(s))^{n-\alpha-1}\Upsilon(s)ds,$$

and

$$ {}^{C}D^{\alpha;\psi}_{\mathtt{v}_2-}\Upsilon(t) = \frac{(-1)^n}{\Gamma(n-\alpha)}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n\int_t^b\psi'(s)(\psi(s)-\psi(t))^{n-\alpha-1}\Upsilon(s)ds,$$

where $n=[\alpha]+1$, $[\alpha]$ *is represent the integer part of the real number* α.

Theorem 86 ([46]). *For* $n\in\mathbb{N},\rho,\delta\geq 1$*, and let there be a real-valued n-times differentiable function* $\Upsilon:[\mathtt{v}_1,\mathtt{v}_2]\to\mathbb{R}$. *Also, assume that* ψ *is differentiable and strictly increasing such that with* $\psi'\in L_1[\mathtt{v}_1,\mathtt{v}_2]$. *If* $\psi^{(n)}$ *is a convex function on* $[\mathtt{v}_1,\mathtt{v}_2]$*, then:*

$$\begin{aligned}&\Gamma(n-\rho+1)\Big({}^{C}D^{\rho-1,\psi}_{\mathtt{v}_1+}\Upsilon\Big)(\lambda)+\Gamma(n-\delta+1)\Big({}^{C}D^{\rho-1,\psi}_{\mathtt{v}_2-}\Upsilon\Big)(\lambda)\\ \leq\ &\frac{[\psi(\lambda)-\psi(\mathtt{v}_1)]^{n-\rho}}{\lambda-\mathtt{v}_1}\Big[(\lambda-\mathtt{v}_1)[\Upsilon^{(n)}(\lambda)\psi(\lambda)-\Upsilon^{(n)}(\mathtt{v}_1)\psi(\mathtt{v}_1)]\end{aligned}$$

$$-(\Upsilon^{(n)}(\lambda) - \Upsilon^{(n)}(\mathrm{v}_1)) \int_{\mathrm{v}_1}^{\lambda} \psi(s)ds\Big]$$
$$+\frac{[\psi(\mathrm{v}_2) - \psi(\lambda)]^{n-\delta}}{\mathrm{v}_2 - \lambda}\Big[(\mathrm{v}_2 - \lambda)[\Upsilon^{(n)}(\Upsilon^{(n)}(\mathrm{v}_2)\psi(\mathrm{v}_2) - \Upsilon^{(n)}(\lambda)\psi(\lambda)]$$
$$-(\Upsilon^{(n)}(\mathrm{v}_2) - \Upsilon^{(n)}(\lambda)) \int_{\lambda}^{\mathrm{v}_2} \psi(s)ds\Big].$$

Theorem 87 ([46]). *Assume that* Υ *and* ψ *are as in Theorem 86. If function* $|\psi^{(n+1)}|$ *is convex on* $[\mathrm{v}_1, \mathrm{v}_2]$*, then:*

$$\Big|\Gamma(n-\rho+1)\big({}^{C}D^{\rho-1,\psi}_{\mathrm{v}_1+}\Upsilon\big)(\lambda) + \Gamma(n-\delta+1)\big({}^{C}D^{\rho-1,\psi}_{\mathrm{v}_2-}\Upsilon\big)(\lambda)$$
$$-[\psi(\lambda) - \psi(\mathrm{v}_1)]^{n-\rho}\Upsilon^{(n)}(\mathrm{v}_1) + (\psi(\mathrm{v}_2) - \psi(\lambda))^{n-\delta}\Upsilon^{(n)}(\mathrm{v}_2)]\Big|$$
$$\leq \frac{1}{2}\Big[(\psi(\lambda) - \psi(\mathrm{v}_1))^{n-\rho}(\lambda - \mathrm{v}_1)|\Upsilon^{(n+1)}(\mathrm{v}_1)|$$
$$+(\psi(\mathrm{v}_2) - \psi(\lambda))^{n-\delta}(\mathrm{v}_2 - \lambda)|\Upsilon^{(n+1)}(\mathrm{v}_2)|\Big]$$
$$+\frac{1}{2}|\Upsilon^{(n+1)}(\lambda)|\Big[(\psi(\lambda) - \psi(\mathrm{v}_1))^{n-\rho}(\lambda - \mathrm{v}_1) + (\psi(\mathrm{v}_2) - \psi(\lambda))^{n-\delta}(\mathrm{v}_2 - \lambda)\Big].$$

6 Conclusions

Our objective in this paper was to present a comprehensive and up to-date review on H-H inequalities for fractional differential operators. We presented results including inequalities of the H-H type for fractional differential operators through various classes of convexity. We considered inequalities on Caputo fractional derivatives, k-Caputo fractional derivatives and Hilfer fractional derivative operators. We think that the current analysis will give a platform for the investigators studying H-H inequality to learn more about previous research on the subject before coming up with new findings.

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