Surveys in Mathematics and its Applications

ISSN 1842-6298 (electronic), 1843-7265 (print)
Volume 18 (2023), 259 – 272

GEOMETRIC DISTRIBUTION SERIES CONNECTED WITH CERTAIN SUBCLASSES OF UNIVALENT FUNCTIONS

Masoumeh Taliyan, Shahram Najafzadeh and Mohammad Reza Azimi

Abstract. In this paper, we consider the class of normalized analytic functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Following this functions, we define the functions whose coefficients are probabilities of the geometric distribution series and other special modes of this series. Also, we consider different special classes of $f(z)$. In the following we consider some lemmas that make connection between defined special classes with the function $f(z)$. Follower of this topic we will consider the theorems that make connection between defined classes with the functions whose coefficients are probabilities of geometric distribution series. Also we define Alexander-type integral operator and find the necessary and sufficient conditions for being this operator to defined general classes.

1 Introduction

Let $\mathcal{A}$ denote the class of normalized functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : z \in \mathbb{C}, |z| < 1\}$, and let S be the subclass of $\mathcal{A}$ consisting of functions of the form (1.1) that are also univalent in U.

Furthermore, for $g \in \mathcal{A}$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \qquad z \in U$$

we define the Hadamard product (or convolution product) of f and g by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \qquad z \in U$$

2020 Mathematics Subject Classification: 30C45; 30C50
Keywords: Analytic functions, Convolution product, Subordinate, Geometric distribution series

If f and g are two analytic functions in the unit disk U, we say that f is subordinate to g, written as $f(z) \prec g(z)$, if there exists a Schwarz function w, which is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z))$, $z \in U$. In addition, if the function g is univalent in U, we say $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$ (See [3, 12]).

In the following, we introduce two subclasses (a), (b) (see [2, 5, 6, 14, 15]).

(a) A function $f \in \mathcal{A}$ is said to be in the class $q - s_p(\Gamma)$ of q-uniformly starlike functions of order Γ if it satisfies the condition

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)} - \Gamma\right) > q\left|\frac{zf'(z)}{f(z)} - 1\right|, \qquad z \in U, \tag{1.2}$$

where $0 \leq \Gamma < 1$ and $q \geq 0$.

(b) A function $f \in \mathcal{A}$ is said to be in the class $q - ucv(\Gamma)$ of q-uniformly convex functions of order Γ if it satisfies the condition

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)} - \Gamma\right) > q\left|\frac{zf''(z)}{f'(z)}\right|, \qquad z \in U, \tag{1.3}$$

where $0 \leq \Gamma < 1$ and $q \geq 0$.

It follows from (1.2) and (1.3) that for a function $f \in \mathcal{A}$ we have the equivalence relation

$$f \in q - ucv(\Gamma) \Longleftrightarrow zf'(z) \in q - s_p(\Gamma).$$

For $\Gamma = 0$ the classes $q - ucv(\Gamma)$ and $q - s_p(\Gamma)$ reduce to the classes $q - ucv$ and $q - s_p$ respectively, (see [9, 10]).

Definition 1. *([17]) For $-1 \leq C < D \leq 1$ and $|\alpha| < \frac{\pi}{2}$, a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{L}(C, D, \alpha)$ if the following subordination condition is met*

$$e^{i\alpha} f'(z) \prec \frac{1 + Cz}{1 + Dz}\cos\alpha + i\sin\alpha.$$

Using the definition of the subordination, we say that a function $f \in \mathcal{A}$ belongs to the class $\mathcal{L}(C, D, \alpha)$ if and only if there exists an analytic function w, satisfying $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that

$$e^{i\alpha} f'(z) = \frac{1 + Cw(z)}{1 + Dw(z)}\cos\alpha + i\sin\alpha, \qquad z \in U,$$

or equivalently

$$\left|\frac{e^{i\alpha}(f'(z) - 1)}{De^{i\alpha} f'(z) - [De^{i\alpha} + (C - D)\cos\alpha]}\right| < 1, \quad z \in U.$$

Different authors investigated special choices of C, D, α in the following subclasses:

(i) $\mathcal{L}(2\theta-1,1,\alpha) := \mathcal{L}(\alpha,\theta)$ $(0 \leq \theta < 1, |\alpha| < \frac{\pi}{2})$ ([8])

(ii) $\mathcal{L}(\Gamma(2\theta-1),\Gamma,0) := \mathcal{R}(\Gamma,\theta)$ $(0 < \Gamma \leq 1, 0 \leq \theta < 1)$ ([7])

(iii) $\mathcal{L}(\Gamma(2\theta-1),\Gamma,0) := \mathcal{D}(\Gamma)$ $(0 < \Gamma \leq 1)$ ([4])

(iv) $\mathcal{L}(C+(D-C)\theta,D,\alpha) := \mathcal{L}(C,D,\alpha,\theta)$ $(-1 \leq C < D \leq 1, |\alpha| < \frac{\pi}{2}, 0 \leq \theta < 1)$ ([1]).

A variable x is said to have the geometric distribution if it takes the values $0,1,2,3,\ldots$ with the probabilities $(1-m), m(1-m), m^2(1-m), m^3(1-m), \ldots$ respectively, where m is called the parameter. Thus

$$p(x=k) = m^k(1-m), \qquad k = 0,1,2,3,\ldots$$

Now, we introduce a power series whose coefficients are probabilities of the geometric distribution:

$$K(m,z) := z + \sum_{n=2}^{\infty} m^{n-1}(1-m)z^n, \qquad z \in U, (0 \leq m \leq 1). \tag{1.4}$$

We note that, by ratio test, the radius of convergence of the above series is infinity.

We will define the functions

$$\begin{aligned} F(m,\lambda,z) : &= (1-\lambda)K(m,z) + \lambda z(K(m,z))' \\ &= z + \sum_{n=2}^{\infty} [1+\lambda(n-1)]\, m^{n-1}(1-m)z^n, \qquad z \in U, \\ &\quad (0 \leq m \leq 1, \lambda \geq 0) \end{aligned} \tag{1.5}$$

and

$$\begin{aligned} N(m,\lambda,\gamma,z) : &= (1-\lambda+\gamma)K(m,z) + (\lambda-\gamma)z(K(m,z))' + \lambda\gamma z^2(K(m,z))'' \\ &= z + \sum_{n=2}^{\infty} [1+(n-1)(\lambda-\gamma+n\lambda\gamma)]\, m^{n-1}(1-m)z^n \qquad z \in U \\ &\quad (0 \leq m \leq 1, \lambda,\gamma \geq 0) \end{aligned} \tag{1.6}$$

and we introduce the linear operator $P_m : \mathcal{A} \to \mathcal{A}$ defined by

$$P_m(f)(z) := K(m,z) * f(z) = z + \sum_{n=2}^{\infty} m^{n-1}(1-m)a_n z^n, \qquad z \in U, (0 \leq m \leq 1) \tag{1.7}$$

In this paper we make some sufficient conditions for the geometric distribution series and other depending series to be in some subclasses of analytic functions. Also, we give conditions for an integral operator related to this series.

2 Preliminaries

Unless otherwise stated, we assume that $-1 \leq C < D \leq 1, |\alpha| < \dfrac{\pi}{2}, 0 \leq m \leq 1, \lambda, \gamma \geq 0$ and $\lambda \geq \gamma$. To prove our results we need the following lemmas.

Lemma 2. *([1, Theorem 4]) If the function $f \in \mathcal{A}$ is of the form (1.1) and*

$$\sum_{n=2}^{\infty} n\,(1+|D|)\,|a_n| \leq (D-C)\cos\alpha$$

then $f \in \mathcal{L}(C,D,\alpha)$.

Lemma 3. *([1, Theorem 1]) If the function $f \in \mathcal{L}(C,D,\alpha)$ is of the form (1.1), then*

$$|a_n| \leq \frac{(D-C)\cos\alpha}{n},$$

for every $n \geq 2$. The estimate is sharp.

Lemma 4. *([9, Theorem 3.3]) If $f \in \mathcal{A}$ and for some q $(0 \leq q < \infty)$ the following inequality holds*

$$\sum_{n=2}^{\infty} n(n-1)|a_n| \leq \frac{1}{q+2},$$

then $f \in q-ucv$. The number $\dfrac{1}{q+2}$ cannot be increased.

Lemma 5. *([16, Theorem 2.1]) If the function $f \in \mathcal{A}$ is of the form (1.1) and*

$$\sum_{n=2}^{\infty} [n(1+q)-(q+\Gamma)]\,|a_n| \leq 1-\Gamma,$$

then $f \in q-s_p(\Gamma)$.

3 Main Results

Theorem 6. *A sufficient condition for the function $K(m,z)$ given by (1.4) to be in the class $\mathcal{L}(C,D,\alpha)$ is*

$$\frac{m}{1-m}+m \leq \frac{(D-C)\cos\alpha}{1+|D|}. \tag{3.1}$$

Proof. To prove that $K(m,z) \in \mathcal{L}(C,D,\alpha)$, according to Lemma 2, it is sufficient to show that

$$\sum_{n=2}^{\infty} n\,(1+|D|)\,|m|^{n-1}|1-m| \leq (D-C)\cos\alpha.$$

In the proof of this theorem we will use the following relation

$$\sum_{n=2}^{\infty} nm^{n-1} = \frac{1-(1-m)^2}{(1-m)^2}.$$

We have

$$\begin{aligned}\sum_{n=2}^{\infty} n\,(1+|D|)\,m^{n-1}(1-m) &= (1-m)(1+|D|)\sum_{n=2}^{\infty} nm^{n-1} \\ &= (1-m)(1+|D|)\frac{1-(1-m)^2}{(1-m)^2} \\ &= (1+|D|)\left[\frac{m}{1-m}+m\right] \\ &\leq (D-C)\cos\alpha,\end{aligned}$$

and from Lemma 2 it follows that

$$K(m,z) \in \mathcal{L}(C,D,\alpha).$$

□

Theorem 7. *A sufficient condition for the function $F(m,\lambda,z)$ defined by (1.5) to be in the class $\mathcal{L}(C,D,\alpha)$ is*

$$\frac{(1+2\lambda)m}{1-m}+m+\frac{2\lambda m^2}{(1-m)^2} \leq \frac{(D-C)\cos\alpha}{1+|D|}. \tag{3.2}$$

Proof. From Lemma 2, to prove that $F(m,\lambda,z) \in \mathcal{L}(C,D,\alpha)$, it is sufficient to show that

$$\sum_{n=2}^{\infty} [n(1+|D|)]\,[1+\lambda(n-1)]\,|m|^{n-1}|1-m| \leq (D-C)\cos\alpha. \tag{3.3}$$

In the proof of this theorem we will use the following relations

$$\begin{aligned}\sum_{n=2}^{\infty} nm^{n-1} &= \frac{1-(1-m)^2}{(1-m)^2} \\ \sum_{n=2}^{\infty} n(n-1)m^{n-1} &= \frac{2m(1-m)^2+2m(1-(1-m)^2)}{(1-m)^3}\end{aligned}$$

We have

$$\begin{aligned}
&\sum_{n=2}^{\infty}[n(1+|D|)]\left[1+\lambda(n-1)\right]m^{n-1}(1-m)\\
&=(1-m)(1+|D|)\left[\sum_{n=2}^{\infty}nm^{n-1}+\lambda\sum_{n=2}^{\infty}n(n-1)m^{n-1}\right]\\
&=(1-m)(1+|D|)\left[\frac{1-(1-m)^2}{(1-m)^2}+\lambda\frac{2m(1-m)^2+2m(1-(1-m)^2)}{(1-m)^3}\right]\\
&=(1+|D|)\left[\frac{1-(1-m)^2}{1-m}+\lambda\frac{2m(1-m)^2+2m(1-(1-m)^2)}{(1-m)^2}\right]\\
&=(1+|D|)\left[\frac{1}{1-m}-(1-m)+\frac{2\lambda m}{(1-m)^2}\right]\\
&=(1+|D|)\left[\frac{m}{1-m}+m+\frac{2\lambda m}{1-m}+\frac{2\lambda m^2}{(1-m)^2}\right]\\
&=(1+|D|)\left[\frac{(1+2\lambda)m}{(1-m)}+m+\frac{2\lambda m^2}{(1-m)^2}\right].
\end{aligned}$$

According to (3.2), from the above identity, we conclude it follows that the inequality (3.3) holds and therefore $F(m,\lambda,z)\in\mathcal{L}(C,D,\alpha)$. □

Theorem 8. *A sufficient condition for the function $N(m,\lambda,\gamma,z)$ given by (1.6) to be in the class $\mathcal{L}(C,D,\alpha)$ is*

$$\frac{(1+2\lambda-2\gamma+4\lambda\gamma)m}{1-m}+m+\frac{(\lambda-\gamma+5\lambda\gamma)2m^2}{(1-m)^2}+\frac{6\lambda\gamma m^3}{(1-m)^3}\leq\frac{(D-C)\cos\alpha}{1+|D|}. \tag{3.4}$$

Proof. To prove that $N(m,\lambda,\gamma,z)\in\mathcal{L}(C,D,\alpha)$, from Lemma 2, it is sufficient to show that

$$\sum_{n=2}^{\infty}n(1+|D|)\left[1+(n-1)(\lambda-\gamma+n\lambda\gamma)\right]|m|^{n-1}|1-m|\leq(D-C)\cos\alpha. \tag{3.5}$$

In the proof of this theorem we will use the following relations

$$\begin{aligned}
\sum_{n=2}^{\infty}(n-1)m^{n-1}&=\frac{m}{(1-m)^2}\\
\sum_{n=2}^{\infty}(n-1)(n-2)m^{n-1}&=\frac{2m^2}{(1-m)^3}\\
\sum_{n=2}^{\infty}(n-1)(n-2)(n-3)m^{n-1}&=\frac{6m^3}{(1-m)^4}.
\end{aligned}$$

We have

$$\begin{aligned}
&\sum_{n=2}^{\infty} n(1+|D|)\left[1+(n-1)(\lambda-\gamma+n\lambda\gamma)\right] m^{n-1}(1-m) \\
&= (1+|D|)(1-m)\left[\sum_{n=2}^{\infty}(1+2\lambda-2\gamma+4\lambda\gamma)(n-1)m^{n-1}\right. \\
&+\sum_{n=2}^{\infty} m^{n-1}+\sum_{n=2}^{\infty}(\lambda-\gamma+5\lambda\gamma)(n-1)(n-2)m^{n-1} \\
&\left.+\sum_{n=2}^{\infty}\lambda\gamma(n-1)(n-2)(n-3)m^{n-1}\right] \\
&= (1+|D|)(1-m)\left[(1+2\lambda-2\gamma+4\lambda\gamma)\frac{m}{(1-m)^2}+\frac{m}{1-m}\right. \\
&\left.+(\lambda-\gamma+5\lambda\gamma)\frac{2m^2}{(1-m)^3}+\lambda\gamma\frac{6m^3}{(1-m)^4}\right] \\
&= (1+|D|)\left[\frac{(1+2\lambda-2\gamma+4\lambda\gamma)m}{1-m}+m+(\lambda-\gamma+5\lambda\gamma)\frac{2m^2}{(1-m)^2}+\frac{6\lambda\gamma m^3}{(1-m)^3}\right] \\
&\leq (D-C)\cos\alpha.
\end{aligned}$$

This means that (3.5) holds, and hence $N(m,\lambda,\gamma,z) \in \mathcal{L}(C,D,\alpha)$. □

Theorem 9. *(i) If the condition*

$$\frac{2m^2}{(1-m)^2}+\frac{3m}{1-m}+m \leq \frac{(D-C)\cos\alpha}{1+|D|} \tag{3.6}$$

holds, then the operator P_m defined by (1.7) maps the class S^* to the class $\mathcal{L}(C,D,\alpha)$, that is $P_m(S^*) \subset \mathcal{L}(C,D,\alpha)$.

(ii) If the condition (3.1) is satisfied, then the operator P_m maps the class $\mathcal{K}$ to the class $\mathcal{L}(C,D,\alpha)$, that is $P_m(\mathcal{K}) \subset \mathcal{L}(C,D,\alpha)$.

Proof. According to Lemma 2, to prove that $P_m(f) \in \mathcal{L}(C,D,\alpha)$, it is sufficient to show that

$$\sum_{n=2}^{\infty} n(1+|D|)|m|^{n-1}|1-m||a_n| \leq (D-C)\cos\alpha \tag{3.7}$$

(i) If $f \in S^*$ has the form (1.1), then the well-known inequality $|a_n| \leq n$ holds for all $n \geq 2$ ([11, 13]) and using (3.6) we obtain that

$$\begin{aligned}
&\sum_{n=2}^{\infty} n(1+|D|)m^{n-1}(1-m)|a_n| \\
&\leq \sum_{n=2}^{\infty} n^2(1+|D|)m^{n-1}(1-m) \\
&= \sum_{n=2}^{\infty} \left((n-1)(n-2)+3(n-1)+1\right)(1+|D|)m^{n-1}(1-m) \\
&= (1-m)(1+|D|)\left[\sum_{n=2}^{\infty}(n-1)(n-2)m^{n-1} + 3\sum_{n=2}^{\infty}(n-1)m^{n-1} + \sum_{n=2}^{\infty} m^{n-1}\right] \\
&= (1-m)(1+|D|)\left[\frac{2m^2}{(1-m)^3} + \frac{3m}{(1-m)^2} + \frac{m}{1-m}\right] \\
&= (1+|D|)\left[\frac{2m^2}{(1-m)^2} + \frac{3m}{1-m} + m\right] \\
&\leq (D-C)\cos\alpha
\end{aligned}$$

that is (3.7) holds, and hence $P_m(f)(z) \in \mathcal{L}(C, D, \alpha)$.

In the proof of upper results we used the following relations:

$$\begin{aligned}
&\sum_{n=2}^{\infty}(n-1)m^{n-1} = \frac{m}{(1-m)^2}, \\
&\sum_{n=2}^{\infty}(n-1)(n-2)m^{n-1} = \frac{2m^2}{(1-m)^3}, \\
&\sum_{n=2}^{\infty} m^{n-1} = \frac{m}{1-m}.
\end{aligned}$$

(ii) If $f \in \mathcal{K}$ is of the form (1.1), then the well-known inequality $|a_n| \leq 1$ holds for

$n \geq 2$ ([11]) and according to (3.1) we have

$$\begin{aligned}
&\sum_{n=2}^{\infty} n(1+|D|)m^{n-1}(1-m)|a_n| \\
&\leq \sum_{n=2}^{\infty} n(1+|D|)m^{n-1}(1-m) \\
&= (1-m)(1+|D|)\sum_{n=2}^{\infty} n(m)^{n-1} \\
&= (1-m)(1+|D|)\frac{1-(1-m)^2}{(1-m)^2} \\
&= (1+|D|)\frac{1-(1-m)^2}{1-m} \\
&= (1+|D|)\left(\frac{m}{1-m}+m\right) \\
&\leq (D-C)\cos\alpha.
\end{aligned}$$

Hence (3.7) holds and therefore $P_m(f) \in \mathcal{L}(C, D, \alpha)$.

□

Theorem 10. *If the condition*

$$\frac{m}{1-m}(D-C)\cos\alpha \leq \frac{1}{q+2} \tag{3.8}$$

holds, then the operator P_m maps the class $\mathcal{L}(C, D, \alpha)$ to the class $q-ucv$, that is $P_m(\mathcal{L}(C, D, \alpha)) \subset q-ucv$.

Proof. If $f \in \mathcal{L}(C, D, \alpha)$ has the form (1.1), since P_m is given by (1.7), using Lemma 4 we need to prove that

$$\sum_{n=2}^{\infty} n(n-1)|m|^{n-1}|1-m||a_n| \leq \frac{1}{q+2}. \tag{3.9}$$

In the proof of this theorem, we will use the following relation:

$$\sum_{n=2}^{\infty}(n-1)m^{n-1} = \frac{m}{(1-m)^2}.$$

We have

$$\begin{aligned}&\sum_{n=2}^{\infty} n(n-1)m^{n-1}(1-m)|a_n|\\&\leq \sum_{n=2}^{\infty}(n-1)m^{n-1}(1-m)(D-C)\cos\alpha\\&=(1-m)(D-C)\cos\alpha\sum_{n=2}^{\infty}(n-1)m^{n-1}\\&=(1-m)(D-C)\cos\alpha\frac{m}{(1-m)^2}\\&=\left(\frac{m}{1-m}\right)(D-C)\cos\alpha\\&\leq \frac{1}{q+2}.\end{aligned}$$

Hence (3.9) holds and consequently $P_m(f) \in q - ucv$. □

Theorem 11. *If the condition*

$$(D-C)\cos\alpha\left[m(q+1)-\frac{q}{m}(1-m)(-m-\ln(1-m)\right]\leq 1 \tag{3.10}$$

holds, then P_m maps the class $\mathcal{L}(C, D, \alpha)$ to the class $q - s_p := q - s_p(0)$, that is $P_m(\mathcal{L}(C, D, \alpha)) \subset q - s_p$.

Proof. If $f \in \mathcal{L}(C, D, \alpha)$ has the power expansion series (1.1) and P_m is given by (1.7), according to Lemma 5 for $\Gamma = 0$ we need to prove that

$$\sum_{n=2}^{\infty}[n(q+1)-q]\,|m|^{n-1}|1-m||a_n|\leq 1. \tag{3.11}$$

In the proof of this theorem, we will use the following relations:

$$\sum_{n=2}^{\infty} m^{n-1}=\frac{m}{1-m},\qquad \sum_{n=2}^{\infty}\frac{m^n}{n}=(-\ln(1-m)-m]\,.$$

We have

$$\begin{aligned}
&\sum_{n=2}^{\infty}[n(q+1)-q]\,|m|^{n-1}|1-m||a_n| \\
&\leq \sum_{n=2}^{\infty}[n(q+1)-q]\left(\frac{(D-C)\cos\alpha}{n}\right)m^{n-1}(1-m) \\
&= (D-C)\cos\alpha\left[(1-m)(q+1)\sum_{n=2}^{\infty}m^{n-1}-q(1-m)\sum_{n=2}^{\infty}\frac{m^{n-1}}{n}\right] \\
&= (D-C)\cos\alpha\left[(1-m)(q+1)\frac{m}{1-m}-\frac{q}{m}(1-m)(-m-\ln(1-m))\right] \\
&= (D-C)\cos\alpha\left[m(q+1)-\frac{q}{m}(1-m)(-m-\ln(1-m))\right] \\
&\leq 1.
\end{aligned}$$

That is (3.11) holds and thus $P_m(f) \in q - s_p$.

In the following two theorems we will obtain analogues results in connection with the function I_m defined by

$$I_m(z) := \int_0^z \frac{K(m,t)}{t}dt, \qquad z \in U.$$

□

Theorem 12. *A sufficient condition for the function I_m to be in the class $\mathcal{L}(C, D, \alpha)$ is*

$$m(1+|D|) \leq (D-C)\cos\alpha. \tag{3.12}$$

Proof. Since

$$I_m(z) = z + \sum_{n=2}^{\infty} m^{n-1}(1-m)\frac{z^n}{n}, \qquad z \in U \tag{3.13}$$

to prove that $I_m \in \mathcal{L}(C, D, \alpha)$, according to Lemma 2, it is sufficient to show that

$$\sum_{n=2}^{\infty}\frac{n(1+D)}{n}|m|^{n-1}|1-m| \leq (D-C)\cos\alpha. \tag{3.14}$$

We have

$$\begin{aligned}&\sum_{n=2}^{\infty}\frac{n(1+|D|)}{n}m^{n-1}(1-m)\\&=(1-m)(1+|D|)\sum_{n=2}^{\infty}m^{n-1}\\&=(1-m)(1+|D|)\frac{m}{1-m}\\&=m(1+|D|)\leq(D-C)\cos\alpha.\end{aligned}$$

Now from (3.14) it follows that

$$I_m\in\mathcal{L}(C,D,\alpha).$$

□

Theorem 13. *A sufficient condition for the function I_m to be in the class $q-s_p(\Gamma)$ is*

$$m(q+1)-\frac{(1-m)}{m}(q+\Gamma)(-m-\ln(1-m))\leq 1-\Gamma. \tag{3.15}$$

Proof. Since I_m has the form (3.13), to prove that $I_m\in q-s_p(\Gamma)$, according to Lemma 5, it is sufficient to prove that

$$\sum_{n=2}^{\infty}\frac{n(q+1)-(q+\Gamma)}{n}|m|^{n-1}|1-m|\leq 1-\Gamma. \tag{3.16}$$

In the proof of this theorem we will use the following relations

$$\begin{aligned}\sum_{n=2}^{\infty}m^{n-1}&=\frac{m}{1-m},\\\sum_{n=2}^{\infty}\frac{m^n}{n}&=-\ln(1-m)-m.\end{aligned}$$

We have

$$\begin{aligned}&\sum_{n=2}^{\infty}\frac{n(q+1)-(q+\Gamma)}{n}m^{n-1}(1-m)\\&=(1-m)(q+1)\sum_{n=2}^{\infty}m^{n-1}-\frac{(1-m)}{m}(q+\Gamma)\sum_{n=2}^{\infty}\frac{m^n}{n}\\&=(1-m)(q+1)\frac{m}{1-m}-\frac{(1-m)}{m}(q+\Gamma)(-\ln(1-m)-m)\\&=m(q+1)-\frac{(1-m)}{m}(q+\Gamma)(-\ln(1-m)-m)\\&\leq 1-\Gamma.\end{aligned}$$

Now from (3.16), we conclude that

$$I_m \in q - s_p(\Gamma).$$

□

References

[1] M. K. Aouf, *On certain subclass of analytic p-valent functions of order alpha*, Rend. Math. Appl. *(7)8*(1988), 89–104. Zbl 0686.30012. MR 0986230.

[2] R. Bharati, R. Parvatham and A. Swaminathan, *On subclasses of uniformly convex functions and corresponding class of starlike functions*, Tamkang J. Math. **28**(1997), 17–32. Zbl 0898.30010. MR 1457247.

[3] T. Bulboaca, *Differential subordinations and superordinations: Recent results*, House of Scientific Book Publ, Cluj-Napoca, 2005.

[4] T. R. Caplinger and W. M. Causey, *A class of univalent functions*, Proc. Amer. Math. Soc. **39**(1973), 357–361. MR0320294. Zbl 0267.30010.

[5] A. W. Goodman, *On uniformly convex functions*, Ann. Polon. Math. **56(1)**(1991), 87–92. MR1097287. Zbl 0744.30010.

[6] A. W. Goodman, *On uniformly starlike functions*, J. Math. Anal. Appl. **155**(1991), 364–370. MR1097287. Zbl 0726.30013.

[7] O. P. Juneja and M. L. Mogra, *A class of univalent0419.30014 functions*, Bull. Sci. Math. **103(4)**(1979), 435–447. MR0548919. Zbl 0419.30014.

[8] S. Kanas and H. M. Srivastava, *Linear operators associated with K-uniformly convex functions*, Integral Transforms Spec. Funct. **9(2)**(2000), 121–132. MR1784495. Zbl 0959.30007.

[9] S. Kanas and A. Wisniowska, *Conic regions and K-uniform convexity*, J. Comput. Appl, Math. **105(1-2)**(1999), 327–336. MR1690599. Zbl 0944.30008.

[10] S. Kanas and A. Wisniowska, *Conic domains and starlike functions*, Rev. Roumaine Math. Pures Appl. **45**(2000), 647–657. MR1836295. Zbl 0990.30010.

[11] K. Löwner, *Untersuchungen über die verzerrung bei Kanformen Abbidungen des Einheitskreises* $|z| < 1$, die durch Funktionen mit nichtverschwindender Ableitung geliefert werden, Leipzig Ber. **69**(1917), 89–106.

[12] S. S. Miller and P. T. Mocanu, *Differential subordination: Theory and applications*, Monographs and Textbooks in Pure and Applied Mathematics 225, Marcel Dekker Inc., New York and Basel, 2000. MR1760285.

[13] R. H. Nevanlinna, *Über die konforme Abbildung von sterngebieten*, Helsingfors Centraltr, 1921.

[14] F. Rϕnning, *On starlike functions associated with parabolic regions*, Ann. Univ. Mariae Curies-Sklodowska Sect. A. **45**(1991), 117–122. MR1322145. Zbl 0769.30011.

[15] F. Ronning, *Uniformly convex functions and a corresponding class of starlike functions*, Proc. Amer. Soc. **118**(1993), 189–196. MR1128729. Zbl 0805.30012.

[16] S. Shams, S. R. Kulkarni and J. M. Jahangiri, *Classes of uniformly starlike and convex functions*, Int. J. Math. Sci. **55**(2004), 2959–2961. MR2145534. Zbl 1067.30033.

[17] L. Shukla and Dashrath, *On a class of univalent functions*, Soochow J. Math. **10**(1984), 117–126. MR0801679. Zbl 0578.30008.

M. Taliyan
Department of Mathematics
Faculty of Sciences, University of Maragheh, 55181-83111, Golshahr, Maragheh, Iran
e-mail: taliyan.ma2018@gmail.com

Sh. Najafzadeh
Department of Mathematics
Payame Noor University, P. O. Box 19395-3697, Tehran, Iran
e-mail: najafzadeh1234@yahoo.ie, shnajafzadeh44@pnu.ac.ir

M. R. Azimi
Department of Mathematics
Faculty of Sciences, University of Maragheh, 55181-83111, Golshahr, Maragheh, Iran
e-mail: mhreza.azimi@gmail.com

License

Received: August 21, 2023; Accepted: October 22, 2023; Published: October 29, 2023.