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NEW DYNAMIC FIXED POINT RESULTS IN MENGER SPACES

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Abstract. The objective of this paper is to generalize and improve some results in fixed point theorems in both complete metric space and Menger space. These results are generalizations of the analogous ones recently proved by Khojasteh [5], Demma [1], Yildirim [13], where we establish a dynamic information about their other fixed points if there exist i.e the distance between two fixed point in case of metric space and their equivalent in probabilistic metric space. Some illustrative examples are furnished, which demonstrate the validity of the hypotheses.

As an application to our main result, we derive a uniqueness fixed point theorem for a self-mapping under strong conditions.

1 Introduction

One of fundamental theorem in fixed point theory is Banach's contraction principle which has many applications in different disciplines, such as computer science, chemistry, biology, physics and any other branches of mathematics.

In the other way, Menger [7] introduced a new generalisation of the metric space, called Menger space. The important development of fixed point theory in this spaces was due to Sehgal and Bharucha-Reid [11] and Schweizer-Sklar [10]. Since then there has been a massive growth of fixed point theorems using certain conditions on the mappings or on the space itself. For some results of fixed and common fixed point in the setting of Menger spaces see [12, 3, 4, 9] and their references.

In this paper we aims to generalized the theorem in [5] in the both metric and Menger spaces and we give some examples to illustrate our studies. Our paper is organized as follows:

The first section, Preliminaries, we recall some basic notions of Menger space, in the second section we present our main results which divided into two parts: in the first one we generalize a theorem in [5] when we found a better estimation than their result and in second part we try to generalize this last result and our previous

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results in [6] to the Menger space case. Some illustrative examples are presented too.

2 Preliminaries

We recall some well known notions and definition of this spaces.

Definition 1. *[2]*
A distribution function (on $[-\infty, +\infty]$) is a function $F : [-\infty, +\infty] \to [0, 1]$ which is left-continuous on $\mathbb{R}$, non-decreasing and $F(-\infty) = 0$, $F(+\infty) = 1$.
We denote by Δ the family of all distribution functions on $[-\infty, +\infty]$.

Definition 2. *[2]*
A distance distribution function $F : [-\infty, +\infty] \to [0, 1]$ is a distribution function with support contained in $[0, +\infty]$.
The family of all distance distribution functions will be denoted by Δ^+. We denote $\mathcal{D}^+ = \{F | F \in \Delta^+, \lim_{t\to+\infty} F(t) = 1\}$.
Since any function from Δ^+ is equal 0 on $[-\infty, 0]$ we can consider the set Δ^+ consisting of non-decreasing functions F defined on $[0, +\infty]$ that satisfy $F(0) = 0$ and $F(+\infty) = 1$.
Moreover, $\mathcal{D}^+$ then consists of non-decreasing functions F defined on $[0, +\infty)$ that satisfy $F(0) = 0$ and $\lim_{t\to+\infty} F(T) = 1$.
The class $\mathcal{D}^+$ will play the important role in the probabilistic fixed point theorems. H is a special element of $\mathcal{D}^+$ defined by

$$H(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ 1 & \text{if } t > 0. \end{cases}$$

If X is a non-empty set, $\mathcal{F} : X \times X \to \mathcal{D}^+$ is called a probabilistic distance on X and $F(x, y)$ is usually denoted by $F_{x,y}$.

Definition 3. *[11] A probabilistic metric space (PM-space) is a pair $(X, \mathcal{F})$ where X is a set and $\mathcal{F}$ is a function defined on $X \times X$ into the set of distribution functions F such that for all x, y and z in X, for all $s, t > 0$*

1. *$F_{x,y}(0) = 0$;*
2. *$F_{x,y}(t) = H(t)$ iff $x = y$;*
3. *$F_{x,y}(t) = F_{y,x}(t)$;*
4. *If $F_{x,y}(t) = 1$ and $F_{y,z}(s) = 1$ then $F_{x,z}(s + t) = 1$.*

Remark 4. Let (X, d) be a metric space. The distribution function $F_{x,y}$ defined by the relation $F_{x,y}(t) = H(t - d(x, y))$ induces a PM-space.

Definition 5. *[2]*

A triangular norm $*$ (t-norm for short) is a binary operation on the unit interval $[0,1]$, which is commutative, associative, non-decreasing in its second component and for all $x \in [0,1]$ $x * 1 = x$.

Remark 6. The monotonicity of a t-norm $*$ in its second component is, together with the commutativity, equivalent to the (joint) monotonicity in both components, i.e., to

$$x_1 * y_1 \le x_2 * y_2 \text{ whenever } x_1 \le x_2 \text{ and } y_1 \le y_2.$$

Definition 7. *[11] A Menger PM-space is a triplet $(X, \mathcal{F}, *)$ where $(X, \mathcal{F})$ is a PM-space and $*$ is a t -norm with the following condition:*

$$F_{x,z}(t+s) \ge F_{x,y}(t) * F_{y,z}(s)$$

for all $x, y, z \in X$ and $s, t > 0$.

This inequality is known as Menger's triangle inequality.

Definition 8. *[8] A sequence $(x_n)_{n\in\mathbb{N}}$ in a Menger PM space $(X, \mathcal{F}, *)$ is said*

1. *to converge to a point x in X if for every $\epsilon > 0$ and $\lambda > 0$, there is an integer n_0 such that $F_{x_n,x}(\epsilon) > 1 - \lambda$ for all $n \ge n_0$.*
2. *to be Cauchy if for each $\epsilon > 0$ and $\lambda > 0$, there is an integer n_0 such that $F_{x_n,x_m}(\epsilon) > 1 - \lambda$ for all $n, m \ge n_0$.*
3. *to be complete if every Cauchy sequence in it converges to a point of it.*

We present the following famous lemma that will help us prove our results later.

3 Main results

The following is the first our main result.

Theorem 9. *Let (X, d) be a complete metric space and let T be a self mapping in X. Suppose there exist four positive real number a, b, c, f, $e \in \mathbb{R}^+$ such that $a \le \min\{c, f\}$ and for all $x, y \in X$*

$$d(Tx, Ty) \le \frac{ad(x, Ty) + bd(y, Tx)}{cd(x, Tx) + fd(y, Ty) + e} d(x, y). \tag{3.1}$$

Then,

1. *T has at least one fixed point $\dot{x} \in X$;*
2. *every Picard sequence $(x_n)_{n\in\mathbb{N}}$ converges to a fixed point;*

3. if T has two distinct fixed points $\dot{x}, \dot{y}$ in X then, $d(\dot{x}, \dot{y}) \geq \dfrac{e}{a+b}$.

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a Picard sequence($x_{n+1} = Tx_n$).

Step 1: Let's show that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

By putting $x = x_{n-1}$ and $y = x_n$ in inequality (3.1) we find,

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \frac{ad(x_{n-1}, x_{n+1})}{cd(x_{n-1}, x_n) + fd(x_n, x_{n+1}) + e} d(x_{n-1}, x_n) \\ &\leq \frac{ad(x_{n-1}, x_n) + ad(x_n, x_{n+1})}{cd(x_{n-1}, x_n) + fd(x_n, x_{n+1}) + e} d(x_{n-1}, x_n) \\ &\leq \frac{ad(x_{n-1}, x_n) + ad(x_n, x_{n+1})}{\min\{c; f\}\big(d(x_{n-1}, x_n) + d(x_n, x_{n+1})\big) + e} d(x_{n-1}, x_n) \end{aligned}$$

We denote that $\theta_n = \dfrac{ad(x_{n-1}, x_n) + ad(x_n, x_{n+1})}{\min\{c; f\}\big(d(x_{n-1}, x_n) + d(x_n, x_{n+1})\big) + e}$ for all $n \in \mathbb{N}$.

Since $a \leq \min\{c, f\}$ then $0 \leq \theta_n < 1$, furthermore, the sequence $(\theta_n)_{n\in\mathbb{N}}$ is decreasing because for all $n \in \mathbb{N}$

$$\begin{aligned} \theta_{n+1} - \theta_n &= \frac{ae\big[d(x_{n+1}, x_{n+2}) - d(x_{n-1}, x_n)\big]}{\big[\min\{c; f\}(d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})) + e\big]} \\ &\quad \times \frac{1}{\big[\min\{c; f\}(d(x_{n-1}, x_n) + d(x_n, x_{n+1})) + e\big]} \\ &< 0. \end{aligned}$$

On the other hand we have,

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \theta_n d(x_{n-1}, x_n) \\ &\leq \theta_n \theta_{n-1} d(x_{n-2}, x_{n-1}) \\ &\vdots \\ &\leq \theta_n \theta_{n-1} \cdots \theta_1 d(x_0, x_1) \\ &\leq \theta_1^n d(x_0, x_1). \end{aligned}$$

By passaging to the limit with $n \to +\infty$, we find

$$\lim_{n\to+\infty} d(x_n, x_{n+1}) = 0.$$

Now for all $n, m \in \mathbb{N}$ such that $m > n$ we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1,m}) \\ &\leq \sum_{k=n}^{m-1} (\theta_k \theta_{k-1} \cdots \theta_1) d(x_0, x_1) \end{aligned}$$

Suppose that $u_k = \theta_k\theta_{k-1}\cdots\theta_1$. Since

$$\lim_{n\to+\infty} \frac{u_{k+1}}{u_k} = 0 \tag{3.2}$$

thus $\sum_{k=1}^{+\infty} u_k$ is convergent. It means that,

$$\lim_{n,m\to+\infty} \sum_{k=n}^{m-1} \theta_k\theta_{k-1}\cdots\theta_1 = 0 \tag{3.3}$$

then $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X.
Since the metric space (X,d) is complete, there exit $\dot{x}\in X$ such that

$$\lim_{n\to+\infty} x_n = \dot{x}.$$

Step 2: Check that $\dot{x}$ is a fixed point of T.
By putting $x=\dot{x}$, $y=x_n$ in inequality (3.1), we find,

$$d(T\dot{x},x_{n+1}) \le \frac{ad(\dot{x},x_{n+1})+bd(x_n,T\dot{x})}{cd(\dot{x},T\dot{x})+fd(x_n,x_{n+1})+e}d(\dot{x},x_n). \tag{3.4}$$

By taking limit on both sides of (3.4), we have $d(T\dot{x},\dot{x})=0$ that mean $T\dot{x}=\dot{x}$.
Step 3: Suppose that T have two fixed points $\dot{x},\dot{y}$ in X. Find the distance between these two fixed points.

By putting $x=\dot{x}$, $y=\dot{y}$ in inequality (3.1), we find,

$$\begin{aligned} d(\dot{x},\dot{y}) &\le \frac{ad(\dot{x},\dot{y})+bd(\dot{y},\dot{x})}{cd(\dot{x},\dot{x})+fd(\dot{y},\dot{y})+e}d(\dot{x},\dot{y}) \\ &\le \frac{(a+b)d(\dot{x},\dot{y})}{e}d(\dot{x},\dot{y}). \end{aligned}$$

Then, $d(\dot{x},\dot{y}) \ge \frac{e}{a+b}$. □

Example 10. Let $X=\{0;1;2\}$ and let $d: X\times X \longrightarrow [0;+\infty)$ be defined by

$$d(0;1)=2; \quad d(1;2)=3; \quad d(0;2)=3.5.$$

$$d(0;0)=d(1;1)=d(2;2)=0$$

$$d(x;y)=d(y;x); \forall x,y\in X.$$

(X,d) is a complete metric space. Let $T: X\longrightarrow X$ be a self mapping defined by

$$T(0)=0, \quad T(1)=1, \quad T(2)=0.$$

We choose $a = 1$, $b = 1$, $c = 1$, $f = 1$, $e = 4$, then, we have

$$d(T(0), T(1)) = 2 \leq \frac{d(0, T(1)) + d(1, T(0))}{d(0, T(0)) + d(1, T(1)) + 4} d(0, 1) = 2,$$

$$d(T(1), T(2)) = 2 \leq \frac{d(1, T(2)) + d(2, T(1))}{d(1, T(1)) + d(2, T(2)) + 4} d(1, 2) = 2,$$

$$d(T(0), T(2)) = 0 \leq \frac{d(0, T(2)) + d(2, T(0))}{d(0, T(0)) + d(2, T(2)) + 4} d(0, 2) = \frac{49}{30}.$$

Therefore, T satisfies all the conditions of Theorem 9. Also, T has two distinct fixed point $\{0, 1\}$ and $d(0; 1) = 2 \geq \frac{e}{a+b} = 2$.

Remark 11. On remark that this estimation is the better one this due to the choosing of the constant a, b, c, f, e.

In the following, we try to generalised our last theorem for the case of Menger spaces. Then, we are looking for dynamic information about the set of fixed points of such mapping.

Theorem 12. *Let $(X, F, \min)$ be a complete Menger space. Let T be a self mapping in X.*
If exists four positives real numbers a, b, c, d (only one of a or b can be null) such that $\max\left\{\frac{a+b}{c+d}; \frac{b}{d}\right\} < 1$ and for all $x, y \in X$, $t > 0$,

$$1 - F_{Tx,Ty}(t) \leq \frac{a \min\{F_{x,Tx}(\frac{t}{2}); F_{y,Ty}(\frac{t}{2})\} + b}{c \min\{F_{x,Ty}(t); F_{y,Tx}(t)\} + d}(1 - F_{x,y}(t)). \tag{3.5}$$

Then,

1. *T has at least a fixed point in X;*
2. *all Picard sequence $(Tx_n)_{n\in\mathbb{N}}$ converges to a fixed point;*
3. *if T has two fixed points $\dot{x}, \dot{y}$ in X then, for all $t > 0$;*

$$F_{\dot{x},\dot{y}}(t) \leq \max\{\frac{a+b-d}{c}; 0\} \quad or \quad F_{\dot{x},\dot{y}}(t) = 1.$$

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a Picard sequence with an arbitrary $x_0 \in X$ and $x_{n+1} = Tx_n$, $n \in \mathbb{N}$.

Step1: Let's show that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

By putting $x = x_{n-1}$, $y = x_n$ in inequality (3.5), we find,

$$\begin{aligned}
1 - F_{Tx_{n-1},Tx_n}(t) &\leq \frac{a\min\{F_{x_{n-1},x_n}(\frac{t}{2}); F_{x_n,x_{n+1}}(\frac{t}{2})\} + b}{c\min\{F_{x_{n-1},x_{n+1}}(t); F_{x_n,x_n}(t)\} + d}(1 - F_{x_{n-1},x_n}(t)) \\
&\leq \frac{aF_{x_{n-1},x_{n+1}}(t) + b}{cF_{x_{n-1},x_{n+1}}(t) + d}(1 - F_{x_{n-1},x_n}(t)) \\
&\leq \max\left\{\frac{a+b}{c+d}; \frac{b}{d}\right\}(1 - F_{x_{n-1},x_n}(t)) \\
&\;\vdots \\
&\leq \left(\max\left\{\frac{a+b}{c+d}; \frac{b}{d}\right\}\right)^n (1 - F_{x_0,x_1}(t)).
\end{aligned}$$

We denote $\theta_n(t) = \dfrac{aF_{x_{n-1},x_{n+1}}(t) + b}{cF_{x_{n-1},x_{n+1}}(t) + d}$.
Let $n, m \in \mathbb{N}$ such that $m \geq n$, we have,

$$F_{x_n,x_m}(t) \geq \min\{F_{x_n,x_{n+1}}(\frac{t}{m-n}); F_{x_{n+1},x_{n+2}}(\frac{t}{m-n}); \cdots; F_{x_{m-1},x_m}(\frac{t}{m-n})\}. \tag{3.6}$$

So,

$$\begin{aligned}
1 - F_{x_n,x_m}(t) &\leq \max\left\{\begin{array}{c} 1 - F_{x_n,x_{n+1}}(\frac{t}{m-n}); 1 - F_{x_{n+1},x_{n+2}}(\frac{t}{m-n}); \\ \cdots; 1 - F_{x_{m-1},x_m}(\frac{t}{m-n}) \end{array}\right\} \\
&\leq \max\left\{\begin{array}{c} \theta_n(\frac{t}{m-n})\theta_{n-1}(\frac{t}{m-n})\cdots\theta_1(\frac{t}{m-n}); \\ \theta_{n+1}(\frac{t}{m-n})\theta_n(\frac{t}{m-n})\cdots\theta_1(\frac{t}{m-n}); \\ \vdots \\ \theta_{m-1}(\frac{t}{m-n})\theta_{m-2}(\frac{t}{m-n})\cdots\theta_1(\frac{t}{m-n}) \end{array}\right\} \\
&\quad \times(1 - F_{x_0,x_1}(\frac{t}{m-n})) \\
&\leq \theta_n(\frac{t}{m-n})\theta_{n-1}(\frac{t}{m-n})\cdots\theta_1(\frac{t}{m-n})(1 - F_{x_0,x_1}(\frac{t}{m-n})) \\
&\leq \max\{\frac{a+b}{c+d}; \frac{b}{d}\}^n(1 - F_{x_0,x_1}(\frac{t}{m-n})) \\
&\leq \max\{\frac{a+b}{c+d}; \frac{b}{d}\}^n
\end{aligned}$$

Taking limit $n, m \to +\infty$ we deduce

$$\lim_{n,m\to+\infty} F_{x_n,x_m}(t) = 1 \quad \forall t > 0. \tag{3.7}$$

Then, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.
As the space X is complete, there exits $\dot{x} \in X$ such that $\lim\limits_{n\to+\infty} x_n = \dot{x}$.

Step 2: Check that $\dot{x}$ is a fixed point of T.
By putting $x = \dot{x}$, $y = x_n$ in inequality (3.5), we find,

$$1 - F_{T\dot{x},Tx_n}(t) \leq \frac{a\min\{F_{\dot{x},T\dot{x}}(\frac{t}{2}); F_{x_n,x_{n+1}}(\frac{t}{2})\} + b}{c\min\{F_{\dot{x},x_{n+1}}(t); F_{x_n,\dot{x}}(t)\} + d}(1 - F_{\dot{x},x_n}(t))$$

Passing to the limit $n \longrightarrow +\infty$,

$$1 - F_{T\dot{x},\dot{x}}(t) \leq \frac{aF_{\dot{x},T\dot{x}}(\frac{t}{2}) + b}{cF_{\dot{x},\dot{x}}(t) + d}(1 - F_{\dot{x},\dot{x}}(t))$$

$$\lim_{n\to+\infty} F_{T\dot{x},\dot{x}}(t) = 1 \quad \forall t > 0.$$

Then, $T\dot{x} = \dot{x}$. So, T has at least one fixed point in X.
Step 3: Let $\dot{x}, \dot{y} \in X$ be two fixed point of T.
By putting $x = \dot{x}$, $y = \dot{y}$ in inequality (3.5), we find,

$$1 - F_{T\dot{x},T\dot{y}}(t) \leq \frac{a + b}{cF_{\dot{x},\dot{y}}(t) + d}(1 - F_{\dot{x},\dot{y}}(t))$$

then,

$$cF^2_{\dot{x},\dot{y}}(t) + (d - c - a - b)F_{\dot{x},\dot{y}}(t) + a + b - d \geq 0 \quad \text{for all } t > 0,$$

then

$$F_{\dot{x},\dot{y}}(t) \leq \max\{\frac{a + b - d}{c}; 0\} \quad \text{or} \quad F_{\dot{x},\dot{y}}(t) = 1.$$

□

Theorem 13. *Let $(X, F, \min)$ be a complete Menger space. Let T be a self mapping in X.*
If exists four positives real numbers a, b, c, d (only one of a or b can be null) such that $a + b < d$ and satisfying the inequality (3.5) then T has a unique fixed point in X.

Proof. The demonstration of the existence of fixed point is mentioned in the lines of proof of theorem 12.

Next we will show the uniqueness of the fixed point of T.

Let $\dot{x}, \dot{y} \in X$ two fixed point of T. Replacing $x = \dot{x}$, $y = \dot{y}$ in inequality (3.5) we find

$$1 - F_{T\dot{x},T\dot{y}}(t) \leq \frac{a + b}{cF_{\dot{x},\dot{y}}(t) + d}(1 - F_{\dot{x},\dot{y}}(t)).$$

Then,

$$cF^2_{\dot{x},\dot{y}}(t) + (d - c - a - b)F_{\dot{x},\dot{y}}(t) + a + b - d \geq 0 \quad \text{for all } t > 0,$$

which means $F_{\dot{x},\dot{y}}(t) = 1$ for all $t > 0$. Then $\dot{x} = \dot{y}$. □

Example 14. Let $X = \{0, 1, 2\}$ be a set associated with the following distributions functions:

$$F_{0,1}(t) = F_{1,2}(t) = \begin{cases} 0 & \text{if } t \leq 2, \\ 1 & \text{if } t > 2. \end{cases}$$

$$F_{0,2}(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ \frac{1}{2} & \text{if } 0 < t \leq 3, \\ 1 & \text{if } t > 3. \end{cases}$$

$$F_{x,x}(t) = H(t) \quad \text{for all}\, x \in X \text{for all} t \in \mathbb{R},$$

$$F_{x,y}(t) = F_{y,x}(t) \quad \text{for all}\, x, y \in X \quad \text{for all}\, t \in \mathbb{R}.$$

$(X, \mathcal{F}, \min)$ is a Menger space because all conditions in definition 3 and definition 7 are satisfies. Conditions 1-3 of definition 3 are obviously verified.

Now, it remains to prove the fourth condition:

- Suppose that $F_{0,1}(t) = 1$ and $F_{0,2}(t) = 1$ then, $t > 2$ and $s > 3$ then, $t + s > 5$, then $F_{1,2}(t + s) = 1$.
- Suppose that $F_{0,2}(t) = 1$ and $F_{1,2}(t) = 1$ then, $t > 3$ and $s > 2$ then, $t + s > 5$, then $F_{0,1}(t + s) = 1$.
- Suppose that $F_{0,1}(t) = 1$ and $F_{1,2}(t) = 1$ then, $t > 2$ and $s > 2$ then, $t + s > 4$, then $F_{0,2}(t + s) = 1$.

Also, the Menger's triangle inequality is satisfied because for all $t, s > 0$,

$$F_{0,2}(t + s) \geq \min\{F_{0,1}(t); F_{1,2}(s)\} \tag{3.8}$$

$$F_{0,1}(t + s) \geq \min\{F_{0,2}(t); F_{1,2}(s)\} \tag{3.9}$$

$$F_{1,2}(t + s) \geq \min\{F_{0,1}(t); F_{0,2}(s)\} \tag{3.10}$$

In addition, $(X, \mathcal{F}, \min)$ is a complete Menger space. Let T be a self mapping defined on X as,

$$T(0) = 0 \quad T(1) = 1 \quad T(2) = 0$$

Now, we check the validity of the inequality (3.5). By choosing $a = c = d = 1$ and $b = \frac{1}{2}$, it is clear that $\dfrac{a + b}{c + d} < 1$ and $\dfrac{b}{d} < 1$. If $x = y$, the inequality (3.5) was verified.

For $x = 0$ and $y = 1$ we have

$$1 - F_{T(0),T(1)}(t) = 1 - F_{0,1}(t) = \begin{cases} 1 & \text{if } t \leq 2, \\ 0 & \text{if } t > 2. \end{cases}$$

$$\begin{aligned}\frac{a\min\{F_{0,T(0)}(\frac{t}{2});F_{1,T(1)}(\frac{t}{2})\}+b}{c\min\{F_{0,T(1)}(t);F_{1,T(0)}(t)\}+d}(1-F_{0,1}(t)) &= \frac{1.5}{F_{0,1}(t)+1}\\ &= \begin{cases}1.5 & \text{if } t\leq 2,\\ 0.75 & \text{if } t>2.\end{cases}\end{aligned}$$

So,

$$1-F_{0,1}(t)\leq \frac{1.5}{F_{0,1}(t)+1} \quad \text{for all}\, t>0$$

For $x=0$ and $y=2$,

$$\begin{aligned}1-F_{T(0),T(2)}(t) &= 1-F_{0,0}(t)\\ &= 0\leq \frac{\min\{F_{0,T(0)}(\frac{t}{2});F_{2,T(2)}(\frac{t}{2})\}+\frac{1}{2}}{\min\{F_{0,T(2)}(t);F_{2,T(0)}(t)\}+1}(1-F_{0,2}(t)).\end{aligned}$$

For $x=1$ and $y=2$,

$$1-F_{T(1),T(2)}(t)=1-F_{1,0}(t)=\begin{cases}1 & \text{if } t\leq 2,\\ 0 & \text{if } t>2.\end{cases}$$

$$\begin{aligned}\frac{a\min\{F_{1,T(1)}(\frac{t}{2});F_{2,T(2)}(\frac{t}{2})\}+b}{c\min\{F_{1,T(2)}(t);F_{2,T(1)}(t)\}+d}(1-F_{1,2}(t)) &= \frac{F_{0,2}(\frac{t}{2})+\frac{1}{2}}{F_{1,2}(t)+1}(1-F_{1,2}(t))\\ &= \begin{cases}\frac{1}{2} & \text{if } t\leq 0,\\ 1 & \text{if } 0<t\leq 2,\\ 0 & \text{if } t>2.\end{cases}\end{aligned}$$

So,

$$1-F_{1,0}(t)\leq \frac{F_{0,2}(\frac{t}{2})+\frac{1}{2}}{F_{1,2}(t)+1}(1-F_{1,2}(t)) \quad \text{for all}\, t>0.$$

Then, all conditions of theorem 12 were verified. Then, T has at least one fixed point in X (exactly it has two fixed point 0 and 1). Over more, $F_{0,1}(t)\leq \frac{1}{2}$ or $F_{0,1}(t)=1$.

Also, every Picard sequence $(Tx_n)_{n\in\mathbb{N}}$ converge to a fixed point.
If $x_0=0$, $Tx_n=0$ for all $n\in\mathbb{N}$, it converge to 0.
If $x_0=1$, $Tx_n=1$ for all $n\in\mathbb{N}$, it converge to 1.
If $x_0=2$, $Tx_n=0$ for all $n\in\mathbb{N}$, it converge to 0.

Open Question. Very interesting results can be obtain in the same frame of Menger space, for the case of multivalued operators.

4 Conclusions

Our paper generalize and discuss some results of related literature, given by Khojasteh [5], Demma [1], Yildirim [13] for both, complete metric space and Menger space. We established a dynamic information about their other fixed points if there exist i.e. the distance between two fixed point in case of metric space and their equivalent in probabilistic metric space. To strength our results some interesting examples are given.

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