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FIXED POINTS IN BICOMPLEX VALUED S-METRIC SPACES WITH APPLICATIONS

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Abstract. This article introduces the idea of bicomplex valued S-metric space and deduces some of its features. Additionally, for bicomplex valued S-metric spaces, some fixed point results of contraction maps are shown to meet various categories of rational inequalities. Moreover, these results generalize certain significant, well-known results. An example is provided to highlight our major result. Furthermore, a theorem guaranteeing the existence of the one and only solution to the linear system of equations was developed using our main result.

1 Introduction

Assuming that $\mathbb{C}_1$ is the set of all complex numbers. Let $z_1, z_2 \in \mathbb{C}_1$ and define a partial order $\precsim$ on $\mathbb{C}_1$ as follows. $z_1 \precsim z_2$ if and only if(or, iff) $Re(z_1) \leq Re(z_2), Im(z_1) \leq Im(z_2)$. It follows that $z_1 \precsim z_2$ if one of the following axioms is fulfilled:

(I) $Re(z_1) = Re(z_2), Im(z_1) < Im(z_2)$,

(II) $Re(z_1) < Re(z_2), Im(z_1) = Im(z_2)$,

(III) $Re(z_1) < Re(z_2), Im(z_1) < Im(z_2)$,

(IV) $Re(z_1) = Re(z_2), Im(z_1) = Im(z_2)$.

In particular we will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (I),(II) and (III) is fulfilled, and we will write $z_1 \prec z_2$ if only (III) is fulfilled. Note that

$$0 \precsim z_1 \precnsim z_2 \Rightarrow |z_1| < |z_2|$$
$$z_1 \precsim z_2, z_2 \prec z_3 \Rightarrow z_1 \prec z_3.$$

Assuming that $\mathbb{C}_0$ and $\mathbb{C}_2$ are the set of all real and bicomplex numbers respectively. Bicomplex numbers are defined by C. Segre [13] as: $\tau = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2$, where

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$a_1, a_2, a_3, a_4 \in \mathbb{C}_0$, and the independent units i_1, i_2 are such that $i_i{}^2 = i_2{}^2 = -1$, and $i_1 i_2 = i_2 i_1$. We denote the set of bicomplex numbers $\mathbb{C}_2$ is defined as:
$\mathbb{C}_2 = \{\tau : \tau = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2, a_1, a_2, a_3, a_4 \in \mathbb{C}_0\}$,
i.e., $\mathbb{C}_2 = \{\tau : \tau = z_1 + i_2 z_2, z_1, z_2 \in \mathbb{C}_1\}$, where $z_1 = a_1 + a_2 i_1 \in \mathbb{C}_1$ and $z_2 = a_3 + a_4 i_1 \in \mathbb{C}_1$.

If $\tau = z_1 + i_2 z_2$, $\nu = w_1 + i_2 w_2 \in \mathbb{C}_2$, then the sum is $\tau \pm \nu = (z_1 + i_2 z_2) \pm (w_1 + i_2 w_2) = (z_1 \pm w_1) + i_2(z_2 \pm w_2)$ and the product is $\tau.\nu = (z_1 + i_2 z_2).(w_1 + i_2 w_2) = (z_1 w_1 - z_2 w_2) + i_2(z_1 w_2 + z_2 w_1)$.

An element $\nu = w_1 + i_2 w_2 \in \mathbb{C}_2$ is nonsingular iff $|w_1{}^2 + w_2{}^2| \neq 0$ and singular iff $|w_1{}^2 + w_2{}^2| = 0$. The inverse of ν is defined as $\nu^{-1} = \frac{w_1 - i_2 w_2}{w_1{}^2 + w_2{}^2}$.

A bicomplex number $\tau = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2 \in \mathbb{C}_2$ is said to be nonsingular if the matrix

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$$

is nonsingular. In that case τ^{-1} exists and it is also nonsingular.

The norm $||.||$ of $\mathbb{C}_2$ is a positive real valued function and $||.|| : \mathbb{C}_2 \to \mathbb{C}_0{}^+$ by $||\tau|| = ||z_1 + i_2 z_2|| = \{|z_1|^2 + |z_2|^2\}^{\frac{1}{2}} = (a_1{}^2 + a_2{}^2 + a_3{}^2 + a_4{}^2)^{\frac{1}{2}}$, where $\tau = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2 = z_1 + i_2 z_2 \in \mathbb{C}_2$.

Define a partial order $\precsim_{i_2}$ on $\mathbb{C}_2$ as follows. For $\tau = z_1 + i_2 z_2$ and $\nu = w_1 + i_2 w_2$ be any two bicomplex numbers. $\tau \precsim_{i_2} \nu$ iff $z_1 \precsim w_1$, and $z_2 \precsim w_2$. It follows that $\tau \precsim_{i_2} \nu$ if one of the following axioms is fulfilled:

(i) $z_1 = w_1$, $z_2 = w_2$,

(ii) $z_1 \prec w_1$, $z_2 = w_2$,

(iii) $z_1 = w_1$, $z_2 \prec w_2$,

(iv) $z_1 \prec w_1$, $z_2 \prec w_2$.

In particular we will write $\tau \precnsim_{i_2} \nu$ if $\tau \precsim_{i_2} \nu$ and $\tau \neq \nu$ and one of (ii),(iii), and (iv) is fulfilled, and we will write $\tau \prec \nu$ if only (iv) is fulfilled. Remember that

(I) $\tau \precsim_{i_2} \nu \Rightarrow ||\tau|| \leq ||\nu||$,

(II) $||\tau + \nu|| \leq ||\tau|| + ||\nu||$,

(III) $||a\tau|| = a||\tau||$, where a is a non negative real number,

(IV) $||\tau\nu|| \leq \sqrt{2}||\tau||||\nu||$, and the equality holds only when atleast one of τ and ν is nonsingular,

(V) $||\tau^{-1}|| = ||\tau||^{-1}$ if $\tau \in \mathbb{C}_2$ is a nonsingular with $0 \prec \tau$,

(VI) $||\frac{\tau}{\nu}|| = \frac{||\tau||}{||\nu||}$, if $\nu \in \mathbb{C}_2$ is a nonsingular.

Complex valued metric spaces were first discussed by A. Azam et al. in [1]. J. Choi et al. proposed the idea of bicomplex valued metric spaces in their paper [3]; various properties were derived, and common fixed point results for mappings meeting a rational inequality were demonstrated. See [2, 4, 5, 7, 14] for a list of recent publications on fixed point theory in bicomplex valued metric spaces.

Definition 1. *[1] Assuming that $G \neq \emptyset$ is a set. The mapping $d : G \times G \to \mathbb{C}_2$ is said to be a bicomplex valued metric if*

(i) $0 \precsim_{i_2} d(\beth, \aleph)$, $\forall \beth, \aleph \in G$,

(ii) $d(\beth, \aleph) = 0$ *iff* $\beth = \aleph$ *in* G,

(iii) $d(\beth, \aleph) = d(\aleph, \beth)$, $\forall \beth, \aleph \in G$,

(iv) $d(\beth, \aleph) \precsim_{i_2} d(\beth, \wp) + d(\wp, \aleph)$, $\forall \beth, \wp, \aleph \in G$.

The pair (G, d) is then referred to as a bicomplex valued metric space.

S. Sedghi et al. first established the idea of S-metric space in [12]. A S-metric is a real valued mapping on G^3, for some set $G \neq \emptyset$, where the map represents the perimeter of the triangle. See [6, 9, 10, 11, 15, 16] for a list of articles on fixed point theory in S-metric spaces.

Definition 2. *[12] Assuming that $G \neq \emptyset$ is a set. The mapping $S : G^3 \to [0, \infty)$ is said to be a S-metric if*

(1) $S(\beth, \aleph, \rho) \geq 0$, *for all* $\beth, \aleph, \rho \in G$ *and*

(2) $S(\beth, \aleph, \rho) = 0$ *iff* $\beth = \aleph = \rho$, *for all* $\beth, \aleph, \rho \in G$; *and*

(3) $S(\beth, \aleph, \rho) \leq S(\beth, \beth, \sigma) + S(\aleph, \aleph, \sigma) + S(\rho, \rho, \sigma)$, *for all* $\beth, \aleph, \rho, \sigma \in G$.

The pair (G, S) is then referred to as a S-metric space.

In order to get at common fixed point results, N. M. Mlaika [8] recently established the concept of complex valued S-metric space. In this research, we generalise the concept of complex valued metric space and S-metric space by extending the codomain of complex valued S-metric to bicomplex numbers. We also propose a new definition of bicomplex valued S-metric space. We also obtain certain S-metric space features with bicomplex values. Additionally, in bicomplex valued S-metric space, we demonstrate certain fixed point results for contraction maps meeting various kinds of rational inequalities. Additionally, by applying our primary conclusion, we show an existence theorem for the unique solution to the linear system of equations, generalise some significant and well-known results, and give an example to explain our major result.

2 Bicomplex Valued S-Metric Spaces

In order to get to conclusions about fixed point theory, this section introduces bicomplex valued S-metric space and some of its characteristics.

Definition 3. *Assuming that $G \neq \emptyset$ is a set. The mapping $S : G^3 \to \mathbb{C}_2$ is called a bicomplex valued S-metric if*

(1) $0 \precsim_{i_2} S(\daleth, \aleph, \rho)$, *for all* $\daleth, \aleph, \rho \in G$ *and*

(2) $S(\daleth, \aleph, \rho) = 0$ *iff* $\daleth = \aleph = \rho$, *for all* $\daleth, \aleph, \rho \in G$; *and*

(3) $S(\daleth, \aleph, \rho) \precsim_{i_2} S(\daleth, \daleth, \sigma) + S(\aleph, \aleph, \sigma) + S(\rho, \rho, \sigma)$, *for all* $\daleth, \aleph, \rho, \sigma \in G$.

The pair (G, S) is then referred to as a bicomplex valued S-metric space(or, BVSMS).

Example 4. *Assuming that $G = (0, \infty)$ and $S(\daleth, \aleph, \rho) = (1 + i_1 + i_2 + i_1 i_2)(|\daleth - \rho| + |\aleph - \rho|)$. Then (G, S) is a BVSMS.*

Definition 5. *Suppose (G, S) is a BVSMS. Assuming that $\{\daleth_n\}$ is a sequence in G.*
(i) $(\daleth_n)_{n=1}^{\infty}$ converges to a point $\daleth$(or $(\daleth_n)_{n=1}^{\infty} \to \daleth$) iff for every $c \in \mathbb{C}_2$ with $0 \prec_{i_2} c$, there exists $n_0 \in \mathbb{N}$(Natural numbers) such that $S(\daleth_n, \daleth_n, \daleth) \prec_{i_2} c$, $\forall\ n \geq n_0$.
(ii) $\{\daleth_n\}$ is called a Cauchy sequence, if for each $c \in \mathbb{C}_2$ with $0 \prec_{i_2} c$, there is an $n_0 \in \mathbb{N}$ such that $S(\daleth_n, \daleth_n, \daleth_m) \prec_{i_2} c$, $\forall\ n, m \geq n_0$.

Definition 6. *Assuming that (G, S) is a BVSMS. Then G is called complete if every Cauchy sequence is convergent in G.*

Lemma 7. *Assuming that (G, S) is a BVSMS. Then a sequence $(\daleth_n)_{n=1}^{\infty}$ converges to a point $\daleth$ iff $||S(\daleth_n, \daleth_n, \daleth)|| \to 0$ as $n \to \infty$.*

Proof. Assuming that $(\daleth_n)_{n=1}^{\infty}$ is a sequence, and $(\daleth_n)_{n=1}^{\infty} \to \daleth \in G$. For $\epsilon \in \mathbb{C}_0$ with $\epsilon > 0$, let $c = \frac{\epsilon}{2} + i_1 \frac{\epsilon}{2} + i_2 \frac{\epsilon}{2} + i_1 i_2 \frac{\epsilon}{2}$. For every $c \in \mathbb{C}_2$ with $0 \prec_{i_2} c$, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, $S(\daleth_n, \daleth_n, \daleth) \prec_{i_2} c$.

$$||S(\daleth_n, \daleth_n, \daleth)|| < ||c|| = \epsilon,\ \forall\ n \geq n_0.$$

It follows that $||S(\daleth_n, \daleth_n, \daleth)|| \to 0$ as $n \to \infty$.
Conversely, assuming that $||S(\daleth_n, \daleth_n, \daleth)|| \to 0$ as $n \to \infty$. Then given $c \in \mathbb{C}_2$ with $0 \prec_{i_2} c$, there exists $\delta \in \mathbb{C}_0$, with $\delta > 0$ such that for $z \in \mathbb{C}_2$

$$||z|| < \delta \Rightarrow z \prec_{i_2} c$$

For this δ, there exists $n_0 \in \mathbb{N}$ such that

$$||S(\daleth_n, \daleth_n, \daleth)|| < \delta,\ \forall\ n \geq n_0.$$

This means that $S(\daleth_n, \daleth_n, \daleth) \prec_{i_2} c, \forall n \geq n_0$. Hence $\daleth_n \to \daleth \in G$. □

Lemma 8. *Assuming that* (G, S) *is a BVSMS. If a sequence* $(\beth_n)_{n=1}^{\infty}$ *converges to* $\beth$ *and a sequence* $(\aleph_n)_{n=1}^{\infty}$ *converges to* $\aleph$*, then* $S(\beth_n, \beth_n, \aleph_n) \to S(\beth, \beth, \aleph)$ *as* $n \to \infty$*.*

Proof. Assuming that $(\beth_n)_{n=1}^{\infty} \to \beth \in H$, and $(\aleph_n)_{n=1}^{\infty} \to \aleph \in G$. For $\epsilon \in \mathbb{C}_0$ with $\epsilon > 0$, let $c = \frac{\epsilon}{2} + i_1\frac{\epsilon}{2} + i_2\frac{\epsilon}{2} + i_1 i_2\frac{\epsilon}{2}$. For each $c \in \mathbb{C}_2$ with $0 \prec_{i_2} c$, there exist $n_0, n_1 \in \mathbb{N}$ such that, $\forall\ n \geq n_0$, we have $S(\beth_n, \beth_n, \beth) \prec_{i_2} \frac{c}{4}$, and for all $n \geq n_1$, we have $S(\aleph_n, \aleph_n, \aleph) \prec_{i_2} \frac{c}{4}$, then for all $n \geq n_2 = \max\{n_0, n_1\}$,

$$\begin{aligned} S(\beth, \beth, \aleph) &\precsim_{i_2} 2S(\beth, \beth, \beth_n) + 2S(\aleph, \aleph, \aleph_n) + S(\aleph_n, \aleph_n, \beth_n) \\ &\prec c + S(\aleph_n, \aleph_n, \beth_n) \end{aligned}$$

implies

$$S(\beth, \beth, \aleph) - S(\beth_n, \beth_n, \aleph_n) \precsim_{i_2} c,$$

$$||S(\beth, \beth, \aleph) - S(\beth_n, \beth_n, \aleph_n)|| \leq ||c|| = \epsilon,$$

and hence $S(\beth_n, \beth_n, \aleph_n) \to S(\beth, \beth, \aleph)$ as $n \to \infty$. □

Lemma 9. *Assuming that* (G, S) *is a BVSMS. Then* $\{\beth_n\}$ *is a Cauchy sequence iff* $||S(\beth_n, \beth_n, \beth_{n+m})|| \to 0$ *as* $n \to \infty$*.*

Proof. Assuming that $\{\beth_n\}$ is a Cauchy sequence. For $\epsilon \in \mathbb{C}_0$ with $\epsilon > 0$, let $c = \frac{\epsilon}{2} + i_1\frac{\epsilon}{2} + i_2\frac{\epsilon}{2} + i_1 i_2\frac{\epsilon}{2}$. For every $c \in \mathbb{C}_2$ with $0 \prec_{i_2} c$, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, $S(\beth_n, \beth_n, \beth_{n+m}) \prec_{i_2} c$.

$$||S(\beth_n, \beth_n, \beth_{n+m})|| < ||c|| = \epsilon,\ \forall\ n \geq n_0.$$

It follows that $||S(\beth_n, \beth_n, \beth_{n+m})|| \to 0$ as $n \to \infty$. Conversely, assuming that $||S(\beth_n, \beth_n, \beth_{n+m})|| \to 0$ as $n \to \infty$. Then given $c \in \mathbb{C}_2$ with $0 \prec_{i_2} c$, there exists $\delta \in \mathbb{C}_0$ with $\delta > 0$ such that for $z \in \mathbb{C}_2$

$$||z|| < \delta \Rightarrow z \prec_{i_2} c$$

For this δ, there exists $n_0 \in \mathbb{N}$ such that

$$||S(\beth_n, \beth_n, \beth_{n+m})|| < \delta,\ \forall\ n \geq n_0.$$

This means that $S(\beth_n, \beth_n, \beth_{n+m}) \prec_{i_2} c, \forall n \geq n_0$. Therefore $\{\beth_n\}$ is a Cauchy sequence. □

Lemma 10. *Assuming that* S *is a bicomplex valued S-metric on* G*, then* $S(\beth, \beth, \aleph) = S(\aleph, \aleph, \beth)$*,* $\forall\ \beth, \aleph \in G$*.*

Proof. By the definition of bicomplex valued S-metric, we have
$S(\beth, \beth, \aleph) \prec_{i_2} 2S(\beth, \beth, \beth) + S(\aleph, \aleph, \beth)$. In view of $S(\beth, \beth, \beth) = 0$, we find that $S(\beth, \beth, \aleph) \prec_{i_2} S(\aleph, \aleph, \beth)$. Similarly, we find that $S(\aleph, \aleph, \beth) \prec_{i_2} S(\beth, \beth, \aleph)$. It follows that $S(\beth, \beth, \aleph) = S(\aleph, \aleph, \beth)$. □

3 Main Results

We will demonstrate a few fixed point theorems in this section for various contraction mappings that satisfy rational inequalities on the BVSMS.

Theorem 11. *Assuming that (G,S) is a complete BVSMS with nonsingular $1+S(\beth,\beth,\aleph)$ and $||1+S(\beth,\beth,\aleph)|| \neq 0$, whenever $\beth,\aleph \in G$. If a map $f: G \to G$ satisfies $S(f(\aleph),f(\aleph),f(\beth)) \precsim_{i_2} \lambda S(\beth,\beth,\aleph)+\frac{\mu S(\beth,\beth,f(\beth))S(f(\aleph),f(\aleph),\aleph)}{1+S(\beth,\beth,\aleph)}$, for all $\beth,\aleph \in G$, where $\lambda,\mu \in (0,1)$ with $\lambda+\sqrt{2}\mu < 1$, then the function f has a unique fixed point(or, UFP).*

Proof. Let $\beth_0 \in G$, and $\beth_1 = f(\beth_0)$. Suppose $\beth_{n+1} = f(\beth_n)$, whenever $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, from

$$\begin{aligned}
S(\beth_n,\beth_n,\beth_{n-1}) &= S(f(\beth_{n-1}),f(\beth_{n-1}),f(\beth_{n-2}))\\
&\precsim_{i_2} \lambda S(\beth_{n-1},\beth_{n-1},\beth_{n-2})\\
&\quad+\frac{\mu S(\beth_{n-2},\beth_{n-2},f(\beth_{n-2}))S(f(\beth_{n-1}),f(\beth_{n-1}),\beth_{n-1})}{1+S(\beth_{n-1},\beth_{n-1},\beth_{n-2})}\\
&= \lambda S(\beth_{n-1},\beth_{n-1},\beth_{n-2})\\
&\quad+\frac{\mu S(\beth_{n-2},\beth_{n-2},\beth_{n-1})S(\beth_n,\beth_n\beth_{n-1})}{1+S(\beth_{n-1},\beth_{n-1},\beth_{n-2})}\\
||S(\beth_n,\beth_n,\beth_{n-1})|| &\leq ||\lambda S(\beth_{n-1},\beth_{n-1}\beth_{n-2})\\
&\quad+\frac{\mu S(\beth_{n-2},,\beth_{n-2},\beth_{n-1})S(\beth_n,\beth_n,\beth_{n-1})}{1+S(\beth_{n-1},\beth_{n-1},\beth_{n-2})}||\\
&\leq \lambda||S(\beth_{n-1},\beth_{n-1},\beth_{n-2})||+\sqrt{2}\mu||S(\beth_n,\beth_n,\beth_{n-1})||,
\end{aligned}$$

so that

$$||S(\beth_n,\beth_n,\beth_{n-1})|| \leq \frac{\lambda}{1-\sqrt{2}\mu}||S(\beth_{n-1},\beth_{n-1},\beth_{n-2})||,$$

Hence, by applying $\alpha = \frac{\lambda}{1-\sqrt{2}\mu}$, we get

$$||S(\beth_n,\beth_n\beth_{n-1})|| \leq \alpha^n||S(\beth_1,\beth_1,\beth_0)||.$$

For every $m,n \in \mathbb{N}$,

$$\begin{aligned}
S(\beth_n,\beth_n,\beth_m) &\precsim_{i_2} S(\beth_n,\beth_n,\beth_{n+1})+S(\beth_n,\beth_n,\beth_{n+1})+S(\beth_m,\beth_m,\beth_{n+1})\\
&\precsim_{i_2} 2S(\beth_n,\beth_n,\beth_{n+1})+S(\beth_{n+1},\beth_{n+1},\beth_m)\\
&\precsim_{i_2} 2(\alpha^n+...+\alpha^{m-1})S(\beth_1,\beth_1,\beth_0),\\
&\precsim_{i_2} 2(\frac{\alpha^n}{1-\alpha})S(\beth_1,\beth_1,\beth_0),\ \text{if } m>n,
\end{aligned}$$

$$||S(\beth_n,\beth_n,\beth_m)|| \leq 2(\frac{\alpha^n}{1-\alpha})||S(\beth_1,\beth_1,\beth_0)||, \text{ if } m>n,$$

By $\alpha \in (0,1)$, $||S(\beth_n,\beth_n,\beth_m)|| \to 0$, as $n,m \to \infty$, we determine that $\{\beth_n\}$ is a Cauchy sequence. Since (G,S) is complete, $\{\beth_n\}$ converges to a point $\wp \in G$. Hence, $f(\beth_n) = \beth_{n+1} \to \wp \in G$ as $n \to \infty$ implies $S(f(\wp),f(\wp),f(\beth_n)) \to S(f(\wp),f(\wp),\wp)$ as $n \to \infty$, because of Lemma 8. Moreover by taking the limit from

$$S(f(\wp),f(\wp),f(\beth_n)) \precsim_{i_2} \lambda S(\beth_n,\beth_n,\wp) + \frac{\mu S(\beth_n,\beth_n,f(\beth_n))S(f(\wp),f(\wp),\wp)}{1+S(\beth_n,\beth_n,\wp)}$$

we obtain

$$||S(f(\wp),f(\wp),f(\beth_n))|| \leq \lambda||S(\beth_n,\beth_n,\wp)|| + \frac{\mu||S(\beth_n,\beth_n,f(\beth_n)S(f(\wp),f(\wp),\wp)||}{||1+S(\beth_n,\beth_n,\wp)||},$$

as $n \to \infty$, we get $S(f(\wp),f(\wp),\wp) = 0$. Therefore $f(\wp) = \wp$. Hence $\wp$ is a fixed point of f.
If, in addition, $f(\rho) = \rho$ for some another fixed point ρ of f, then

$$\begin{aligned} S(\wp,\wp,\rho) = S(f(\wp),f(\wp),f(\rho)) &\precsim_{i_2} \lambda S(\wp,\wp,\rho) + \frac{\mu S(\wp,\wp,f(\wp))S(f(\rho),f(\rho),\rho)}{1+S(\wp,\wp,\rho)} \\ &\precsim_{i_2} \lambda S(\wp,\wp,\rho). \end{aligned}$$

Therefore $||S(\wp,\wp,\rho)|| = 0$ and it is implies that $\rho = \wp$. Hence f has a UFP. □

Example 12. *Assuming that $G = \{0,\frac{1}{2},2\}$ and $S(\beth,\aleph,\rho) = (1+i_2)(|\beth-\rho|+|\aleph-\rho|)$, where $\beth,\aleph,\rho \in G$. Then (G,S) is a complete BVSMS. Define a map $f:(G,S) \to (G,S)$ by $f(0) = 0$, $f(\frac{1}{2}) = 0$, and $f(2) = \frac{1}{2}$. Then, f satisfies the inequality $S(f(\aleph),f(\aleph),f(\beth)) \precsim_{i_2} \lambda S(\beth,\beth,\aleph) + \frac{\mu S(\beth,\beth,f(\beth))S(f(\aleph),f(\aleph),\aleph)}{1+S(\beth,\beth,\aleph)}$ for $\lambda = \frac{1}{3}$ and $\mu = \frac{1}{6}$. Hence f has a UFP zero in G, because of Theorem 11.*

Theorem 13. *Assuming that (G,S) is a complete BVSMS with nonsingular $1+S(\beth,\beth,\aleph)$ and $||1+S(\beth,\beth,\aleph)|| \neq 0$, whenever $\beth,\aleph \in G$. If a map $f:(G,S) \to (G,S)$ satisfies $S(f(\aleph),f(\aleph),f(\beth)) \precsim_{i_2} \lambda[S(\beth,\beth,f(\beth)) + S(f(\aleph),f(\aleph),\aleph)]$ $+ \frac{\mu S(\beth,\beth,f(\beth))S(f(\aleph),f(\aleph),\aleph)}{1+S(\beth,\beth,\aleph)}$, for all $(\beth,\aleph) \in G$, where $\lambda,\mu \in (0,1)$ with $2\lambda+\sqrt{2}\mu < 1$, then the function $f:G \to G$ has a UFP.*

Proof. Let $\beth_0 \in G$, and $\beth_1 = f(\beth_0)$. Suppose $\beth_{n+1} = f(\beth_n)$, whenever $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, from

$$\begin{aligned}
S(\beth_n,\beth_n,\beth_{n-1}) &= S(f(\beth_{n-1}),f(\beth_{n-1}),f(\beth_{n-2}))\\
&\precsim_{i_2} \lambda[S(\beth_{n-1},\beth_{n-1},f(\beth_{n-1}))+S(\beth_{n-2},\beth_{n-2},f(\beth_{n-2}))]\\
&\quad+\frac{\mu S(\beth_{n-2},\beth_{n-2},f(\beth_{n-2}))S(f(\beth_{n-1}),f(\beth_{n-1}),\beth_{n-1})}{1+S(\beth_{n-1},\beth_{n-1},\beth_{n-2})}\\
&= \lambda[S(\beth_{n-1},\beth_{n-1},\beth_n)+S(\beth_{n-2},\beth_{n-2},\beth_{n-1})]\\
&\quad+\frac{\mu S(\beth_{n-2},\beth_{n-2},\beth_{n-1})S(\beth_n,\beth_n\beth_{n-1})}{1+S(\beth_{n-1},\beth_{n-1},\beth_{n-2})}\\
||S(\beth_n,\beth_n,\beth_{n-1})|| &\leq ||[S(\beth_{n-1},\beth_{n-1},\beth_n)+S(\beth_{n-2},\beth_{n-2},\beth_{n-1})]\\
&\quad+\frac{\mu S(\beth_{n-2},,\beth_{n-2},\beth_{n-1})S(\beth_n,\beth_n,\beth_{n-1})}{1+S(\beth_{n-1},\beth_{n-1},\beth_{n-2})}||\\
&\leq \lambda||[S(\beth_{n-1},\beth_{n-1},\beth_n)+S(\beth_{n-2},\beth_{n-2},\beth_{n-1})]||\\
&\quad+\sqrt{2}\mu||S(\beth_n,\beth_n,\beth_{n-1})||,
\end{aligned}$$

we conclude that

$$||S(\beth_n,\beth_n,\beth_{n-1})|| \leq \frac{\lambda}{1-\lambda-\sqrt{2}\mu}||S(\beth_{n-1},\beth_{n-1},\beth_{n-2})||,$$

Hence, by applying $\alpha = \frac{\lambda}{1-\lambda-\sqrt{2}\mu}$, we get

$$||S(\beth_n,\beth_n,\beth_{n-1})|| \leq \alpha^n||S(\beth_{n-1},\beth_{n-1},\beth_{n-2})||$$

For every $m, n \in \mathbb{N}$,

$$\begin{aligned}
S(\beth_n,\beth_n,\beth_m) &\precsim_{i_2} S(\beth_n,\beth_n,\beth_{n+1})+S(\beth_n,\beth_n,\beth_{n+1})+S(\beth_m,\beth_m,\beth_{n+1})\\
&\precsim_{i_2} 2S(\beth_n,\beth_n,\beth_{n+1})+S(\beth_{n+1},\beth_{n+1},\beth_m)\\
&\precsim_{i_2} 2(\alpha^n+...+\alpha^{m-1})S(\beth_1,\beth_1,\beth_0),\\
&\precsim_{i_2} 2(\frac{\alpha^n}{1-\alpha})S(\beth_1,\beth_1,\beth_0),\ \text{if } m>n,
\end{aligned}$$

$$||S(\beth_n,\beth_n,\beth_m)|| \leq 2(\frac{\alpha^n}{1-\alpha})||S(\beth_1,\beth_1,\beth_0)||,\ \text{if } m>n,$$

By $\alpha \in (0,1)$, $||S(\beth_n,\beth_n,\beth_m)|| \to 0$, as $n, m \to \infty$, we determine that $\{\beth_n\}$ is a Cauchy sequence. Since (G, S) is complete, $\{\beth_n\}$ converges to a point $\wp \in G$. Hence, $f(\beth_n) = \beth_{n+1} \to \wp \in G$ as $n \to \infty$ implies $S(f(\wp), f(\wp), f(\beth_n)) \to S(f(\wp), f(\wp), \wp)$ as $n \to \infty$, because of Lemma 8. Moreover by taking the limit from

$$\begin{aligned}
S(f(\wp),f(\wp),f(\beth_n)) &\precsim_{i_2} \lambda[S(\beth_n,\beth_n,f(\beth_n))+S(f(\wp),f(\wp),\wp)]\\
&\quad+\frac{\mu S(\beth_n,\beth_n,f(\beth_n))S(f(\wp),f(\wp),\wp)}{1+S(\beth_n,\beth_n,\wp)}
\end{aligned}$$

we obtain

$$\begin{aligned}||S(f(\wp),f(\wp),f(\beth_n))|| &\leq \lambda[||S(\beth_n,\beth_n,f(\beth_n))+S(\wp,\wp,f(\wp))||] \\ &\quad +\frac{\mu||S(\beth_n,\beth_n,f(\beth_n)S(f(\wp),f(\wp),\wp)||}{||1+S(\beth_n,\beth_n,\wp)||},\end{aligned}$$

as $n\to\infty$, we get $S(f(\wp),f(\wp),\wp)=0$. Therefore $f(\wp)=\wp$. Hence $\wp$ is a fixed point of f.
If, in addition, $f(\rho)=\rho$ for some another fixed point ρ of f, then

$$\begin{aligned}S(\wp,\wp,\rho)=S(f(\wp),f(\wp),f(\rho)) &\precsim_{i_2} \lambda[S(\wp,\wp,f(\wp))+S(f(\rho),f(\rho),\rho)] \\ &\quad +\frac{\mu S(\wp,\wp,f(\wp))S(f(\rho),f(\rho),\rho)}{1+S(\wp,\wp,\rho)}\end{aligned}$$

Therefore $||S(\wp,\wp,\rho)||=0$ and it is implies that $\rho=\wp$. Hence f has a UFP. $\square$

Theorem 14. *Assuming that (G,S) be a complete BVSMS with nonsingular $1+S(\beth,\beth,f(\beth))+S(f(\aleph),f(\aleph),\aleph)$ and $||1+S(\beth,\beth,f(\beth))+S(f(\aleph),f(\aleph),\aleph)||\neq 0$, whenever $\beth,\aleph\in G$. If a map $f:G\to G$ satisfies $S(f(\aleph),f(\aleph),f(\beth))\precsim_{i_2}\lambda[S(\beth,\beth,\aleph)+S(\beth,\beth,f(\beth))+S(f(\aleph),f(\aleph),\aleph)]+\frac{\mu S(\beth,\beth,f(\beth))S(f(\aleph),f(\aleph),\aleph)}{1+S(\beth,\beth,f(\beth))+S(f(\aleph),f(\aleph),\aleph)}$, for all $\beth,\aleph\in G$, where $\lambda,\mu\in(0,1)$ with $3\lambda+\sqrt{2}\mu<1$, then the function f has a UFP.*

Proof. Let $\beth_0\in G$, and $\beth_1=f(\beth_0)$. Suppose $\beth_{n+1}=f(\beth_n)$, whenever $n\in\mathbb{N}$. For every $n\in\mathbb{N}$, from

$$\begin{aligned}S(\beth_n,\beth_n,\beth_{n-1}) &= S(f(\beth_{n-1}),f(\beth_{n-1}),f(\beth_{n-2})) \\ &\precsim_{i_2} \lambda[S(\beth_{n-1},\beth_{n-1},\beth_{n-2}) \\ &\quad +S(\beth_{n-1},\beth_{n-1},f(\beth_{n-1}))+S(\beth_{n-2},\beth_{n-2},f(\beth_{n-2}))] \\ &\quad +\frac{\mu S(\beth_{n-2},\beth_{n-2},f(\beth_{n-2}))S(f(\beth_{n-1}),f(\beth_{n-1}),\beth_{n-1})}{1+S(\beth_{n-2},\beth_{n-2},f(\beth_{n-2}))+S(f(\beth_{n-1}),f(\beth_{n-1}),\beth_{n-1})} \\ &= \lambda[S(\beth_{n-1},\beth_{n-1},\beth_{n-2}) \\ &\quad +S(\beth_{n-1},\beth_{n-1},\beth_n)+S(\beth_{n-2},\beth_{n-2},\beth_{n-1})] \\ &\quad +\frac{\mu S(\beth_{n-2},\beth_{n-2},\beth_{n-1})S(\beth_n,\beth_n\beth_{n-1})}{1+S(\beth_{n-2},\beth_{n-2},\beth_{n-1})+S(\beth_n,\beth_n,\beth_{n-1})} \\ ||S(\beth_n,\beth_n,\beth_{n-1})|| &\leq ||\lambda[S(\beth_{n-1},\beth_{n-1},\beth_{n-2}) \\ &\quad +S(\beth_{n-1},\beth_{n-1},\beth_n)+S(\beth_{n-2},\beth_{n-2},\beth_{n-1})] \\ &\quad +\frac{\mu S(\beth_{n-2},,\beth_{n-2},\beth_{n-1})S(\beth_n,\beth_n,\beth_{n-1})}{1+S(\beth_{n-2},\beth_{n-2},\beth_{n-1})+S(\beth_n,\beth_n,\beth_{n-1})}|| \\ &\leq \lambda||[S(\beth_{n-1},\beth_{n-1},\beth_{n-2})+S(\beth_{n-1},\beth_{n-1},\beth_n) \\ &\quad +S(\beth_{n-2},\beth_{n-2},\beth_{n-1})]||+\sqrt{2}\mu||S(\beth_n,\beth_n,\beth_{n-1})||,\end{aligned}$$

we conclude that

$$||S(\beth_n,\beth_n,\beth_{n-1})|| \leq \frac{2\lambda}{1-\lambda-\sqrt{2}\mu}||S(\beth_{n-1},\beth_{n-1},\beth_{n-2})||,$$

Hence, by applying $\alpha = \frac{\lambda}{1-\lambda-\sqrt{2}\mu}$, we get

$$||S(\beth_n,\beth_n,\beth_{n-1})|| \leq \alpha^n||S(\beth_{n-1},\beth_{n-1},\beth_{n-2})||$$

For every $m, n \in \mathbb{N}$,

$$\begin{aligned} S(\beth_n,\beth_n,\beth_m) &\precsim_{i_2} S(\beth_n,\beth_n,\beth_{n+1}) + S(\beth_n,\beth_n,\beth_{n+1}) + S(\beth_m,\beth_m,\beth_{n+1}) \\ &\precsim_{i_2} 2S(\beth_n,\beth_n,\beth_{n+1}) + S(\beth_{n+1},\beth_{n+1},\beth_m) \\ &\precsim_{i_2} 2(\alpha^n + ... + \alpha^{m-1})S(\beth_1,\beth_1,\beth_0), \\ &\precsim_{i_2} 2(\frac{\alpha^n}{1-\alpha})S(\beth_1,\beth_1,\beth_0), \text{ if } m > n, \end{aligned}$$

$$||S(\beth_n,\beth_n,\beth_m)|| \leq 2(\frac{\alpha^n}{1-\alpha})||S(\beth_1,\beth_1,\beth_0)||, \text{ if } m > n,$$

By $\alpha \in (0,1)$, $||S(\beth_n,\beth_n,\beth_m)|| \to 0$, as $n, m \to \infty$, we determine that $\{\beth_n\}$ is a Cauchy sequence. Since (G,S) is complete, $\{\beth_n\}$ converges to a point $\wp \in G$. Hence, $f(\beth_n) = \beth_{n+1} \to \wp \in G$ as $n \to \infty$ implies $S(f(\wp),f(\wp),f(\beth_n)) \to S(f(\wp),f(\wp),\wp)$ as $n \to \infty$, because of Lemma 8. Moreover by taking the limit from

$$\begin{aligned} S(f(\wp),f(\wp),f(\beth_n)) &\precsim_{i_2} \lambda[S(\beth_n,\beth_n,\wp) + S(\beth_n,\beth_n,f(\beth_n)) + S(f(\wp),f(\wp),\wp)] \\ &\quad + \frac{\mu S(\beth_n,\beth_n,f(\beth_n))S(f(\wp),f(\wp),\wp)}{1 + S(\beth_n,\beth_n,f(\beth_n)) + S(f(\wp),f(\wp),\wp)} \end{aligned}$$

we obtain

$$\begin{aligned} ||S(f(\wp),f(\wp),f(\beth_n))|| &\leq \lambda[||S(\beth_n,\beth_n,\wp) + S(\beth_n,\beth_n,f(\beth_n)) + S(\wp,\wp,f(\wp))||] \\ &\quad + \frac{\mu||S(\beth_n,\beth_n,f(\beth_n)S(f(\wp),f(\wp),\wp)||}{||1 + S(\beth_n,\beth_n,f(\beth_n)) + S(f(\wp),f(\wp),\wp)||}, \end{aligned}$$

as $n \to \infty$, we get $S(f(\wp),f(\wp),\wp) = 0$. Therefore $f(\wp) = \wp$. Hence $\wp$ is a fixed point of f.
If, in addition, $f(\rho) = \rho$ for some another fixed point ρ of f, then

$$\begin{aligned} S(\wp,\wp,\rho) = S(f(\wp),f(\wp),f(\rho)) &\precsim_{i_2} \lambda[S(\wp,\wp,\rho) + S(\wp,\wp,f(\wp)) + S(f(\rho),f(\rho),\rho)] \\ &\quad + \frac{\mu S(\wp,\wp,f(\wp))S(f(\rho),f(\rho),\rho)}{1 + S(\wp,\wp,f(\wp)) + S(f(\rho),f(\rho),\rho)} \end{aligned}$$

Therefore $||S(\wp,\wp,\rho)|| = 0$ and it is implies that $\rho = \wp$. Hence f has a UFP. □

4 Applications

Using main theorem 11, we prove an existence theorem for the unique solution of the linear system of equations.

Theorem 15. *Assuming that* $G = \mathbb{C}^n$ *is BVSMS with the metric* $S(\beth, \aleph, \rho) = \sum_{j=1}^{n} (1 + i_2)(|\beth - \rho| + |\aleph - \rho|)$*, whenever* $\beth, \aleph, \rho \in G$*. If* $\sum_{j=1}^{n} |\lambda_{ij}| \precsim_{i_2} \lambda < 1$*, whenever* $i = 1, 2, ..., n$*, then the linear system*

$$\begin{cases} b_1 = a_{11}\beth_1 + a_{12}\beth_2 + ... + a_{1n}\beth_n \\ b_2 = a_{21}\beth_1 + a_{22}\beth_2 + ... + a_{2n}\beth_n \\ \vdots \\ b_n = a_{n1}\beth_1 + a_{n2}\beth_2 + ... + a_{nn}\beth_n \end{cases}$$

of n *linear equations in* n *unknown has a unique solution.*

Proof. Define $f : G \to G$ by $f(\beth) = A\beth + b$, whenever $\beth = (\beth_1, \beth_2, ..., \beth_n) \in \mathbb{C}^n, b = (b_1, b_2, ..., b_n) \in \mathbb{C}^n$, and $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$

Now,

$$\begin{aligned} S(f\beth, f\beth, f\aleph,) &= \sum_{j=1}^{n} (1 + i_2)(|\lambda_{ij}(\beth - \aleph)| + |\lambda_{ij}(\beth - \aleph)|) \\ &\precsim_{i_2} \sum_{j=1}^{n} |\lambda_{ij}| \left[\sum_{j=1}^{n} (1 + i_2)(|\beth - \aleph| + |\beth - \aleph|) \right] \\ &\precsim_{i_2} \lambda \sum_{j=1}^{n} (1 + i_2)(|\beth - \aleph| + |\beth - \aleph|) \\ &= \lambda S(\beth, \beth, \aleph) \\ &= \lambda S(\beth, \beth, \aleph) + \frac{\mu S(\beth, \beth, f(\beth)) S(f(\aleph), f(\aleph), \aleph)}{1 + S(\beth, \beth, \aleph)}. \end{aligned}$$

All of Theorem 11's requirements are then fulfilled with $\lambda = \frac{1}{6}, \mu = 0$ and $\lambda + \sqrt{2}\mu < 1$. As a result, there is only one solution to the linear system of equations. □

5 Conclusions

All fixed point theorems of bicomplex valued S-metric spaces can be regarded as generalizations of fixed point theorems of bicomplex valued metric spaces, S-metric spaces and complex valued metric spaces. Therefore, studies of fixed point results in bicomplex valued S-metric spaces are significant.

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