Surveys in Mathematics and its Applications

ISSN 1842-6298 (electronic), 1843-7265 (print)
Volume 18 (2023), 343 – 363

EXISTENCE OF POSITIVE SOLUTIONS OF A TERMINAL VALUE PROBLEM FOR FOURTH-ORDER DIFFERENTIAL EQUATIONS

Mohammed Dahmane and Mohammed Derhab

Abstract. We are concerned with the existence of positive solutions of a terminal value problem for a class of fourth-order differential equation. Our arguments are based to establish sufficient conditions which guarantee the existence of at least one positive solution by constructing a cone on which a positive operator is defined. We then apply a theorem of Guo-Krasonelskiì to prove the existence of positive solutions of our problem.

1 Introduction

The objective of this paper is the study of the existence of positive solutions for a terminal value problem concerning a fourth order differential equation. More precisely, we consider the following problem

$$\begin{cases} u^{(4)}(x) = f(x, u(x), u''(x)), \ x \in \mathbb{R}^+, \\ \lim\limits_{x\to+\infty} u(x) = \xi, \\ \lim\limits_{x\to+\infty} u^{(i)}(x) = 0 \text{ for } i = 1, 2, 3, \end{cases} \tag{1.1}$$

where $f : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}$, is continuous and $\xi > 0$.

Boundary value problems on infinite intervals arise naturally in the study of radially symmetric solutions of nonlinear elliptic equations, population dynamics, geophysics and various physical phenomena, such as the theory of drain flows, plasma physics, heat transfer between solids and gasses, nuclear physics, unsteady flow of gas through a semi-infinite porous media and in determining the electrical potential in an isolated neutral atom (see [1], [4], [5], [7], [8], [16], [20], [21], [22] and the references therein).

Terminal value problems for differential equations on unbounded domains have been studied by several authors using Banach and Schauder fixed point theorems,

2020 Mathematics Subject Classification: 47N20; 34B40; 34B18; 47H10
Keywords: Fourth order differential equation; positive solutions; Guo-Krasnoselskii theorem; cone preserving map.
This work was supported by a a MESRS-DRS grant C00L03UN130120190002.

critical point theory, the upper and lower solutions method, the method of successive approximations, the coincidence degree theory of Mawhin and fixed point theorems in cones (see [2], [7], [[14]-[15]], [[18]-[19]], [[24]-[25]] and the references therein).

In [18], the authors studied the following problem

$$\begin{cases} -u''(x) = f(x, u(x)), \ x \in \mathbb{R}^+, \\ \lim\limits_{x \to +\infty} u(x) = \xi, \end{cases} \tag{1.2}$$

where $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$, is continuous function and $\xi > 0$. By using the well–known Guo–Krasnoselskii fixed point theorem, the authors proved that the problem (1.2) admits at least one positive solution. The aim of this paper is to improve and generalize the results obtained in [18]. We note also that the results obtained in [18] is true if we suppose the additional condition $\lim\limits_{x \to +\infty} u'(x) = 0$ in (1.2) because if we consider for example the function u defined by $u(x) = \frac{\sin((x+1)^2)}{x+1}$, for all $x \in \mathbb{R}^+$, we have $\lim\limits_{x \to +\infty} u(x) = 0$ but $\lim\limits_{x \to +\infty} u'(x)$ does not exist.

The plan of this paper is organized as follows. In Section 2, we give some definitions and preliminary results that will be used in the remainder of this paper. The main result is presented and proved in Section 3, followed by an example in Section 4 illustrating the application of our result.

2 Preliminaries

In this section we give some definitions and preliminary results that will be used in the remainder of this paper.

2.1 Integration of order $n \in \mathbb{N}^*$ on the half-axis

Definition 1. *[23, Chapter 1 page 15]The Euler gamma function Γ is defined by*

$$\Gamma(x) = \int_0^{+\infty} e^{-t} t^{x-1} dt,$$

where $x > 0$.

Remark 2. *[23, Chapter 1 page 16]For all $n \in \mathbb{N}^*$, we have $\Gamma(n) = (n-1)!$.*

Definition 3. *[23, Chapter 1 page 17]The beta function B is defined by*

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt,$$

where $x > 0$ and $y > 0$.

Remark 4. *[23, Chapter 1 page 17]The function B is connected with the gamma function Γ by the relation*

$$B\left(x,y\right)=\frac{\Gamma\left(x\right)\Gamma\left(y\right)}{\Gamma\left(x+y\right)},$$

for all $x>0$ and $y>0$.

Definition 5. *[23, Chapter 2 page 99] Let $n\in\mathbb{N}^{*}$ and let $g:\mathbb{R}^{+}\to\mathbb{R}$ be a function. The integral of order n on the half-axis of g is defined by*

$$\left(I_{-}^{n}g\right)\left(x\right)=\frac{1}{\Gamma\left(n\right)}\int_{x}^{+\infty}\left(t-x\right)^{n-1}g\left(t\right)dt,\ \text{for all } x\in\mathbb{R}^{+}.$$

Now we state the Fubini's theorem

Theorem 6. *[23, Theorem 1.1]Let $P_1=[r_1,r_2]$, $P_2=[r_3,r_4]$, $-\infty\leq r_1<r_2\leq+\infty$, $-\infty\leq r_3<r_4\leq+\infty$, and let $h\left(x,y\right)$ be a measurable function defined on $P_1\times P_2$. If at least one of the integrals*

$$\int_{P_1}dx\int_{P_2}h\left(x,y\right)dy,\ \int_{P_2}dy\int_{P_1}h\left(x,y\right)dx,\ \iint_{P_1\times P_2}h\left(x,y\right)dxdy,$$

is absolutely convergent then they coincide.

Lemma 7. *Let $g:\mathbb{R}^{+}\to\mathbb{R}$ be a function such that $\int_{x}^{+\infty}\left(t-x\right)^{m+n-1}g\left(t\right)dt$ is absolutely convergent for all $x\in\mathbb{R}^{+}$, where $n\in\mathbb{N}^{*}$ and $m\in\mathbb{N}^{*}$, then we have*

$$I_{-}^{m}(I_{-}^{n}g)\left(x\right)=\left(I_{-}^{m+n}g\right)\left(x\right),\ \text{for all } x\in\mathbb{R}^{+}.$$

Proof. By using the definition of the integrals of order n and m on the half-axis and from Theorem 6, we have

$$\begin{aligned}
I_{-}^{m}(I_{-}^{n}g)\left(x\right) &= \frac{1}{\Gamma\left(m\right)}\int_{x}^{+\infty}\left(t-x\right)^{m-1}\left(I_{-}^{n}g\right)\left(t\right)dt\\
&= \frac{1}{\Gamma\left(m\right)\Gamma\left(n\right)}\int_{x}^{+\infty}\left(t-x\right)^{m-1}\int_{t}^{+\infty}\left(\tau-t\right)^{n-1}g\left(\tau\right)d\tau dt\\
&= \frac{1}{\Gamma\left(m\right)\Gamma\left(n\right)}\int_{x}^{+\infty}g\left(\tau\right)\int_{x}^{\tau}\left(t-x\right)^{m-1}\left(\tau-t\right)^{n-1}dtd\tau.
\end{aligned}$$

Now if we put the change of variables $t = x + y(\tau - x)$, we obtain

$$\begin{aligned} I_-^m(I_-^n g)(x) &= \frac{1}{\Gamma(m)\Gamma(n)} \int_x^{+\infty} (\tau - x)^{m+n-1} g(\tau) \int_0^1 y^{m-1}(1-y)^{n-1} dy d\tau \\ &= \frac{B(m,n)}{\Gamma(m)\Gamma(n)} \int_x^{+\infty} (\tau - x)^{m+n-1} g(\tau) d\tau \\ &= \frac{1}{\Gamma(m+n)} \int_x^{+\infty} (\tau - x)^{m+n-1} g(\tau) d\tau \\ &= \left(I_-^{m+n} g\right)(x). \end{aligned}$$

□

Lemma 8. *Let $n \in \mathbb{N}^*$ and let $g : \mathbb{R}^+ \to \mathbb{R}$ be a continuous function such that $g \in L^1(\mathbb{R}^+)$, then we have*

$$\frac{d^n}{dx^n}\left(I_-^n g\right)(x) = (-1)^n g(x), \text{ for all } x \in \mathbb{R}^+.$$

Proof. For $n = 1$, we have

$$\begin{aligned} \frac{d}{dx}\left(I_-^1 g\right)(x) &= \frac{d}{dx} \int_x^{+\infty} g(t) dt \\ &= -g(x). \end{aligned}$$

Assume for fixed $n \in \mathbb{N}^*$, we have

$$\frac{d^n}{dx^n}\left(I_-^n g\right)(x) = (-1)^n g(x),$$

and we prove that

$$\frac{d^{n+1}}{dx^{n+1}}\left(I_-^{n+1} g\right)(x) = (-1)^{n+1} g(x).$$

By using the hypothesis of recurrence and Lemma7, we have

$$\begin{aligned} \frac{d^{n+1}}{dx^{n+1}}\left(I_-^{n+1} g\right)(x) &= \frac{d}{dx}\left(\frac{d^n}{dx^n} I_-^n(I_-^1 g)\right)(x) \\ &= \frac{d}{dx}\left((-1)^n I_-^1 g\right)(x) \\ &= (-1)^{n+1} g(x). \end{aligned}$$

In conclusion, we have

$$\forall n \in \mathbb{N}^*, \ \frac{d^n}{dx^n}\left(I_-^n g\right)(x) = (-1)^n g(x), \text{ for all } x \in \mathbb{R}^+.$$

□

Lemma 9. *Let $n \in \mathbb{N}^*$ and let $g : \mathbb{R}^+ \to \mathbb{R}$ be a function such that*

(i) $g \in C^n(\mathbb{R}^+,\mathbb{R})$ *such that* $I_-^n g^{(n)} \in L^1(\mathbb{R}^+,\mathbb{R})$;

(ii) $\lim\limits_{x\to+\infty} g(x) = \mu$ *with* $\mu \in \mathbb{R}$ *and* $\lim\limits_{x\to+\infty} g^{(i)}(x) = 0$ *for* $i = 1, ..., n-1$.

Then, we have

$$\left(I_-^n g^{(n)}\right)(x) = (-1)^{n+1}\left[\mu - g(x)\right].$$

Proof. For $n = 1$, we have

$$\begin{aligned} \left(I_-^1 g'\right)(x) &= \int_x^{+\infty} g'(t)\,dt \\ &= \lim_{x\to+\infty} g(x) - g(x) \\ &= \mu - g(x). \end{aligned}$$

Assume for fixed $n \in \mathbb{N}^*$, we have

$$\left(I_-^n g^{(n)}\right)(x) = (-1)^{n+1}\left[\mu - g(x)\right],$$

and we prove that

$$\left(I_-^{n+1} g^{(n+1)}\right)(x) = (-1)^{n}\left[\mu - g(x)\right].$$

By using Lemma 7 and the hypothesis of recurrence, we have

$$\begin{aligned} \left(I_-^{n+1} g^{(n+1)}\right)(x) &= \left(I_-^1 I_-^n g^{(n+1)}\right)(x) \\ &= I_-^1 (-1)^{n+1}\left[\lim_{x\to+\infty} g'(x) - g'(x)\right] \\ &= I_-^1 (-1)^{n+2} g'(x) \\ &= (-1)^{n+2} I_-^1 g'(x) \\ &= (-1)^{n+2}\left[\mu - g(x)\right]. \end{aligned}$$

Which implies that

$$\forall n \in \mathbb{N}^*, \ \left(I_-^n g^{(n)}\right)(x) = (-1)^{n+1}\left[\mu - g(x)\right].$$

□

2.2 Compactness criterion in an unbounded interval

Let

$$C_l = \left\{ u \in C\left(\mathbb{R}^+, \mathbb{R}\right), \lim_{x \to +\infty} u(x) \text{ exists} \right\}.$$

Note that $(C_l, \|.\|_0)$ is a Banach space, where

$$\|u\|_0 = \sup_{x \in \mathbb{R}^+} |u(x)|.$$

Definition 10. *[2]) A set $\tilde{A} \subset C_l$ is called equiconvergent at $+\infty$ if all functions u in $\tilde{A}$ have finite limits at $+\infty$ and in addition,*

$$\forall \varepsilon > 0, \; \exists \, \delta(\varepsilon) > 0, \forall x > \delta(\varepsilon), \text{ we have } \left| u(x) - \lim_{x \to +\infty} u(x) \right| < \varepsilon.$$

Definition 11. *A set $\tilde{A} \subset C_l$ is called equicontinuous on each compact interval of $\mathbb{R}$ if for each $\varepsilon > 0$, there exists $\delta_1(\varepsilon) > 0$ such that, for all functions u in $\tilde{A}$ and all x_1 and x_2 in $\mathbb{R}^+$, the condition $|x_2 - x_1| < \delta_1(\varepsilon)$ implies $|u(x_2) - u(x_1)| < \varepsilon$.*

The following proposition gives the compactness criterion of Corduneanu.

Proposition 12. *([2, Lemma 1 page 149] or [6, Page 62]) A set $\tilde{A}$ of C_l is relatively compact if and only if*

(i) $\tilde{A}$ is uniformly bounded in C_l;

(ii) $\tilde{A}$ is equicontinuous on each compact interval of $\mathbb{R}^+$;

(iii) $\tilde{A}$ is equiconvergent at $+\infty$.

Now let
$$C_l^2 = \left\{ u \in C^2\left(\mathbb{R}^+, \mathbb{R}\right), \lim_{x \to +\infty} u(x) \text{ exists and } \lim_{x \to +\infty} u^{'}(x) = \lim_{x \to +\infty} u^{''}(x) = 0 \right\}.$$
Note that $\left(C_l^2, \|.\|_2\right)$ is a Banach space, where

$$\|u\|_2 = \max\left(\|u\|_0, \|u'\|_0, \|u''\|_0\right).$$

We have the following result

Proposition 13. *A set $\widetilde{B}$ of C_l^2 is relatively compact if and only if*

(i) $\widetilde{B}$ is uniformly bounded in C_l^2;

(ii) the functions belonging to the sets $\left\{ v : \; v(x) = u(x), \, u \in \widetilde{B} \right\}$, $\left\{ w : \; w(x) = u'(x), \, u \in \widetilde{B} \right\}$ and $\left\{ \theta : \; \theta(x) = u''(x), \, u \in \widetilde{B} \right\}$ are equicontinuous on each compact interval of $\overline{I}$;

(iii) the functions from $\left\{v: \ v(x) = u(x), \ u \in \widetilde{B}\right\}$,
$\left\{w: \ w(x) = u'(x), \ u \in \widetilde{B}\right\}$ *and* $\left\{\theta: \ \theta(x) = u''(x), \ u \in \widetilde{B}\right\}$ *are equiconvergent at* $+\infty$.

Proof. The proof is similar to that of Theorem 2.3 in [27], so it is omitted. □

Definition 14. *[10] Let E be a real Banach space. A nonempty convex $K \subset E$ is called a cone if it satisfies the following two conditions:*

(i) $u \in K$, $\lambda \geq 0$ *implies* $\lambda u \in K$;

(ii) $u \in K$, $-u \in K$ *implies* $u = 0$.

Theorem 15. *([10, Theorem 2.3.4]) Let K be a cone in a real Banach space E. Let Ω_1, Ω_2 be two open and bounded subsets of E such that*

$$0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2.$$

Assume that

$$T : K \cap \left(\overline{\Omega_2} \setminus \Omega_1\right) \longrightarrow K$$

is completely continuous operator and satisfies one of the following conditions
(i) $\|Tu\| \leq \|u\|$ *for* $u \in K \cap \partial\Omega_1$, *and* $\|Tu\| \geq \|u\|$ *for* $u \in K \cap \partial\Omega_2$.
(ii) $\|Tu\| \geq \|u\|$ *for* $u \in K \cap \partial\Omega_1$, *and* $\|Tu\| \leq \|u\|$ *for* $u \in K \cap \partial\Omega_2$.
Then T has at least one fixed point in $K \cap \left(\overline{\Omega_2} \setminus \Omega_1\right)$.

Definition 16. *We say that u is a solution for the problem (1.1) if* $u \in C^4(\mathbb{R}^+, \mathbb{R})$ *and u satisfies (1.1).*

We assume first that the nonlinearity f satisfies the following hypothesis

(H) $|f\ (x, u, v)| \leq a(x) L_1(u) + b(x) L_2(v) + c(x)$, where L_1, L_2 , a, b and c : $\mathbb{R}^+ \longrightarrow \mathbb{R}^+$ are continuous functions and L_1, L_2 are increasing such that

$$\begin{aligned} A &= \left(I_-^4 a\right)(0) = \frac{1}{3!}\int_0^{+\infty} t^3 a(t)\,dt < +\infty, \\ B &= \left(I_-^4 b\right)(0) = \frac{1}{3!}\int_0^{+\infty} t^3 b(t)\,dt < +\infty, \end{aligned}$$

and

$$C = \left(I_-^4 c\right)(0) = \frac{1}{3!}\int_0^{+\infty} t^3 c(t)\,dt < +\infty.$$

Remark 17. *In the rest of this paper, we use the notations*

$$\begin{aligned} A_i &= \left(I_-^{4-i}a\right)(0) \ \text{for } i=1,2 \\ B_i &= \left(I_-^{4-i}b\right)(0) \ \text{for } i=1,2, \end{aligned}$$

and

$$C_i = \left(I_-^{4-i}c\right)(0) \ \text{for } i=1,2.$$

3 Main results

First, we define the following sets

$$E = \left\{u \in C_l^2, \ u \text{ bounded}\right\},$$

$$K = \{u \in E, \ u(x) \geq 0\},$$

and

$$K_m = \{u \in K, \ \|u\|_2 < m\},$$

where $m > 0$.

Remark 18. *It is not difficult to verify that $(E, \|.\|_2)$ is a Banach space and K is a cone in E.*

We define the operator T by

$$\begin{aligned} T: \ K_m &\longrightarrow C^2(\mathbb{R}^+, \mathbb{R}) \\ u &\longmapsto (Tu)(x) = \xi + \tfrac{1}{3!}\int_x^{+\infty} (t-x)^3 f(t, u(t), u''(t))\, dt. \end{aligned}$$

Lemma 19. *The operator T is well defined.*

Proof. Let $u \in K_m$, then by assumption (H), we have

$$\begin{aligned} (Tu)(x) &= \xi + \frac{1}{3!}\int_x^{+\infty} (t-x)^3 f\left(t, u(t), u''(t)\right) dt \\ &\leq \xi + \frac{1}{3!}\int_x^{+\infty} (t-x)^3 \left|f\left(t, u(t), u''(t)\right)\right| dt \\ &\leq \xi + \frac{1}{3!}\int_x^{+\infty} (t-x)^3 \left[a(t) L_1(u(t)) + b(t) L_2\left(u''(t)\right) + c(t)\right] dt \\ &\leq \xi + \left(I_-^4 a\right)(x) L_1(m) + \left(I_-^4 b\right)(x) L_2(m) + \left(I_-^4 c\right)(x) \\ &\leq \xi + \left(I_-^4 a\right)(0) L_1(m) + \left(I_-^4 b\right)(0) L_2(m) + \left(I_-^4 c\right)(0) \\ &\leq \xi + AL_1(m) + BL_2(m) + C < +\infty. \end{aligned}$$

Also, we have

$$\begin{aligned}(Tu)''(x) &= \int_x^{+\infty}(t-x)\,f\big(t,u(t),u''(t)\big)\,dt\\ &\le A_2L_1(m)+B_2L_2(m)+C_2\\ &< +\infty,\end{aligned}$$

and it is not difficult to prove that $(Tu)''$ is continuous on $\overline{I}$. □

Lemma 20. *Assume that the hypothesis (H) is satisfied, then a function $u\in\overline{K}_m$ is a solution for the problem (1.1) if and only if it is a fixed point of the operator T.*

Proof. Let $u\in\overline{K}_m$ a fixed point of T that is

$$u(x)=(Tu)(x)=\ \xi+\frac{1}{3!}\int_x^{+\infty}(t-x)^3 f\big(t,u(t),u''(t)\big)\,dt,\ \ x\in\mathbb{R}^+.$$

We have

$$u(x)=(Tu)(x)=\xi+\big(I_-^4\widetilde{g}\big)(x),$$

where

$$\widetilde{g}(x)=f\big(x,u(x),u''(x)\big).$$

From Lemma 8, we obtain

$$u^{(4)}(x)=f\big(x,u(x),u''(x)\big).$$

On the other hand, we have

$$\begin{aligned}u(x) &= \xi+\frac{1}{3!}\int_x^{+\infty}(t-x)^3 f\big(t,u(t),u''(t)\big)\,dt\\ &= \xi+\frac{1}{3!}\int_0^{+\infty}\chi_{[x,+\infty[}(t)\,(t-x)^3\,\widetilde{g}(t)\,dt,\end{aligned}$$

where $\chi_{[x,+\infty[}$ is the characteristic function defined for all $x\in\overline{I}$ by

$$\chi_{[x,+\infty[}(t)=\begin{cases}1 \text{ if } t\in[x,+\infty[,\\ 0 \text{ if } t\notin[x,+\infty[.\end{cases}$$

Which implies that

$$|u(x)-\xi|\le\frac{1}{3!}\int_0^{+\infty}\chi_{[x,+\infty[}(t)\,t^3\left|f\big(t,u(t),u''(t)\big)\right|dt,$$

and by using the hypothesis (H) and the Lebesgue Dominated Convergence Theorem, we obtain

$$\lim_{x\to+\infty} |u(x) - \xi| = 0.$$

Which means that

$$\lim_{x\to+\infty} u(x) = \xi.$$

Similarly, we can prove that

$$\begin{aligned}
\lim_{x\to+\infty} u'(x) &= \lim_{x\to+\infty} (Tu)'(x) \\
&= \lim_{x\to+\infty} \frac{1}{2} \int_x^{+\infty} (t-x)^2 f\left(t, u(t), u''(t)\right) dt \\
&= 0,
\end{aligned}$$

and

$$\begin{aligned}
\lim_{x\to+\infty} u''(x) &= \lim_{x\to+\infty} (Tu)''(x) \\
&= \lim_{x\to+\infty} \int_x^{+\infty} (t-x) f\left(t, u(t), u''(t)\right) dt \\
&= 0.
\end{aligned}$$

Suppose now that u is a solution of problem (1.1) in $\overline{K}_m$, then by applying the operator I_-^4 to the both sides of the equation in (1.1), we have

$$\left(I_-^4 u^{(4)}\right)(x) = \left(I_-^4 \widetilde{g}\right)(x).$$

From Lemma 8, we obtain

$$u(x) - \xi = \frac{1}{3!} \int_x^{+\infty} (t-x)^3 f\left(t, u(t), u''(t)\right) dt.$$

That is

$$u(x) = (Tu)(x).$$

Which means that u is a fixed point of T in $\overline{K}_m$. □

Lemma 21. *Suppose the hypothesis (H) is satisfied, and there exists a constant $M > 0$ such that*

$$AL_1(M) + BL_2(M) + C \leq \xi,$$

then

$$T\left(\overline{K}_M\right) \subset K.$$

Proof. Let $u \in \overline{K}_M$ and $x \in \overline{I}$, then by using the hypothesis (H), one has

$$\begin{aligned}
(Tu)(x) &\geq \xi - \frac{1}{6}\int_x^{+\infty}(t-x)^3\left[a(t)L_1(u(t)) + b(t)L_2\left(u''(t)\right) + c(t)\right]dt \\
&\geq \xi - L_1(M)\left(I_-^4 a\right)(x) - L_2(M)\left(I_-^4 b\right)(x) - \left(I_-^4 c\right)(x) \\
&\geq \xi - L_1(M)\left(I_-^4 a\right)(0) - L_2(M)\left(I_-^4 b\right)(0) - \left(I_-^4 c\right)(0) \\
&\geq \xi - [L_1(M)A + L_2(M)B + C] \\
&\geq 0.
\end{aligned}$$

□

Theorem 22. *Suppose the condition (H) is satisfied, and there exist $M_1 > 0$ and $M_2 > 0$ such that*

(H1) $M_1 < \xi < M_2$,

(H2) $AL_1(M_1) + BL_2(M_1) + C \leq \xi - M_1$,

(H3) $AL_1(M_2) + BL_2(M_2) + C \leq \min(\xi, M_2 - \xi)$.

(H4) $A_1L_1(M_2) + B_1L_2(M_2) + C_1 \leq 2M_2$,

(H5) $A_2L_1(M_2) + B_2L_2(M_2) + C_2 \leq M_2$.

Then the problem (1.1) admits at least one positive solution such that

$$M_1 \leq \|u\|_2 \leq M_2.$$

Proof. Since $AL_1(M_2) + BL_2(M_2) + C \leq \xi$, and using Lemma 21, we have

$$T\left(\overline{K}_{M_2} \backslash K_{M_1}\right) \subset K$$

Now, we are going to prove that the operator T is completely continuous. The proof will be given in three steps.

Claim 1: The set $T\left(\overline{K}_{M_2} \backslash K_{M_1}\right)$ is uniformly bounded.

Let $u \in \overline{K}_{M_2} \backslash K_{M_1}$, then for all $x \in \mathbb{R}^+$, we have

$$\begin{aligned}
|(Tu)(x)| &= \left|\xi + \frac{1}{3!}\int_x^{+\infty}(t-x)^3 f\left(t, u(t), u''(t)\right)dt\right| \\
&\leq |\xi| + \frac{1}{3!}\int_x^{+\infty}(t-x)^3\left|f\left(t, u(t), u''(t)\right)\right|dt \\
&\leq \xi + L_1(M_2)\left(I_-^4 a\right)(x) + L_2(M_2)\left(I_-^4 b\right)(x) + \left(I_-^4 c\right)(x) \\
&\leq \xi + L_1(M_2)\left(I_-^4 a\right)(0) + L_2(M_2)\left(I_-^4 b\right)(0) + \left(I_-^4 c\right)(0) \\
&= \xi + AL_1(M_2) + BL_2(M_2) + C.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}\left|(Tu)'(x)\right| &= \left|\frac{1}{2}\int\limits_x^{+\infty}(t-x)^2 f\left(t,u(t),u''(t)\right)dt\right| \\ &\leq A_1L_1(M_2)+B_1L_2(M_2)+C_1,\end{aligned}$$

and

$$\begin{aligned}\left|(Tu)''(x)\right| &= \left|\int\limits_x^{+\infty}(t-x) f\left(t,u(t),u''(t)\right)dt\right| \\ &\leq A_2L_1(M_2)+B_2L_2(M_2)+C_2.\end{aligned}$$

Which means that the set $T\left(\overline{K}_{M_2}\backslash K_{M_1}\right)$ is uniformly bounded.

Claim 2 The set $T\left(\overline{K}_{M_2}\backslash K_{M_1}\right)$ is equiconvergent at $+\infty$.

By using the Lebesgue Dominated Convergence Theorem (see the proof of Lemma 20), one has

$$\lim_{x\to+\infty}(Tu)(x)=\xi.$$

Which means that

$$\forall\varepsilon>0,\ \exists\,\delta_2(\varepsilon)>0,\forall x>\delta_2(\varepsilon),\ \left|(Tu)(x)-\lim_{x\to+\infty}(Tu)(x)\right|<\varepsilon. \tag{3.1}$$

Similarly, we have

$$\forall\varepsilon>0,\ \exists\,\widetilde{\delta}(\varepsilon)>0,\forall x>\widetilde{\delta}(\varepsilon),\ \left|(Tu)'(x)\right|<\varepsilon, \tag{3.2}$$

and

$$\forall\varepsilon>0,\ \exists\,\widehat{\delta}(\varepsilon)>0,\forall x>\widehat{\delta}(\varepsilon),\ \left|(Tu)''(x)\right|<\varepsilon. \tag{3.3}$$

Then from (3.1), (3.2) and (3.3), it follows that the set $T\left(\overline{K}_{M_2}\backslash K_{M_1}\right)$ is equiconvergent at $+\infty$

Claim 3 The set $T\left(\overline{K}_{M_2}\backslash K_{M_1}\right)$ is equicontinuous on each compact interval of $\mathbb{R}^+$.

Let $u\in T\left(\overline{K}_{M_2}\backslash K_{M_1}\right)$ and $0\leq x_1\leq x_2$, then we have

$$\begin{aligned}&|(Tu)(x_2)-(Tu)(x_1)| \\ &= \tfrac{1}{3!}\left|\int\limits_{x_2}^{+\infty}(t-x_2)^3\widetilde{g}(t)\,dt-\int\limits_{x_1}^{+\infty}(t-x_1)^3\widetilde{g}(t)\,dt\right| \\ &= \tfrac{1}{3!}\left|\int\limits_{x_2}^{+\infty}((t-x_2)^3-(t-x_1)^3)\widetilde{g}(t)\,dt-\int\limits_{x_1}^{x_2}(t-x_1)^3\widetilde{g}(t)\,dt\right| \\ &\leq \tfrac{1}{3!}\int\limits_{x_2}^{+\infty}((t-x_2)^3-(t-x_1)^3)\left|\widetilde{g}(t)\right|dt+\tfrac{1}{3!}\int\limits_{x_1}^{x_2}(t-x_1)^3\left|\widetilde{g}(t)\right|dt.\end{aligned}$$

That is

$$\left|(Tu)(x_2)-(Tu)(x_1)\right| \le F_1(x_1,x_2)+F_2(x_1,x_2), \tag{3.4}$$

where

$$F_1(x_1,x_2)=\frac{1}{3!}\int_{x_2}^{+\infty}\left[(t-x_2)^3-(t-x_1)^3\right]\left|\widetilde{g}(t)\right|dt,$$

and

$$F_2(x_1,x_2)=\frac{1}{3!}\int_{x_1}^{x_2}(t-x_1)^3\left|\widetilde{g}(t)\right|dt.$$

On the other hand, we have

$$\begin{aligned}
&F_1(x_1,x_2)\\
&\le \frac{1}{3!}\int_0^{+\infty}((t-x_2)^3-(t-x_1)^3)(a(t)L_1(u(t))+b(t)L_2(u''(t))+c(t))dt\\
&\le \frac{L_1(M_2)}{3!}\int_0^{+\infty}((t-x_2)^3-(t-x_1)^3)a(t)\,dt\\
&+\frac{L_2(M_2)}{3!}\int_0^{+\infty}((t-x_2)^3-(t-x_1)^3)b(t)\,dt\\
&+\frac{1}{3!}\int_0^{+\infty}((t-x_2)^3-(t-x_1)^3)c(t)\,dt.
\end{aligned}$$

Since

$$\int_0^{+\infty}(t-x_1)^3\,a(t)\,dt \le \int_0^{+\infty}t^3a(t)\,dt<+\infty,$$

and

$$\lim_{x_1\to x_2}(t-x_1)^3\,a(t)=(t-x_2)^3\,a(t),$$

then by using the Lebesgue Dominated Convergence Theorem, it follows that

$$\lim_{x_1\to x_2}\int_0^{+\infty}(t-x_1)^3\,a(t)\,dt=\int_0^{+\infty}(t-x_2)^3\,a(t)\,dt.$$

Similarly, we have

$$\lim_{x_1\to x_2}\int_0^{+\infty}(t-x_1)^3\,b(t)\,dt=\int_0^{+\infty}(t-x_2)^3\,b(t)\,dt,$$

and

$$\lim_{x_1\to x_2}\int_0^{+\infty}(t-x_1)^3\,c(t)\,dt=\int_0^{+\infty}(t-x_2)^3\,c(t)\,dt.$$

Then if we put by definition $\delta_3(\varepsilon) = \frac{\varepsilon}{3}(L_1(M_2) + L_2(M_2) + 1)$, we obtain

$$\forall \varepsilon > 0,\ \exists\ \delta_3(\varepsilon) > 0,\ (|x_1 - x_2| < \delta_3(\varepsilon) \Rightarrow F_1(x_1, x_2)) < \varepsilon). \tag{3.5}$$

On the other hand, we have

$$\begin{aligned} \int_{x_1}^{x_2} (t - x_1)^3 \ \ |\widetilde{g}(t)|\, dt &\leq \sup_{t\in\mathbb{R}^+} |\widetilde{g}(t)| \int_{x_1}^{x_2} (t - x_1)^3\ \ dt \\ &\leq \sup_{t\in\mathbb{R}^+} |\widetilde{g}(t)|\, (x_2 - x_1)^4. \end{aligned}$$

Then if we put by definition $\delta_4(\varepsilon) = 6\left(\frac{\varepsilon}{\sup_{t\in\overline{I}} |\widetilde{g}(t)| + 1}\right)^{\frac{1}{4}}$, we obtain

$$\forall \varepsilon > 0,\ \exists\ \delta_4(\varepsilon) > 0,\ (|x_1 - x_2| < \delta_4(\varepsilon) \Rightarrow F_2(x_1, x_2)) < \varepsilon). \tag{3.6}$$

Then from (3.4), (3.5) and (3.6) and if we put $\delta_5(\varepsilon) = \min(\delta_3(\varepsilon), \delta_4(\varepsilon))$, we obtain

$$\forall \varepsilon > 0,\ \exists\ \delta_5(\varepsilon) > 0,\ (|x_1 - x_2| < \delta_5(\varepsilon) \Rightarrow |(Tu)(x_2) - (Tu)(x_1)|) < \varepsilon).$$

Similarly, we can prove that

$$\forall \varepsilon > 0,\ \exists\ \delta_6(\varepsilon) > 0,\ \left(|x_1 - x_2| < \delta_6(\varepsilon) \Rightarrow \left|(Tu)'(x_2) - (Tu)'(x_1)\right|\right) < \varepsilon\right),$$

and

$$\forall \varepsilon > 0,\ \exists\ \delta_7(\varepsilon) > 0,\ \left(|x_1 - x_2| < \delta_7(\varepsilon) \Rightarrow \left|(Tu)''(x_2) - (Tu)''(x_1)\right|\right) < \varepsilon\right).$$

Then it follows that the set $T\left(\overline{K}_{M_2} \backslash K_{M_1}\right)$ is equicontinuous on each compact interval of $\mathbb{R}^+$ and from **Claim** 1, **Claim 2** and **Claim 3** and by using proposition 13 we deduce that the set $T\left(\overline{K}_{M_2} \backslash K_{M_1}\right)$ is relatively compact.

Now, we are going to prove that the operator T is continuous. Indeed, let $u \in T\left(\overline{K}_{M_2} \backslash K_{M_1}\right)$ and (u_n) an arbitrary sequence in $T\left(\overline{K}_{M_2} \backslash K_{M_1}\right)$ with $\lim\limits_{n\to+\infty} u_n = u$, then we have $\lim\limits_{n\to+\infty} u_n(x) = u(x)$ and $\lim\limits_{n\to+\infty} u_n''(x) = u''(x)$ for $x \in \mathbb{R}^+$ and by using the Lebesgue Dominated Convergence Theorem, we have

$$\lim_{n\to+\infty} \frac{1}{3!} \int_x^{+\infty} (t - x)^3 f\left(t, u_n(t), u_n''(t)\right) dt = \frac{1}{3!} \int_x^{+\infty} (t - x)^3 f\left(t, u(t), u''(t)\right) dt,$$

and then for all $x \in \overline{I}$, we have the pointwise convergence

$$\lim_{n\to+\infty} (Tu_n)(x) = (Tu)(x).$$

Now, we are going to prove that

$$\lim_{n \to +\infty} Tu_n = Tu.$$

Let (v_n) be a subsequence of (Tu_n). Since $T\left(\overline{K}_{M_2} \backslash K_{M_1}\right)$ is relatively compact, then there exists a subsequence (v_{n_k}) of (v_n) and a function $y \in E$ such that $\lim\limits_{n_k \to +\infty} v_{n_k} = y$. By uniqueness of the limit we have $y = Tu$. which means that $\lim\limits_{n \to +\infty} Tu_n = Tu$.

Now, let $u \in K$ with $\|u\|_2 = M_1$, then $0 \leq u^{(i)}(x) \leq M_1$, for $i = 0, 1, 2$ and all $x \in \mathbb{R}^+$.

Since by the hypotheses (H2) and (H), we have

$$AL_1(M_1) + BL_2(M_1) + C \leq \xi - M_1,$$

and

$$f\ \left(x, u(x), u''(x)\right) \leq a(x) L_1(u(x)) + b(x) L_2\left(u''(x)\right) + c(x),$$

we obtain

$$\begin{aligned}
(Tu)(x) &= \xi + \frac{1}{3!}\int_x^{+\infty} (t-x)^3 f\left(t, u(t), u''(t)\right) dt \\
&\geq \xi - \frac{1}{3!}\int_x^{+\infty} (t-x)^3 \left[a(t) L_1(u(t)) + b(t) L_2\left(u''(t)\right) + c(t)\right] dt \\
&\geq \xi - \frac{1}{3!}\int_0^{+\infty} (t-x)^3 \left[a(t) L_1(u(t)) + b(t) L_2\left(u''(t)\right) + c(t)\right] dt \\
&\geq \xi - \left[AL_1(M_1) + BL_2(M_1) + C\right] \\
&\geq M_1 = \|u\|_2 .
\end{aligned}$$

Which implies that

$$\|Tu\|_2 \geq \|u\|_2 .$$

Similarly let $u \in K \cap \left(\overline{K}_{M_2} \backslash K_{M_1}\right)$ with $\|u\|_2 = M_2$, then $0 \leq u^{(i)}(x) \leq M_2$, for $i = 0, 1, 2$ and all $x \in \mathbb{R}^+$.

Since by the hypotheses (H3) and (H), we have

$$AL_1(M_2) + BL_2(M_2) + C \leq \min(\xi, M_2 - \xi),$$

and

$$f\ \left(x, u(x), u''(x)\right) \geq -(a(x) L_1(u(x)) + b(x) L_2\left(u''(x)\right) + c(x)),$$

we obtain

$$\begin{aligned}
(Tu)(x) &= \xi + \frac{1}{3!}\int_x^{+\infty} (t-x)^3 f\left(t, u(t), u''(t)\right) dt \\
&\leq \xi + \frac{1}{3!}\int_x^{+\infty} (t-x)^3 \left[a(t) L_1(u(t)) + b(t) L_2\left(u''(t)\right) + c(t)\right] dt \\
&\leq \xi + \frac{1}{3!}\int_0^{+\infty} (t-x)^3 \left[a(t) L_1(u(t)) + b(t) L_2\left(u''(t)\right) + c(t)\right] dt \\
&\leq \xi + \left[AL_1(M_2) + BL_2(M_2) + C\right] \\
&\leq \xi + \min(\xi, M_2 - \xi) \\
&\leq M_2 = \|u\|_2 .
\end{aligned}$$

That is

$$(Tu)(x) \leq M_2 = \|u\|_2 , \text{ for all} x \in \overline{I}. \tag{3.7}$$

Similarly, we have

$$\begin{aligned}
(Tu)'(x) &= \frac{1}{2}\int_x^{+\infty} (t-x)^2 f\left(t, u(t), u''(t)\right) dt \\
&\leq \frac{1}{2}\int_x^{+\infty} (t-x)^3 \left[a(t) L_1(u(t)) + b(t) L_2\left(u''(t)\right) + c(t)\right] dt \\
&\leq \frac{1}{2}\int_0^{+\infty} (t-x)^2 \left[a(t) L_1(u(t)) + b(t) L_2\left(u''(t)\right) + c(t)\right] dt \\
&\leq \frac{A_1L_1(M_2) + B_1L_2(M_2) + C_1}{2} \\
&\leq \frac{2M_2}{2} \\
&= M_2 = \|u\|_2 .
\end{aligned}$$

That is

$$(Tu)'(x) \leq M_2 = \|u\|_2 , \text{ for all} x \in \mathbb{R}^+, \tag{3.8}$$

and

$$\begin{aligned}(Tu)''(x) &= \int_x^{+\infty} (t-x)\, f\left(t, u(t), u''(t)\right) dt \\ &\leq \int_x^{+\infty} (t-x)\left[a(t)\, L_1(u(t)) + b(t)\, L_2\left(u''(t)\right) + c(t)\right] dt \\ &\leq \int_0^{+\infty} (t-x)^2 \left[a(t)\, L_1(u(t)) + b(t)\, L_2\left(u''(t)\right) + c(t)\right] dt \\ &\leq A_2 L_1(M_2) + B_2 L_2(M_2) + C_2 \\ &= M_2 = \|u\|_2 .\end{aligned}$$

That is

$$(Tu)''(x) \leq M_2 = \|u\|_2 , \text{ for all} x \in \mathbb{R}^+. \tag{3.9}$$

Then from (3.7), (3.8) and (3.9), it follows that

$$\|Tu\|_2 \leq \|u\|_2 ,$$

and consequently by Theorem 15, we obtain that the problem (1.1) admits at least one positive solution u such that

$$M_1 \leq \|u\|_2 \leq M_2.$$

□

3.1 Application

In this section, we give an example illustrating the application of our result.

$$\begin{cases} u^{(4)}(x) = \frac{1}{2}\left(e^{-x} - e^{-u(x)} - \dfrac{e^{(u''(x))^2}}{1+(u''(x))^2}\right), \ x \in \mathbb{R}^+, \\ \lim\limits_{x\to+\infty} u(x) = \sqrt{2}, \\ \lim\limits_{x\to+\infty} u^{(i)}(x) = 0 \text{ for } i = 1,2,3. \end{cases} \tag{3.10}$$

If we put by definition

$$f(x,u,v) = \frac{1}{2}\left(e^{-x} - e^{-u} - \frac{e^{v^2}}{1+v^2}\right).$$

We have

$$|f(x,u,v)| \leq a(x)\, L_1(u) + b(x)\, L_2(v) + c(x),$$

where
$$L_1(u) = 1 - e^{-u}, L_2(v) = 1 - \frac{1}{1+v^2},$$
and
$$a(x) = b(x) = c(x) = \frac{e^{-x}}{6}.$$
Also, we have
$$A = B = C = \frac{1}{3!}\int_0^{+\infty} \frac{1}{6} t^3 e^{-t} dt = \frac{1}{6},$$
$$A_1 = B_1 = C_1 = \frac{1}{2}\int_0^{+\infty} \frac{1}{6} t^2 e^{-t} dt = \frac{1}{6},$$
and
$$A_2 = B_2 = C_2 = \int_0^{+\infty} \frac{1}{6} t e^{-t} dt = \frac{1}{6}.$$
We are going to prove that the terminal value problem (3.10) has at least one positive solution u such that
$$M_1 = 1 \le \|u\|_2 \le 2 = M_2.$$
First it is easy to see that the functions L_1 and L_2 are increasing on $\mathbb{R}^+$.
On the other hand, we have
$$AL_1(M_1) + BL_2(M_1) + C = \frac{1}{6}\left(1 - \frac{1}{e}\right) + \frac{1}{6}\left(1 - \frac{1}{3}\right) + \frac{1}{6} = 0.38313,$$
and
$$\xi - M_1 = \sqrt{2} - 1 = 0.41421.$$
Which means that
$$AL_1(M_1) + BL_2(M_1) + C \le \xi - M_1.$$
Similarly, we have
$$AL_1(M_2) + BL_2(M_2) + C = 0.45893 \le \min(\xi, M_2 - \xi) = 0.58579,$$
$$A_1L_1(M_2) + B_1L_2(M_2) + C_1 = 0.45893 \le 4,$$
and
$$A_2L_1(M_2) + B_2L_2(M_2) + C_2 = 0.45893 \le 2.$$
Then by Theorem 22, it follows that the problem (3.10) admits at least one positive solution u such that
$$1 \le \|u\|_2 \le 2.$$

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Mohammed Dahmane
Dynamic Systems and Applications Laboratory,
Department of Mathematics, Faculty of Sciences,
University Abou-Bekr Belkaid Tlemcen, B.P.119, Tlemcen, 13000, Algeria.
e-mail: dahmo_md@yahoo.fr

Mohammed Derhab
Dynamic Systems and Applications Laboratory,
Department of Mathematics, Faculty of Sciences,
University Abou-Bekr Belkaid Tlemcen, B.P.119, Tlemcen, 13000, Algeria.
e-mail: derhab@yahoo.fr

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Received: August 24, 2021; Accepted: December 30, 2023;
Published: December 31, 2023.