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ALMOST PERIODIC AND QUASI-PERIODIC FUNCTIONS. A BRIEF SURVEY AND SOME APPLICATIONS

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Abstract. In the present work, we study almost-periodic and quasi-periodic functions, their main properties and some related results. We also present almost periodic functions with parameters, which we adapt to the quasi-periodic framework. We then show that these spaces are isomorphic to other spaces of almost periodic functions (or quasi-periodic functions) with values in Banach spaces. We shall study in particular the superposition operators between these spaces. We introduce then a Sobolev-like space of almost periodic functions and analyze its main properties. We illustrate by some applications especially on convolution operation and almost periodic solutions of linear differential equations

1 Introduction

The periodic and, more general, the oscillatory functions/motions appeared in Science and Engineering and other fields of knowledge, have conducted to the development of classical Fourier Analysis of periodic functions and their associated series.

While the first traces of this branch of classical analysis can be found in the Mathematics of the XVIII-th century (Euler, for instance), it is the XIX-th century that contains significant results, which stimulated substantially the birth of new theories, contributing vigorously to the new concepts of Modern Analysis (Set Theory, Real variables including Measure and Integral). The Fourier Analysis, as developed until the third decade of the XX-th century, has known a strong impulse due to the emerging of the concept of Almost Periodicity, due to H. Bohr (1923-25), and successfully continued to the present day. It is also true that the topics of classical Fourier Analysis have also kept the attention of many leading mathematicians, after the birth of almost periodic functions.

The well known treatises of N.K. Bary (Pergamon, 1964) and A. Zygmund (Cambridge Univ. Press, 2002) contain a wealth of results and information about the periodic functions and their Fourier series, specially obtained before the introduction

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of the methods of Functional Analysis.

Naturally, almost periodic functions (a.p.) appear as soon as several simultaneous periodic movements are interlaced (for example two springs of different elasticities attached to two different masses). Indeed, they are not functions which are nearly periodic, but are functions which have many almost periods. In particular, almost periodic functions are not disturbances of periodic functions, but can be seen as superpositions of periodic functions which have no common period. They formalize oscillations whose frequencies can not be reduced to a single basic frequency. Therefore, the concept of almost periodic oscillations is an idea worth considering because it corresponds better to the physical phenomena since the periodicity is too utopic.

Usually, in the courses of mechanics (as an example), we encounter some two-dimensional differential systems of the form $x' = Ax + f(t)$, where A is a 2×2 matrix with purely imaginary eigenvalue and $f(\cdot)$ is a periodic exterior force. For the periodic case, it is well established that when this forced system possesses a periodic oscillation, then the period of this oscillation is exactly the period of the exterior force.

The concept of almost periodicity had several leading contributors to its beginning period. In his famous treatise *Nouvelles Méthodes de la Mécanique Céleste* (1893), H. Poincaré considered the problem of developing a function in a series of sine functions, namely $f(t) = \displaystyle\sum_{k=1}^{\infty} f_k \sin \lambda_k t$, for $t \in \mathbb{R}$, where λ_k's are arbitrary real numbers, not necessarily like $\lambda_k = k\omega$, $k \in \mathbb{N}$, $\omega > 0$. Poincaré has succeeded to obtain the coefficients f_k, $k \geq 1$, simultaneously introducing the mean value of a function on the whole real line.

Using the complex notations, which became common with the new concept of almost periodicity, the previous formula can be rewritten as $f(t) = \displaystyle\sum_{k=1}^{\infty} f_k \exp(i\lambda_k t)$, where $t \in \mathbb{R}$, $f_k \in \mathbb{C}$ and $\lambda_k \in \mathbb{R}$, $k \geq 1$. The coefficients f_k are determined by the formulas $f_k = \displaystyle\lim_{\ell\to\infty} (2\ell)^{-1} \int_{-\ell}^{\ell} f(t) \exp(-i\lambda_k t)dt$, in which the Poincaré's mean value (i.e., on an infinite interval) appears: $\mathcal{M}(g) = \displaystyle\lim_{\ell\to\infty} (2\ell)^{-1} \int_{-\ell}^{\ell} g(t)dt$, where g is locally integrable on $\mathbb{R}$, under the assumption that the limit exists as a finite number.

It is known that most concepts related to almost periodicity, including the Fourier exponents and coefficients are based on the mean value.

In 1933, Bochner [4] defined and studied the almost periodic functions with values in Banach spaces. He showed that these functions include certain earlier generalizations of the notion of almost periodic functions. Some extensions of Bohr's concept have been introduced, most notably by A. S. Besicovitch, V. V. Stepanov H. Weyl and Eberlein. One can remark that, speaking about Stepanov, Weyl or Besicovitch metrics implicitly means to deal with the related quotient spaces, because otherwise we should rather speak about Stepanov, Weyl or Besicovitch..

Other early contributors, preceding the period initiated by H. Bohr [6, 7, 8], include P. Bohl (1893) and E. Esclangon (1919) who dealt with what was later called quasi-periodic functions, a special case of almost periodicity. They have investigated oscillatory functions with a finite number of frequencies, the periodic case being concerned with only one basic frequency $(2\pi/\omega)$, ω-period. Some methods encountered to P. Bohl, but particularly to E. Esclangon, have been adapted to the general case of almost periodicity by H. Bohr.

In this direction, almost periodicity has been recently an attractive topic for many researchers and works. We can refer for instance to [1, 3, 5, 13, 17, 18, 19, 15, 21, 22, 25] and the references therein. For instance, as concerning the framework of the dissipative differential equations, we can refer to the books of Fink [16]. We refer also the reader to [10, 12] which deal with the KAM aspects.

As aforementioned, the notion and properties of almost periodic functions, either in their initial, or in generalized form turned out to be of great importance in various fields of analysis, function theory, topology, and applied mathematics. The necessity of a manuscript giving a concise and systematic exposition of the fundamentals of the theory of almost periodic functions was becoming more and more obvious. In particular, the spaces of almost periodic in Bohr's sense and in Besicovitch sense have shown a great efficiency in studying of differential equations and partial differential equations, especially because Sobolev-like spaces of function have been introduced and used to study the almost and the quasi periodicity of solutions for many differential systems.

This work is organized in four sections. In Section 2, we shall introduce the almost periodic functions and the quasi-periodic ones. We study and recall their fundamental properties. In Section 3, we extend the study to the framework with parameters. We analyzed the almost periodic and the quasi-periodic functions with parameters. Namely, we study the properties of the superposition operators between the introduced functional spaces. Section 4 is devoted to the study of the space of almost periodic and quasi periodic functions in the sense of Besicovitch. We introduce especially a space of Sobolev-type analogous to the classical Sobolev space for the periodic case. In a final fourth Section, we illustrate by some applications. Namely, we deal with convolution product and its link to the simplest differential equation $x'(t) + \alpha x(t) = f(t)$, when f is taken as an almost periodic forcing term and we deal with the linear autonomous differential system $x'(t) = Ax(t) + f(t)$, where A is a constant square matrix and f is an almost periodic function.

2 Almost periodic and quasi periodic functions

Throughout this manuscript, we denote by $\mathbb{N}$ or by $\mathbb{Z}_+$ the set of all positive integers and by $\mathbb{Z}$ the set of all integers. The set of real numbers is denoted by $\mathbb{R}$ and that of all complex numbers is denoted by $\mathbb{C}$. We use the notation $\Re(z)$ to denote the real part of the complex number z.

$(\mathbb{E}, |\cdot|_{\mathbb{E}})$ denotes a Banach space. When $\mathbb{E}$ is a Hilbert space, we often use the same notation, replacing $\mathbb{E}$ by $\mathbb{H}$.

Here and in the sequel, we use the notation $e_\nu : \mathbb{R}^k \to \mathbb{C}$ (when $\nu \in \mathbb{R}^k$, where $k \in \mathbb{N}$) to denote the function defined for all $x \in \mathbb{R}^k$, by $e_\nu(x) := e^{i\nu\cdot x}$.

As usually, we denote by $\mathcal{C}^0$ the class of continuous functions and by BC the set of bounded continuous functions.

For $k \in \mathbb{N}$, $\mathcal{C}^k$ the set of all functions which are k-times differentiable and its k-the differential is continuous. We denote by $\mathcal{C}^\infty$ the set of indefinitely differentiable functions. We denote also by $L^1(\mathbb{R}, \mathbb{C})$ the space of integrable function defined on $\mathbb{R}$, that is, the set of all functions $f : \mathbb{R} \to \mathbb{C}$ such that $\displaystyle\int_{\mathbb{R}} |f(t)|dt$ is finite.

2.1 Space of almost periodic functions

Definition 1. *A continuous function $f : \mathbb{R} \longrightarrow \mathbb{E}$ is said to be almost-periodic (in Bohr's sense) if for all $\varepsilon > 0$, there exists $\ell > 0$, such that for all $a \in \mathbb{R}$, we can find a number $\tau \in [a, a + \ell]$ verifying for all $x \in \mathbb{R}$,*

$$|f(x + \tau) - f(x)|_{\mathbb{E}} < \varepsilon.$$

Such a number τ is called an almost period attached to ε, and ℓ is said inclusion length attached to ε.

Theorem 2. *[1, VIII, p.9] Let $f : \mathbb{R} \to \mathbb{E}$ be a continuous function and let denote by $Trans(f)$ the following set : $Trans(f) := \{\tau_a f := f(\cdot + a) \quad : \quad a \in \mathbb{R}\}$.*
The function f is almost periodic if and only if the set $Trans(f)$ is relatively compact in the set $BC(\mathbb{R}, \mathbb{E})$ of bounded continuous functions from $\mathbb{R}$ to $\mathbb{E}$ endowed with the sup *norm on $\mathbb{R}$.*

As a consequence, we have immediately :

Proposition 3. *$f : \mathbb{R} \to \mathbb{E}$ is an almost periodic function if and only if it verifies the normality property, that is, from any sequence of translated $(\tau_{h_n} f)_n$, we can extract a sub-sequence uniformly converging on $\mathbb{R}$.*

We denote $AP^0(\mathbb{E})$ (or equivalently $AP^0(\mathbb{R}, \mathbb{E})$ as well as $AP(\mathbb{E})$ or $AP(\mathbb{R}, \mathbb{E})$) the set of all almost periodic functions from $\mathbb{R}$ into $\mathbb{E}$.

We notice that a trigonometric polynomial from $\mathbb{R}$ to $\mathbb{E}$ is a function from $\mathbb{R}$ to $\mathbb{E}$ having the following form: $P(t) = \sum_{j=1}^{N} a_j e^{i\lambda_j t}$, where $a_j \in \mathbb{E}$ and $\lambda_j \in \mathbb{R}$. In this writing, we carry the following resolution: $a_j e^{it\lambda_j} = e^{it\lambda_j} a_j$. When $\mathbb{E}$ is real, in order to get $P(t) \in \mathbb{E}$, we should have: for each j, there exists k such that $\lambda_k = -\lambda_j$ and $a_k = \bar{a}_j$ (complex conjugation). We denote by $TP(\mathbb{R}, \mathbb{E})$ the set of all trigonometric polynomials from $\mathbb{R}$ to $\mathbb{E}$. It is then clear that $TP(\mathbb{R}, \mathbb{E})$ is a vector subspace of the space of bounded continuous functions. For all these notions and questions, we refer the reader to [1, IX, p.10.].

Theorem 4. *[1, I, p.15.] The function $f : \mathbb{R} \to \mathbb{E}$ is almost periodic if and only if it satisfies the following approximation property : f is the uniform limit of a sequence of trigonometric polynomials, i.e. there exists a double sequence $(a_{n,p})_{(n,p)\in\mathbb{N}\times\mathbb{Z}}$ where for all n, $(a_{n,p})_{p\in\mathbb{Z}}$ is nearly zero, such that:*

$$\lim_{n\to+\infty} \left\| f - \sum_{p\in\mathbb{Z}} a_{n,p} e_p \right\|_\infty = 0.$$

Here, $\|f\|_\infty := \sup_{t\in\mathbb{R}} |f(t)|_{\mathbb{E}}$.

In other words, $f \in AP^0(\mathbb{E})$ if and only if for all $\varepsilon > 0$, we can find a $P \in TP(\mathbb{R}, \mathbb{E})$ such that $\sup_{t\in\mathbb{R}} |f(t) - P(t)|_{\mathbb{E}} \le \varepsilon$.

Remark 5.
1. *Each continuous periodic function is an almost periodic function.*
2. *Each trigonometric polynomial is an almost periodic function, that is, $TP(\mathbb{R}, \mathbb{E}) \subset AP^0(\mathbb{R}, \mathbb{E})$.*

Proposition 4 can be expressed differently as : " $AP^0(\mathbb{E})$ is the closure for the uniform norm of the vector space generated by the set of periodic continuous functions".

Then, equipped with the norm $\|f\|_\infty = \sup_{t\in\mathbb{R}} |f(t)|_{\mathbb{E}}$ is in particular a space of Banach.

Proposition 6. *[1, III, IV, V, VII, p.5–7.]*
$AP^0(\mathbb{E})$ has the following properties :

1. *Each function of $AP^0(\mathbb{E})$ is uniformly continuous and has a relatively compact image.*

2. *If $\mathbb{E}_i$, $i = 1, 2$ are two Banach spaces, and if $f \in AP^0(\mathbb{E}_1)$ and $F : \mathbb{E}_1 \longrightarrow \mathbb{E}_2$ is uniformly continuous on $\overline{f(\mathbb{R})}$, then, $F \circ f \in AP^0(\mathbb{E}_2)$. In particular, if $f \in AP^0(\mathbb{E})$, then, $[t \mapsto |f(t)|_{\mathbb{E}}] \in AP^0(\mathbb{R})$.*

3. *If $f \in AP^0(\mathbb{E})$ and $\varphi \in AP^0(\mathbb{K})$, then, $\varphi.f \in AP^0(\mathbb{E})$.*

4. *If for $i = 1, \cdots, p$, $f_i \in AP^0(\mathbb{E}_i)$, where $\mathbb{E}_i$ are Banach spaces, then, the p-uplet $(f_1, f_2, \cdots, f_p)$ belongs to $AP^0\left(\prod_{1 \leq i \leq p} \mathbb{E}_i\right)$. Especially, for all $\varepsilon > 0$, we can find a translation number common to all the functions $f_1, \cdots, f_p$.*

2.2 Fourier analysis of almost-periodic functions

To begin with, when $f \in AP^0(\mathbb{E})$, $\dfrac{1}{2T}\displaystyle\int_{-T}^{T} f(t)dt$ tends to a finite limit in $\mathbb{E}$ as T goes to $+\infty$, [1, II, p.21.].

Proposition 7. *Let $f \in AP^0(\mathbb{E})$. Then, the following limit exists in $\mathbb{E}$ and does not depend on $a \in \mathbb{R}$:*

$$\lim_{T \to +\infty} \frac{1}{T} \int_a^{a+T} f(t)dt.$$

This limit is called the mean of f and is denoted by $\mathcal{M}\{f\}$ or $\mathcal{M}\{f(t)\}_t$. It satisfies, in addition, the following property for all $L \in \mathbb{E}'$,

$$L \cdot_{\mathbb{E}' \times \mathbb{E}} \mathcal{M}\{f\} = \mathcal{M}\left\{L \cdot_{\mathbb{E}' \times \mathbb{E}} f\right\}.$$

Remarks 8. *1. The integral considered here is, like many others in this article, the integral of a valued function in a Banach space. We will always assume that they are integrals in the sense of Bochner-Lebesgue, or integrals in the strong sense (on these questions, see [14, 20]).*

2. When f is continuous and T-periodic, we have

$$\mathcal{M}\{f\} = \frac{1}{T} \int_0^T f(t)dt.$$

Now, for any $\lambda \in \mathbb{R}$ and for $f \in AP^0(\mathbb{E})$, the function $t \mapsto f(t)e^{-i\lambda t}$ belongs to $AP^0(\mathbb{E})$. This implies that this last function has a mean and then we have the following definition :

Definition 9. *For $f \in AP^0(\mathbb{E})$ and $\lambda \in \mathbb{R}$, we define the coefficient of Fourier-Bohr with order λ of f by :*

$$a_\lambda(f) = \mathcal{M}\{e_{-\lambda}.f\}.$$

Remark 10. *The Fourier-Bohr coefficients belong to the complexification $\mathbb{E}_{\mathbb{C}}$ of $\mathbb{E}$. When $\mathbb{K} = \mathbb{C}$, $\mathbb{E}_{\mathbb{C}} = \mathbb{E}$, which makes Fourier analysis easier to expose on $\mathbb{C}$. When the underlying scalar field is $\mathbb{R}$, we have the additional relations verified for all $\lambda \in \mathbb{R}$: $a_{-\lambda}(f) = \overline{a_\lambda(f)}$.*

Definition 11. *Let $f \in AP^0(\mathbb{E})$. We define :*

$$\Lambda(f) := \{\lambda \in \mathbb{R} \quad : \quad a_\lambda(f) \neq 0\}\,.$$

We notice that the set $\Lambda(f)$ is (at most) countable, [1, III, p22.].

Proposition 12. *[1, III, p.29.] Let $f \in AP^0(\mathbb{E})$. Then f can be developed in Fourier-Bohr series as*

$$f \sim \sum_{\lambda \in \mathbb{R}} a_\lambda e_\lambda,$$

where the convergence take place in quadratic mean (within the meaning of summable families).

Besides, if $\mathbb{E} = \mathbb{H}$, Parseval's relation is valid

$$\mathcal{M}\{|f|^2_{\mathbb{H}}\} = \sum_{\lambda \in \mathbb{R}} |a_\lambda|^2_{\mathbb{H}}.$$

When in particular $\mathbb{E} = \mathbb{R}$, we have also, [2, propriété 5, p.353, corollaire 2, p.357.],

Proposition 13.
1. *If $f \in AP^0(\mathbb{R})$ and $\lim_{t \to +\infty} f(t) = \ell$ ($\ell \in \mathbb{R}$), (respectively $-\infty$), then for all $t \in \mathbb{R}$, $f(t) = \ell$.*
2. *If $f \in AP^0(\mathbb{R})$ and if for all $t \in \mathbb{R}$, $f(t) \geq 0$, then $\mathcal{M}\{f(t)\}_t \geq 0$ and $\mathcal{M}\{f(t)\}_t = 0$ if and only if for all $t \in \mathbb{R}$, $f(t) = 0$.*

Remarks 14.
1. *The Fourier-Bohr series development of an almost periodic function is uniquely determined, in the sense that if*

$$f \sim \sum_{\lambda \in \mathbb{R}} \alpha_\lambda e_\lambda$$

then, $a_\lambda(f) = \alpha_\lambda$, for all λ.

2. *The space that performs the harmonic synthesis is not $AP^0(\mathbb{H})$ (it shall be introduced later).*

2.3 Bochner's polynomials.

Following Féjer's method on Cesaro-summability of Fourier series, Bochner explicitly constructed a sequence of trigonometric polynomials converging uniformly to a given function almost periodic.

The following exposition clarifies the presentation of [1, pp.25–29]. We may refer the reader also to [13, Theorem 6.15 p.152].

Let us take a function $f \in AP^0(\mathbb{E})$.

We define for $n \in \mathbb{Z}_+^*$ the kernel of Féjer :

$$K_n(t) := \sum_{\nu=-n}^{n} \left(1 - \frac{|\nu|}{n}\right) e^{-i\nu t} = \frac{1}{n} \left(\frac{\sin \frac{nt}{2}}{\sin \frac{t}{2}}\right)^2 .$$

Take also a family $B = (\beta_k)_{k \in I}$ (I denotes here an interval of $\mathbb{Z}_+$ starting by 1) $\mathbb{Z}$-linearly independent so that any element of $\Lambda(f)$ is a linear combination with rational coefficients of elements of B.

We define the Bochner polynomial as :

$$Q_{n_1 \cdots n_k}^{\alpha_1 \cdots \alpha_k}(f)(t) := \mathcal{M}\left\{ f(s+t) \prod_{j=1}^{k} K_{n_j}(\alpha_j s) \right\}_s .$$

When B is infinite, we set :

$$f_m := Q_{(m!)^2 \cdots (m!)^2}^{\frac{\beta_1}{m!} \cdots \frac{\beta_m}{m!}}(f)$$

and if B is finite with cardinal q, we set for $m \geq q$:

$$f_m := Q_{(m!)^2 \cdots (m!)^2}^{\frac{\beta_1}{m!} \cdots \frac{\beta_q}{m!}}(f).$$

So, we prove that the sequence $(f_m)_m$ converges uniformly to f.

Proposition 15. *Let $(a_\lambda)_{\lambda \in \mathbb{R}}$ be a summable family of $\mathbb{E}$ elements. Then the following sum :*

$$\sum_{\lambda \in \mathbb{R}} a_\lambda e_\lambda$$

has a meaning and defines an almost-periodic function.

Proof. The summability of a family is equivalent to its absolute summability. Since $|e_\lambda| = 1$, the existence of the sum is obvious. We denote by f the function defined by this sum.

Let $(A_n)_n$ be an increasing sequence of finite subsets of $\mathbb{R}$ whose union forms $\Lambda(f)$ (such a sequence exists because $\Lambda(f)$ is at most countable). We denote so :

$$x_n = \sum_{\lambda \in A_n} a_\lambda e_\lambda .$$

The sequence $(x_n)_n$ is a sequence of trigonometric polynomials converging uniformly to f, since:

$$\|f - x_n\|_\infty \leq \sum_{\lambda \notin A_n} |a_\lambda|_{\mathbb{E}} \longrightarrow 0 \quad \text{as} \quad n \to +\infty.$$

We deduce that f is almost periodic.

□

2.4 Differentiation and integration of almost-periodic functions

When a periodic function is differentiable, its derivative is automatically periodic.

As far as almost-periodic functions are concerned, this is not true, since nothing ensures that the derivative is uniformly continuous, which is necessary to be almost-periodic. Indeed, a remarkable result ensures that this condition is sufficient:

Proposition 16. *[1, VI, p.6.] Let $f \in AP^0(\mathbb{E})$ be a differentiable. Then, the derivative is almost-periodic if and only if it is uniformly continuous.*

Notation 17. *We denote, for all integer $k \in \mathbb{N} \cup \{+\infty\}$:*

$$AP^k(\mathbb{E}) := \Big\{ f \in C^k(\mathbb{R}, \mathbb{E}) : \quad \forall j \leq k, f^{(j)} \in AP^0(\mathbb{E}) \Big\}.$$

For a continuous periodic function, the zero-mean condition ensures that the primitives are almost-periodic.

For almost-periodic functions, this condition is not enough, as the following example shows :

Example 18. *Let us consider the function f defined as :*

$$f := \sum_{n \geq 1} \frac{e_{1/n}}{n^{1,1}}.$$

This function is almost periodic, due to Proposition 15, and its mean value is equal to zero, by uniqueness of the development. If its primitives were almost-periodic, their expansions would be (up to a constant):

$$\sum_{n \geq 1} \frac{e_{1/n}}{n^{0,1}}$$

which is not an almost periodic function since Parseval relation does not occur here.

In fact, the condition of relative compactness of the image of the primitives, which is necessary, is not automatically satisfied. It is remarkable that this condition is sufficient:

Proposition 19. *[1, I, II, p.53–55] Let $f \in AP^0(\mathbb{E})$ and F be a primitive of f. One of the following conditions ensures that the primitives are almost-periodic :*

1. *The rank of F is relatively compact.*
2. *F is bounded and $\mathbb{E}$ is uniformly convex, cf. [1, p.49.] and [11, p.51.] Uniformly convex spaces are reflexive; the spaces ℓ^p and L^p with $1 < p < +\infty$ and the Hilbert spaces are uniformly convex.*

The first statement is due to Bochner and the second one is due to Amerio, cf. [1, Chapter 4]. This chapter of this book contains other theorems.

2.5 Quasi-periodic functions

We have seen that if $f \in AP^0(\mathbb{E})$, the set $\Lambda(f)$ is at most countable. We denote by $Mod(f)$ the $\mathbb{Z}$-module that it generates, *i.e.* :

$$Mod(f) := \left\{ \sum_{p=0}^{n} k_p \lambda_p : \quad n \in \mathbb{N},\ k_p \in \mathbb{Z},\ \lambda_p \in \Lambda(f) \right\}.$$

Definition 20. *Let $k \in \mathbb{N}$ and $f \in AP^k(\mathbb{E})$. The function f is said to be a quasi-periodic function (up to the order k), and we denote $f \in QP^k(\mathbb{E})$ (or $QP^k(\mathbb{R}, \mathbb{E})$) (or $QP(\mathbb{E})$ or $QP(\mathbb{R}, \mathbb{E})$ if $k = 0$), when the family $Mod(f)$ is free of finite type, that is, there exists an integer $p \in \mathbb{N} \setminus \{0\}$ and $u_1, \cdots, u_p \in \mathbb{R}$ such that*

$$Mod(f) = \mathbb{Z}u_1 + \cdots + \mathbb{Z}u_p.$$

and $(\sum_i k_i u_i = 0, \quad k_i \in \mathbb{Z})$ implies that for all i, $\quad k_i = 0$.

Notation 21. *Let $f \in QP^0(\mathbb{E})$ and $\omega := (\omega_1, \cdots, \omega_m) \in \mathbb{R}^m$ such that its components form a basis on $\mathbb{Z}$ of $Mod(f)$ i.e. $Mod(f) = \mathbb{Z}\langle\omega\rangle$. So, we denote : $f \in QP^k_\omega(\mathbb{E}) = QP^k_\omega(\mathbb{R}, \mathbb{E})$ (or if $k = 0$, $QP_\omega(\mathbb{E}) = QP_\omega(\mathbb{R}, \mathbb{E})$).*

3 Almost-periodic and quasi-periodic functions with parameters

We shall recall here the definition of an almost periodic function uniformly with respect to a parameter and we shall prove the isomorphism with a space of almost periodic functions with values in a Banach space.

We adapt these considerations to the quasi periodic frame.

For two non empty sets X and Y, the notation Y^X is used to denote the set of all functions from X into Y. We denote also by $\mathcal{K}(X)$ the set of all compacts included in X.

3.1 General remarks on almost-periodic functions with parameters

Let X be a Banach space and P a subset of X satisfying one of the two conditions :

- P is compact (it will be denoted P instead K).
- There exists an exhaustive sequence of compacts, that is, an increasing sequence of compacts non empty $(K_n)_{n\geq 1}$ such that $P = \bigcup_{n\geq 1} K_n$ and for all n, $K_n \subset \operatorname{Int} K_{n+1}$.

  Here, for a set A, $\operatorname{Int} A$ denotes its interior, that is, the biggest open subset contained in A.

In the second case, we will say that P is countable at infinity.

Remark 22. *Every non-empty open set Ω of $\mathbb{R}^p$ is countable to infinity. We can indeed take:*

$$K_n := \{x \in \mathbb{R}^p \quad : \quad d(x, \Omega^c) \geq 1/n \quad \text{and} \quad |x| \leq n\}.$$

We take a function $f \in C^0(\mathbb{R}, P, \mathbb{E})$ and we define the Nemytskii operator built on f from $P^{\mathbb{R}}$ to $\mathbb{E}^{\mathbb{R}}$ as being for all $\varphi \in P^{\mathbb{R}}$

$$\mathcal{N}_f(\varphi) := [t \mapsto f(t, \varphi(t))] \in \mathbb{E}^{\mathbb{R}}.$$

Even if for all $\alpha \in P$, the function $f(\cdot, \alpha)$ is almost periodic, the Nemytskii operator does not always associate an almost-periodic periodic.

We need a certain uniformity with respect to α on the choice of the ℓ of the definition (see [16] and especially [26]). This is why we retain the following definition, whose natural aspect will be analyzed later from the point of view of functional spaces.

Definition 23. *[26] Let Q be a nonempty subset of X and $f \in C^0(\mathbb{R}, Q, \mathbb{E})$. We say that f is almost-periodic in t, uniformly with respect to α if :*

for all $\varepsilon > 0$, for all $K \in \mathcal{K}(Q)$, there exists $\ell > 0$ such that for all $a \in \mathbb{R}$, there exists $\tau \in [a, a + l)$, such that

$$\sup_{t \in \mathbb{R}} \sup_{\alpha \in K} |f(t + \tau, \alpha) - f(t, \alpha)|_{\mathbb{E}} \leq \varepsilon.$$

The class of all such functions shall be denoted by $APU(\mathbb{R}, Q, \mathbb{E})$.

This class of function responds well to the problem previously posed and we will see later, that in a certain sense, it is minimal.

Proposition 24. *[26, p.16.] Let Q be a non-empty subset of X and $f \in APU(\mathbb{R}, Q, \mathbb{E})$, and $\varphi \in AP^0(X)$ such that $\varphi(\mathbb{R}) \subset Q$. So :*

$$[t \mapsto f(t, \varphi(t))] \in AP^0(\mathbb{E}).$$

Proposition 25. *[26, Theorems 2.1 and 2.4, p.7, p.13] Each element of $APU(\mathbb{R}, K, \mathbb{E})$ is uniformly continuous and bounded.*

3.2 Almost-periodic functions with a compact set of parameters

Let us assume here that $P = K$ is a compact.

The space $APU(\mathbb{R}, K, \mathbb{E})$ is equipped with the norm :

$$\|f\|_{APU(\mathbb{R},K,\mathbb{E})} := \sup_{(t,\alpha) \in \mathbb{R} \times K} |f(t, \alpha)|_{\mathbb{E}}$$

and as it will be proved, we get a Banach space.

We shall show that we can consider $APU(\mathbb{R}, K, \mathbb{E})$ as a space of almost periodic functions with values in a Banach space.

Proposition 26. *Let K be a compact set of X. The map*

$$\Phi_K : APU(\mathbb{R}, K, \mathbb{E}) \to AP^0(C^0(K, \mathbb{E}))$$

defined by

$$\Phi_K(f) := [t \mapsto f(t, \cdot)]$$

is well-defined and is an isometric isomorphism of Banach spaces.

Proof. **Existence of Φ_K.**

Let $f \in APU(\mathbb{R}, K, \mathbb{E})$. It is uniformly continuous, hence continuous, and then $f(t, \cdot) \in C^0(K, \mathbb{E})$ for all $t \in \mathbb{R}$. Moreover, since f is uniformly continuous, given $\varepsilon > 0$:

$$\exists \eta > 0, \quad \big[\max\{|t - t'|; |\alpha - \alpha'|\} \leq \eta\big] \Longrightarrow [|f(t, \alpha) - f(t', \alpha')|_{\mathbb{E}} \leq \varepsilon]$$

and taking $\alpha' = \alpha$, then the 'sup' on α, we obtain :

$$(|t - t'| \leq \eta) \Longrightarrow (\|f(t, \cdot) - f(t', \cdot)\|_{C^0(K,\mathbb{E})} \leq \varepsilon).$$

Thus, $\Phi_K(f) \in C^0(\mathbb{R}, C^0(K, \mathbb{E}))$.

Let us now prove that it is almost periodic. Since $f \in APU(\mathbb{R}, K, \mathbb{E})$, by definition, we have :

for all $\varepsilon > 0$, there exists $l > 0$, such that for all $a \in \mathbb{R}$, there exists $\tau \in [a, a + l)$, such that

$$\sup_{t\in\mathbb{R}} \sup_{\alpha\in K} |f(t + \tau, \alpha) - f(t, \alpha)|_{\mathbb{E}} \leq \varepsilon.$$

We deduce that for all $\varepsilon > 0$, there exists $l > 0$ such that for all $a \in \mathbb{R}$, there exists $\tau \in [a, a + l)$ such that

$$\sup_{t\in\mathbb{R}} \|f(t + \tau, \cdot) - f(t, \cdot)\|_{C^0(K,\mathbb{E})} \leq \varepsilon.$$

Hence, Φ_K is well-defined.

Φ_K is linear and isometric (hence injective and continuous). These two points are obvious, the second is resulting from:

$$\|\Phi_K(f)\|_{AP^0(C^0(K,\mathbb{E}))} = \sup_{t\in\mathbb{R}} \|f(t, \cdot)\|_{C^0(K,\mathbb{E})} = \sup_{t\in\mathbb{R}} \sup_{\alpha\in K} |f(t, \alpha)|_{\mathbb{E}} = \|f\|_{APU(\mathbb{R},K,\mathbb{E})}.$$

Φ_K is surjective (then bijective, and so it is bi-continuous since it is isometric).

Let $\phi \in AP^0(C^0(K,\mathbb{E}))$. To simplify, we denote ϕ_t instead of $\phi(t)$, and it then belongs to $C^0(K,\mathbb{E})$.

The natural candidate to verify $\Phi_K(f) = \phi$ is $f(t,\alpha) := \phi_t(\alpha)$. We just have to show that we have an element of $APU(\mathbb{R},K,\mathbb{E})$.

Let's start with continuity. Fix $(t_0,\alpha_0) \in \mathbb{R} \times K$. We have :

$$\begin{aligned}|f(t,\alpha) - f(t_0,\alpha_0)|_{\mathbb{E}} &\leq |f(t,\alpha) - f(t_0,\alpha)|_{\mathbb{E}} + |f(t_0,\alpha) - f(t_0,\alpha_0)|_{\mathbb{E}} \\ &\leq \|\phi_t - \phi_{t_0}\|_{C^0(K,\mathbb{E})} + |\phi_{t_0}(\alpha) - \phi_{t_0}(\alpha_0)|_{\mathbb{E}}.\end{aligned}$$

Let $\varepsilon > 0$. Since ϕ is continuous :

there exists $\delta > 0$, such that

$$(|t - t_0| \leq \delta) \Longrightarrow (\|\phi_t - \phi_{t_0}\|_{C^0(K,\mathbb{E})} \leq \varepsilon/2)$$

and as ϕ_{t_0} is continuous :

there exists $\delta' > 0$, such that

$$(|\alpha - \alpha_0| \leq \delta') \Longrightarrow (|\phi_{t_0}(\alpha) - \phi_{t_0}(\alpha_0)|_{\mathbb{E}} \leq \varepsilon/2).$$

Finally, for $(t,\alpha) \in \mathbb{R}\times K$ close enough to (t_0,α_0), we have $|f(t,\alpha) - f(t_0,\alpha_0)|_{\mathbb{E}} \leq \varepsilon$, and so is shown the continuity.

Since ϕ is almost-periodic, we get that :

for any $\varepsilon > 0$, there exists $l > 0$ such that for all $a \in \mathbb{R}$, there exists $\tau \in [a, a+l)$ such that

$$\|\phi_{t+\tau} - \phi_t\|_{C^0(K,\mathbb{E})} \leq \varepsilon$$

and so for all $\varepsilon > 0$, there exists $l > 0$ such that for all $a \in \mathbb{R}$, there exists $\tau \in [a, a+l)$, such that

$$\sup_{t\in\mathbb{R}} \sup_{\alpha\in K} |f(t+\tau,\alpha) - f(t,\alpha)|_{\mathbb{E}} \leq \varepsilon.$$

ϕ_K is then surjective. As the target space is a Banach and Φ_K is bi-continuous, $APU(\mathbb{R},K,\mathbb{E})$ is indeed a Banach space.

The proposition is thus completely proved.

□

We deduce immediately from this proposition that the general properties on almost periodic functions can be adapted without problem to $APU(\mathbb{R},K,\mathbb{E})$.

To cite just one example, here is a density statement of trigonometric polynomials with continuous coefficients :

Proposition 27. *Let $f \in APU(\mathbb{R},K,\mathbb{E})$. Then, there exists a family of elements of $C^0(K,\mathbb{E})$, $(a_\lambda^{(n)})_{(n,\lambda)\in\mathbb{N}\times\mathbb{R}}$, such that*

- *for all n, $(a_\lambda^{(n)})_{\lambda\in\mathbb{R}}$ is nearly zero*
- *we have*

$$\lim_{n\to+\infty}\sup_{(t,\alpha)\in\mathbb{R}\times K}\left|f(t,\alpha)-\sum_{\lambda\in\mathbb{R}}a_\lambda^{(n)}(\alpha)e_\lambda(t)\right|_{\mathbb{E}}=0.$$

Proof. $\Phi_K(f)$ can be approximated by trigonometric polynomials with coefficients in $C^0(K,\mathbb{E})$.

So, we can find a family $(a_\lambda^{(n)})_{(n,\lambda)\in\mathbb{N}\times\mathbb{R}}$ of elements of $C^0(K,\mathbb{E})$ such that

$$\text{for all } n\in\mathbb{N},\quad (a_\lambda^{(n)})_{\lambda\in\mathbb{R}} \text{ is nearlyzero}$$

and

$$\lim_{n\to+\infty}\sup_{t\in\mathbb{R}}\left\|\Phi_K(f)(t,\cdot)-\sum_{\lambda\in\mathbb{R}}a_\lambda^{(n)}(\cdot)e_\lambda(t)\right\|_{C^0(K,\mathbb{E})}=0.$$

As Φ_K is an isometry, the second relation gives

$$\lim_{n\to+\infty}\sup_{(t,\alpha)\in\mathbb{R}\times K}\left|f(t,\alpha)-\sum_{\lambda\in\mathbb{R}}a_\lambda^{(n)}(\alpha)e_\lambda(t)\right|_{\mathbb{E}}=0,$$

which achieves the proof. □

3.3 Almost periodic functions with an infinitely countable set of parameters

We fix once and for all an increasing family of non-empty compact sets $(K_n)_{n\geq 1}$ such that $P=\bigcup_n K_n$ and for all n, $K_n\subset \operatorname{Int} K_{n+1}$.

We start with an efficient lemma for the sequel :

Lemma 28. *Let K be a compact of P. Then, there exists $n\in\mathbb{N}\setminus\{0\}$ such that $K\subset\operatorname{Int} K_n$.*

Proof. We have $K\subset\bigcup\limits_{n\geq 1}K_n$, so by virtue of the hypothesis on the sequence of compacts :

$$K\subset\bigcup_{n\geq 2}\operatorname{Int} K_n.$$

But, we have an open covering of K, so by Borel-Lebesgue, there exists an integer $n\geq 2$ such that

$$K\subset\bigcup_{2\leq p\leq n}\operatorname{Int} K_p=\operatorname{Int} K_n,$$

which ends the proof. □

We endow $APU(\mathbb{R}, P, \mathbb{E})$ with a locally convex topological vector space structure with filtering countable basis of semi-norms $(p_n)_{n\geq 1}$, where

$$p_n(f) := \sup_{(t,\alpha)\in\mathbb{R}\times K_n} |f(t,\alpha)|_{\mathbb{E}} = \left\| f_{|\mathbb{R}\times K_n} \right\|_{APU(\mathbb{R},K_n,\mathbb{E})}.$$

We immediately see that this topology is separated.

Recall that this topology can also be defined by the metric d, where

$$d(f;g) := \sum_{n\geq 1} \frac{1}{2^n} \frac{p_n(f-g)}{1+p_n(f-g)}.$$

Lemma 29. *$APU(\mathbb{R}, P, \mathbb{E})$ is a Fréchet space.*

Proof. We shall use the metric structure.

Let $(f_p)_p$ be a Cauchy sequence in $APU(\mathbb{R}, P, \mathbb{E})$ endowed with d.

So, for all $\varepsilon > 0$, we can find P_0 such that for any $p \geq P_0$, and for all q,

$$d(f_{p+q}; f_p) < \varepsilon.$$

In particular, if we set $n \geq 1$ and if we choose $\varepsilon < 2^{-n}$, we obtain that there exists P_0, such that for all $p \geq P_0$, for any q,

$$\frac{p_n(f_{p+q} - f_p)}{1 + p_n(f_{p+q} - f_p)} < \varepsilon$$

and so, there exists P_0 such that for all $p \geq P_0$, and all q,

$$p_n(f_{p+q} - f_p) < \frac{2^n\varepsilon}{1 - 2^n\varepsilon}.$$

The sequence $(f_{p_{|\mathbb{R}\times K_n}})_p$ is therefore of Cauchy in the Banach space $APU(\mathbb{R}, K_n, \mathbb{E})$ and so that it converges to an element $f^{(n)} \in APU(\mathbb{R}, K_n, \mathbb{E})$.

By uniqueness of the limit, we have if $m \geq n$, $f^{(m)}_{|\mathbb{R}\times K_n} = f^{(n)}$, so it is legitimate to set

$$f(t,\alpha) := f^{(n)}(t,\alpha) \quad \text{ if } \alpha \in K_n$$

(such a K_n exists because of 28).

Let $(t_0, \alpha_0) \in \mathbb{R} \times P$ and n be such that $\alpha_0 \in \operatorname{Int} K_n$. So, $\mathbb{R} \times \operatorname{Int} K_n$ is an open neighborhood of (t_0, α_0) in $\mathbb{R} \times P$, and on this neighborhood f equals to the continuous function $f^{(n)}$.

We deduce that f is continuous at (t_0, α_0), hence everywhere.

Now, let K be a fixed compact. By Lemma 28, there exists N such that $K \subset K_N$. Since $f^{(N)}$ is almost periodic, we can write

$$\forall \varepsilon > 0, \exists l > 0, \forall a \in \mathbb{R}, \exists \tau \in [a; a+l), \ \sup_{t\in\mathbb{R}} \sup_{\alpha\in K_N} |f(t+\tau,\alpha) - f(t,\alpha)|_{\mathbb{E}} \leq \varepsilon,$$

so that, we have for all $K \in \mathcal{K}(P)$ and all $\varepsilon > 0$, there exists $l > 0$ such that for all $a \in \mathbb{R}$, there exists $\tau \in [a; a + l)$,

$$\sup_{t\in\mathbb{R}} \sup_{\alpha\in K} |f(t+\tau,\alpha) - f(t,\alpha)|_{\mathbb{E}} \leq \varepsilon,$$

Thus, $f \in APU(\mathbb{R}, P, \mathbb{E})$.

□

Similarly, we endow $AP^0(C^0(P, \mathbb{E}))$ with a structure of a locally convex topological vector space structure with a filtering countable basis of semi-norms $(\pi_n)_{n\geq 1}$, where :

$$\pi_n(f) := \sup_{t\in\mathbb{R}} \|f(t,\cdot)\|_{C^0(P,\mathbb{E})}.$$

We see that this topology is separated and that it can also be defined by the metric D, where :

$$D(f; g) := \sum_{n\geq 1} \frac{1}{2^n} \frac{\pi_n(f-g)}{1+\pi_n(f-g)}.$$

We can also prove that

Lemma 30. *$AP^0(C^0(P, \mathbb{E}))$ is a Fréchet space.*

We give the Fréchet version of Proposition 26 :

Proposition 31. *The map $\Phi_P : APU(\mathbb{R}, P, \mathbb{E}) \to AP^0(C^0(P, \mathbb{E}))$ defined by :*

$$\Phi_P(f) := [t \longmapsto f(t,\cdot)]$$

is an isometric isomorphism of Fréchet spaces.

Proof. Let $\alpha \in P$, and n be an integer such that $\alpha \in \operatorname{Int} K_n$. We then have:

$$\Phi_P(f)(t,\alpha) = \Phi_{K_n}(t)(t,\alpha)$$

which justifies the existence of Φ_P, shows its linear and isometric character. It only remains to prove the surjectivity.

Given $\phi \in AP^0(C^0(P, \mathbb{E}))$, we set

$$f(t,\alpha) := \Phi_{K_n}^{-1}(\phi)(t,\alpha)$$

where n is any integer such that $\alpha \in \operatorname{Int} K_n$ (the definition of f does not depend on the choice of n).

By construction, for all n, $f_{|\mathbb{R}\times K_n} \in APU(\mathbb{R}, K_n, \mathbb{E})$, and since any compact K is injected into a K_n (cf. 28), we can conclude that $f \in APU(\mathbb{R}, P, \mathbb{E})$.

Finally, the fact that $\Phi_P(f) = \phi$ is immediate by construction.

□

3.4 Extension to the quasi-periodic framework

3.4.1 Quasi-periodic functions with parameters

For the sake of simplicity, the results are stated and proved in this paragraph with a compact set of parameters, the results canonically are naturally extending to the infinitely countable frame.

We denote by $\mathcal{M}_1$ the mean operator on $AP^0(C^0(K,\mathbb{E}))$ and we define on $APU(\mathbb{R},K,\mathbb{E})$ the mean operator with values in $C^0(K,\mathbb{E})$ as being

$$\mathcal{M}_2 := \mathcal{M}_1 \circ \Phi_K.$$

It is a continuous linear operator of norm 1 as the composition of two such operators. We shall denote $\mathcal{M}_2\{f\}$ or $\mathcal{M}_2\{f(t,\alpha)\}_{(t,\alpha)}$ its value for $f \in APU(\mathbb{R},K,\mathbb{E})$.

We define so $\tilde{a}_\lambda(f) \in C^0(K,\mathbb{E})$ for all $\lambda \in \mathbb{R}$ by

$$\tilde{a}_\lambda(f) := \mathcal{M}_2\{f(t,\alpha)e_{-\lambda}(t)\}_{(t,\alpha)}.$$

In particular, the following relation holds true :

$$\tilde{a}_\lambda(f)(\alpha) = a_\lambda\left(\Phi_K(f(\cdot,\alpha))\right).$$

We also define the following evaluation operators in $\beta \in K$:

- $ev^1_\beta : APU(\mathbb{R},K,\mathbb{E}) \longrightarrow AP^0(\mathbb{E})$ by :
$$ev^1_\beta(f) := f(\cdot,\beta).$$
- $ev^2_\beta : C^0(K,\mathbb{E}) \longrightarrow \mathbb{E}$ by :
$$ev^2_\beta(\varphi) := \varphi(\beta).$$

These operators are visibly continuous linear operators of norm 1.

The evaluation commutes with the averaging operators. More specifically :

Lemma 32. *We have :*
$$ev^2_\beta \circ \mathcal{M}_2 = \mathcal{M} \circ ev^1_\beta.$$

Proof. We have to show that $ev^2_\beta \circ \mathcal{M}_1 \circ \Phi_K = \mathcal{M} \circ ev^1_\beta$, or, as Φ_K is an isomorphism (isometric) :
$$ev^2_\beta \circ \mathcal{M}_1 = \mathcal{M} \circ ev^1_\beta \circ \Phi_K^{-1}.$$

Each member of the equality is a continuous linear operator (of norm 1). So, it suffices to prove the equality on a dense subset.

Let us take as a dense subset of the vector space of trigonometric polynomials with values in $C^0(K,\mathbb{E})$. Such a polynomial has the form

$$f(t) = \sum_{\lambda\in\mathbb{R}} a_\lambda e_\lambda, \quad a_\lambda \in C^0(K,\mathbb{E}),$$

the sum here is being finite.

We then calculate

$$ev_\beta^2 \circ \mathcal{M}_1(f) = ev_\beta^2(a_0(\cdot)) = a_0(\beta)$$

and :

$$\mathcal{M} \circ ev_\beta^1 \circ \Phi_K^{-1}(f) = \mathcal{M} \circ ev_\beta^1 \circ \Phi_K^{-1}\left([(t,\alpha) \mapsto \sum_{\lambda\in\mathbb{R}} a_\lambda(\alpha)e_\lambda(t)]\right)$$

$$= \mathcal{M}\left\{\sum_{\lambda\in\mathbb{R}} a_\lambda(\beta)e_\lambda\right\} = a_0(\beta).$$

The two members are therefore identical.

□

Lemma 33. *Let $\lambda \in \mathbb{R}$. Then, the following two assertions are equivalent :*

1. *There exists $\alpha \in K$ such that $a_\lambda(f(\cdot,\alpha)) \neq 0$.*

2. $\tilde{a}_\lambda(f) \neq 0$.

Proof. This lemma follows immediately from the previous one applied to the function $(t,\alpha) \mapsto f(t,\alpha)e_{-\lambda}(t)$ and with $\beta = \alpha$.

□

By virtue of this lemma, we have the equality

$$\{\lambda \in \mathbb{R} \quad : \quad \tilde{a}_\lambda(f) \neq 0\} = \bigcup_{\alpha\in K} \Lambda(f(\cdot,\alpha)).$$

We note $\Lambda(f)$ this set (for the infinitely countable case, we must take the union of the sets defined by restriction at each $\mathbb{R}\times K_n$), and $Mod(f)$ the $\mathbb{Z}$-module it generates.

Given a $\mathbb{Z}$-module of finite type M, we then set :

$$QPU_M(\mathbb{R},K,E) := \{f \in APU(\mathbb{R},K,E) \quad : \quad Mod(f) \subset M\}$$

and we abbreviate $QPU_{\mathbb{Z}\langle\omega\rangle}(\mathbb{R},K,E)$ in $QPU_\omega(\mathbb{R},K,E)$.

3.4.2 Main result statement

From the lemma 33, the statements 26 and 31 immediately adapt to the quasi-periodic framework by stating it as follows :

Proposition 34. *The map* $\Phi_K : QPU_\omega(\mathbb{R}, K, \mathbb{E}) \to QP^0_\omega(C^0(K, \mathbb{E}))$ *defined by*

$$\Phi_K(f) := [t \longmapsto f(t, \cdot)]$$

is an isometric isomorphism of Banach spaces.

The map $\Phi_P : QPU_\omega(\mathbb{R}, P, \mathbb{E}) \to QP^0_\omega(C^0(P, \mathbb{E}))$ *defined by :*

$$\Phi_P(f) := [t \longmapsto f(t, \cdot)]$$

is an isometric isomorphism of Fréchet spaces.

4 Besicovitch spaces and Sobolev-type spaces

We shall construct spaces playing an analogous role in the almost-periodic cases and quasi-periodic to the space L^2 and to the Sobolev spaces.

The main references are [22, Chap. I] and [3, Chap. II] for the finite dimension case.

4.1 Besicovitch spaces

If f is a function of the Lebesgue space $L^1_{loc}(\mathbb{R}, \mathbb{R})$ of locally intergable functions on $\mathbb{R}$, we denote by :

$$\overline{\mathcal{M}}\{f\} := \limsup_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)dt.$$

If $f \in \mathcal{L}^1_{loc}(\mathbb{R}, \mathbb{R})$, the temporal mean of f, when it exists, is defined as :

$$\mathcal{M}\{f\} = \mathcal{M}\{f(t)\}_t := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)dt.$$

When $p \in \mathbb{N}$ and $p \geq 1$, we denote by $\mathcal{B}^p(\mathbb{E})$ the closure of $AP^0(\mathbb{E})$ in $\mathcal{L}^p_{loc}(\mathbb{R}, \mathbb{R})$ for the semi-norm $f \mapsto \left(\overline{\mathcal{M}}\{|f|^p\}\right)^{1/p}$.

The elements of $\mathcal{B}^2(\mathbb{E})$ are called almost periodic functions in the sense of Besicovitch.

If $f \in \mathcal{B}^p(\mathbb{E})$, $(\mathcal{M}\{|f|^p\})^{1/p}$ exists in $\mathbb{R}$ and if $g, h \in \mathcal{B}^p(\mathbb{E})$, we denote by $g \sim_p h$ the equivalence relation defined as

$$\overline{\mathcal{M}}\{|g - h|^p\} = 0.$$

The equivalence class of f with respect to the equivalence relation $\sim_p$ shall be denoted by $[f]_p$. The quotient space associated is denoted by $B^p(\mathbb{E}) : \mathcal{B}^p(\mathbb{E})/\sim_p$. For skae of simplicity, we shall confuse often f with its equivalence class.

Proposition 35. *[22, Chap. I] The space* $B^p(\mathbb{E})$ *equipped with the norm* $\|[f]\|_{B^p} = (\mathcal{M}\{|f|^p\})^{1/p}$ *is a Banach space.*

Let take $p = 2$. The space $\mathcal{B}^2(\mathbb{E})$ will be the closure of $AP^0(\mathbb{E})$ in $\mathcal{L}^2_{loc}(\mathbb{E})$ for the semi-norm $f \mapsto \overline{\mathcal{M}}\{|f|^2\}^{1/2}$. We denote $f \sim_2 g$ if $\overline{\mathcal{M}}\{|f-g|^2\} = 0$.

The quotient of $\mathcal{B}^2(\mathbb{E})$ for this equivalence relation is denoted $B^2(\mathbb{E})$ and is called the space of almost-periodic functions in the sense of Besicovitch.

We shall use the notation $[f]_2$ to denote the equivalence class with respect to this equivalence relation and whose representing function is f. We confuse often f with its equivalence class.

When we do the same operation by replacing $AP^0(\mathbb{E})$ by $QP^0_\omega(\mathbb{E})$, we obtain a space noted $B^2_\omega(\mathbb{E})$.

These spaces are complete, and if moreover $\mathbb{E} = \mathbb{H}$, they are Hilbert spaces when equipped with the scalar product $< u; v >_{B^2} = \mathcal{M}\{u \cdot_{\mathbb{H}} v\}$. We denote by $\|\cdot\|_2$ its associated norm.

Proposition 36. *[22, Chap. I] The space $B^2(\mathbb{H})$ is a Hilbert space for the inner product $< u; v >_{B^2} = \mathcal{M}\{u \cdot_{\mathbb{H}} v\}$.*

It is also the closure of the space $AP^0(\mathbb{H})$ for the same inner product.

One can of course expand the elements of these spaces in Fourier series, and if $\mathbb{E} = \mathbb{H}$, their expansion satisfies the Parseval relation.

Proposition 37. *[22] The following assertions are true.*

1. *Each function $f \in B^2(\mathbb{E})$ admits a Fourier series expansion:*

$$f \sim_2 \sum_{\lambda \in \mathbb{R}} a(f;\lambda) e_\lambda$$

and if $\mathbb{E} = \mathbb{H}$ is a Hilbert space, f satisfies the following Parseval relation

$$\mathcal{M}\{|f|^2_{\mathbb{H}}\} = \sum_{\lambda \in \mathbb{R}} |a(f;\lambda)|^2_{\mathbb{H}}.$$

2. *Each function $f \in B^2_\omega(\mathbb{E})$ has a Fourier series expansion :*

$$f \sim_2 \sum_{\lambda \in \mathbb{Z}\langle\omega\rangle} a(f;\lambda) e_\lambda$$

and if $\mathbb{E} = \mathbb{H}$ is a Hilbert space, f verifies the following Parseval relation :

$$\mathcal{M}\{|f|^2_{\mathbb{H}}\} = \sum_{\lambda \in \mathbb{Z}\langle\omega\rangle} |a(f;\lambda)|^2_{\mathbb{H}}.$$

Better than that, the following Riesz-Fisher-Besicovitch theorem shows that these spaces, in fact, carry out the harmonic synthesis, and therefore play a role analogous to $L^2((0,T),\mathbb{H})$ for periodic functions :

Theorem 38. *[3, 22]* **(Riesz-Fisher-Besicovitch)** *The following maps are isometric isomorphisms of Hilbert spaces.*

1. $\Phi_1 : B^2(\mathbb{H}) \longrightarrow \ell^2(\mathbb{R}, \mathbb{H})$ *defined by :*

$$\Phi_1(f) := (a(f; \lambda))_{\lambda \in \mathbb{R}}.$$

2. $\Phi_2 : B^2_\omega(\mathbb{H}) \longrightarrow \ell^2(\mathbb{Z} < \omega >, \mathbb{H})$ *defined by :*

$$\Phi_2(f) := (a(f; \lambda))_{\lambda \in \mathbb{Z}<\omega>}.$$

4.2 Sobolev-type spaces

The main purpose, here, is to define a notion of weak derivative on Besicovitch spaces. This derivative was namely inspired from the works of Vo-Khac, [23, 24].

For a Hilbert space $\mathbb{H}$ of finite dimension, we consider the infinitesimal generator of the group $(\tau_r)_r$ of translations in $B^2(\mathbb{H})$. This generator is denoted by ∇ and its domain is denoted $B^{1,2}(\mathbb{H})$. This is the Sobolev-type space. It corresponds to the space of Sobolev $H^1((0, T), \mathbb{H})$ of the periodic case. It is a complete space and we endow $B^{1,2}(\mathbb{H})$ with the scalar product :

$$< u; v >_{B^{1,2}} = < u; v >_{B^2} + < \nabla u; \nabla v >_{B^2},$$

which makes it a Hilbert space.

Proposition 39. *The following properties hold true.*

1. *If* $u \in B^{1,2}(\mathbb{E})$ *and* $r \in \mathbb{R}$, *then* $\nabla(\tau_r u) \sim_2 \tau_r(\nabla u)$.
2. *If* $u \in AP^1(\mathbb{E})$, *then* $u \in B^{1,2}(\mathbb{E})$ *and* $\nabla u \sim_2 \dot{u}$.
3. *If* $k \in \mathbb{N} \cup \{+\infty\}$, *then* $AP^k(\mathbb{E})$ *is dense in* $B^{1,2}(\mathbb{E})$.
4. *If* $u \in B^{1,2}(\mathbb{E})$, *then for all* λ, $\quad a(\nabla u; \lambda) = i\lambda a(u; \lambda)$ *and* $\mathcal{M}\{\nabla u\} = 0$.

Proof. 1. This assertion follows immediately from the continuity of τ_r and because the operator τ_r commutes with $(\tau_s - id)/s$.

2. By the second inequality of the mean, we have:

$$|u(t + r) - u(t) - u'(t)r| \leq |r| \sup \Big\{ |u'(s) - u'(t)|, \; s \in [t, t + r] \Big\}.$$

So, we have

$$\left| \frac{1}{r} \Big(u(t + r) - u(t) \Big) - u'(t) \right| \leq \sup \Big\{ |u'(s) - u'(t)|, \; s \in [t, t + r] \Big\}.$$

Since $u' \in AP^0(\mathbb{E})$, u' is uniformly continuous and thus

$$\lim_{r\to 0}\left\|\frac{1}{r}\Big(\tau_r(u)-u\Big)-u'\right\|_\infty = 0.$$

We deduce that

$$\lim_{r\to 0}\left\|\frac{1}{r}\Big(\tau_r([u]_2)-[u]_2\Big)-[u']_2\right\|_2 = 0.$$

This shows that $u \in B^{1,2}(\mathbb{E})$ and $\nabla u \sim_2 u'$.

3. Take $f \in B^{1,2}(\mathbb{E})$ such that $f(t) \sim_2 \sum_\lambda a(f;\lambda)e^{i\lambda t}$ and $\nabla f(t) \sim_2 \sum_\lambda a(f;\lambda)e^{i\lambda t}$.

   Consider an injective sequence $(\lambda_k)_{k\geq 1}$ with values in $(0,\infty)$ such that the elements of $\Big\{\lambda_k;\ k\geq 1\Big\}$ are from the spectrum of f.

   Set
   $$P_N(t) = a(f;0) + \sum_{k=1}^{N} a(f;\lambda_k)e^{i\lambda_k t} + \sum_{k=1}^{N} a(f;-\lambda_k)e^{-i\lambda_k t}.$$

   Then, P_N belongs to $AP^\infty(\mathbb{E})$ and $|P_N - f|_2$ and $|\nabla(P_N)-\nabla f|_2$ tend to zero as N tends to ∞ and so that the sequence $(P_N)_N$ converges to f in $B^{1,2}(\mathbb{E})$.

4. We have $a\Big((\tau_r(u)-u);\lambda\Big) = \Big((e^{i\lambda r}-1)/r\Big)a(u;\lambda)$.

   Since $a(\cdot;\lambda)$ is a continuous linear form on $B^2(\mathbb{E})$, we obtain that $a(\nabla u;\lambda) = i\lambda a(u;\lambda)$, when r tends to zero.

   We argue by the same manner with $\lambda = 0$ to obtain that $\mathcal{M}\{\nabla u\} = 0$.

$\square$

Remark 40. *The assertion* **4.** *of the previous proposition indicates that to differentiate a function of $B^{1,2}(\mathbb{E})$, it suffices to differentiate term by term the series of Fourier*

We have the following essential property, which follows from the Riesz-Fischer Theorem :

Proposition 41. *Let $u \in B^2(\mathbb{H})$. We have :*
$u \in B^{1,2}(\mathbb{H})$ if and only if $\sum_{\lambda\in\mathbb{R}} \left(1+|\lambda|^2\right)|a_\lambda(u)|^2_{\mathbb{H}} < +\infty$.

From there, we canonically define the spaces $B^{j,2}(\mathbb{E})$ by iteration, and the spaces $B^{j,2}_\omega(\mathbb{E}) := B^{j,2}(\mathbb{E}) \cap B^2_\omega(\mathbb{E})$.

The results already shown can then be extended naturally to $B^{1,2}_\omega(\mathbb{E})$.

5 Applications : Almost periodic solutions.

5.1 Convolution.

In this paragraph, we shall discuss a method of constructing functions in $AP^0(\mathbb{C})$ based on the operation of convolution .

Take a function $K \in L^1(\mathbb{R}, \mathbb{C})$, that is, such that $\int_{\mathbb{R}} |K(s)|ds < \infty$.

We are going to prove that the operator $T : f \mapsto \int_{\mathbb{R}} K(s)f(t-s)ds$ sends the space of almost periodic functions into the same space.

Proposition 42. *The T takes the space of trigonometric polynomials into itself, that is, for any $P \in TP(\mathbb{R}, \mathbb{C})$, we have $T(P) \in TP(\mathbb{R}, \mathbb{C})$.*

Proof. Take a trigonometric polynomial $P(t) = \sum_{k=1}^{n} a_k e^{i\lambda_k t}$, where $a_k \in \mathbb{C}$ and $\lambda_k \in \mathbb{R}$, for all $k = 1, \cdots, n$.

Then, it is not difficult to show that

$$\int_{\mathbb{R}} K(s)T(t-s)ds = \sum_{k=1}^{n} \left[a_k \int_{\mathbb{R}} K(s)e^{-i\lambda_k s}ds \right] e^{i\lambda_k t}.$$

Therefore, $T(P) \in TP(\mathbb{R}, \mathbb{C})$, since it can be written as $T(P)(t) = \sum_{k=1}^{n} b_k e^{-i\lambda_k t}$, where $b_k := a_k \int_{\mathbb{R}} K(s)e^{-i\lambda_k s}ds \in \mathbb{C}$, as soon as $K \in L^1(\mathbb{R}, \mathbb{C})$.

□

Now, we have :

Proposition 43. *If $f \in AP^0(\mathbb{C})$, then $T(f) \in AP^0(\mathbb{C})$.*

Proof. Let $f \in AP^0(\mathbb{C})$. Then, by Proposition 4, there exists a sequence $(P_n)_{n\geq 1} \in TP(\mathbb{C})^{\mathbb{N}}$ such that $(P_n)_n$ uniformly converges to f, as n tends to ∞

Due to Proposition 42, we have $T(P_n) \in TP(\mathbb{C})$, for all $n \geq 1$.

On the other hand, we have :

$$\begin{aligned}
&\Big| \int_{\mathbb{R}} K(s)f(t-s)ds - \int_{\mathbb{R}} K(s)P_n(t-s)ds \Big| \\
&= \Big| \int_{\mathbb{R}} K(s) \Big| f(t-s) - P_n(t-s) \Big| ds \Big| \\
&\leq \int_{\mathbb{R}} |K(s)| \Big| f(t-s) - P_n(t-s) \Big| ds \\
&\leq \Big\| K \Big\|_{L^1(\mathbb{R},\mathbb{C})} \Big\| f - P_n \Big\|_{\infty}.
\end{aligned}$$

Since $\|f - P_n\|_\infty$ tends to 0 as n tends to ∞ and $\|K\|_{L^1(\mathbb{R},\mathbb{C})}$ is finite as K is an integrable function and since the previous inequality holds for all $t \in \mathbb{R}$, we deduce that, taking the limit as n tends to ∞, the sequence of trigonometric polynomials $\left(\int_{\mathbb{R}} K(s)P_n(t-s)\right)_{n\in\mathbb{N}}$ converges uniformly to $\int_{\mathbb{R}} K(s)f(t-s)ds$.

Due to Proposition 4, we conclude that $T(f) \in AP^0(\mathbb{C})$.

□

A possible choice for the function K is that we take

$$K(t) = e^{-\alpha t}\chi_{[0,\infty)},$$

where $\alpha > 0$ is a fixed real and $\chi_{[0,\infty)}$ denotes the indicator function of the interval $[0,\infty)$, that is, the function taking the value 1 on this interval and 0 outside. This means that $K(t) = e^{-\alpha t}$ if $t \geq 0$ and $K(t) = 0$ if $t < 0$.

This function leads to the expression

$$T(f) = e^{-\alpha t}\int_{-\infty}^{t} e^{\alpha s}f(s)ds \tag{5.1}$$

It is easy to check that this last expression is the (only) almost periodic solution to the differential equation $x'(t) + \alpha x(t) = f(t)$.

5.2 Linear oscillations.

Let us consider the linear differential systems of the form

$$x'(t) = Ax(t) + f(t), \quad t \in \mathbb{R}, \tag{5.2}$$

where $x : \mathbb{R} \to \mathbb{C}^n$ is the unknown function, $A \in \mathcal{L}(\mathbb{C}^n, \mathbb{C}^n)$ a square matrix with complex entries and f a given function (the forcing term). f is taken here belonging to one of the introduced spaces of almost periodic functions.

We are only considering in this paragraph the case of constant coefficients. The variable coefficient case where A depends on t is more difficult and needs other materials to be investigated, [17].

To the form 5.2, we can reduce higher-order scalar equations of the form

$$y^n + a_1 y^{n-1} + \cdots + a_n y = g(t),$$

where for all $k = 1, \cdots, n$, a_k are in $\mathbb{R}$ or $\mathbb{C}$ and the notation y^k deserves for the k-th derivative of the function y. It is important to mention that such equations, particularly if $n = 2$, can describe linear oscillations in mechanical, electrical, and other kinds of physical and engineering systems.

Now, let us analyze the algebraic background of these systems. From linear algebra theory, it is well known that by means of a linear transformation on x of the form

$$x = Tu, \quad x, u \in \mathbb{C}^n, \tag{5.3}$$

where $T \in \mathcal{L}(\mathbb{C}^n, \mathbb{C}^n)$ such that $\det(T) \neq 0$, the equation (5.3) can be reduced to the special following form

$$u' = Bu + \tilde{f}(t) \tag{5.4}$$

with an upper-triangular matrix B :

$$B = T^{-1}AT. \tag{5.5}$$

The vector function $\tilde{f}(t)$ is given by

$$\tilde{f}(t) = T^{-1}f(t), \quad t \in \mathbb{R}. \tag{5.6}$$

It is also not difficult to show that the diagonal elements of B are the characteristic numbers or eigenvalues of the matrix A :

$$B = \begin{bmatrix} \lambda_1 & b_{12} & \dots & b_{1,n} \\ 0 & \lambda_2 & b_{23}\dots & b_{2,n} \\ \vdots & \ddots & & \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

The eigenvalues λ_k, $k = 1, \cdots, n$, satsify the algebraic equation

$$\det(\lambda_k I - A) = 0, \quad k = 1, 2, \cdots, n. \tag{5.7}$$

Since $\det(\lambda I - B) = \det(\lambda I - A)$, this justifies the claim about the nature of the diagonal elements of the matrix B.

We shall now investigate the problem of almost periodicity of the solutions of the differential system (5.2), when we suppose that the forcing term f is an almost periodic function.

Proposition 44. *If x is a solution of the equation (5.2) with $f \in AP^0(\mathbb{C}^n)$, then x belongs to $AP^0(\mathbb{C}^n$ if and only if x belongs to $BC(\mathbb{C}^n)$.*

Proof. Since $AP^0(\mathbb{C}^n) \subset BC(\mathbb{C}^n)$, the necessary condition is straightforward.

We shall now prove the sufficiency of the condition $x \in BC(\mathbb{C}^n)$.

For this aim, we are going to use the reduced form of the system (5.2), that is,

$$\begin{cases} u_1' = & \lambda_1 u_1 + b_{12}u_2 + \cdots + b_{1n}u_n + \tilde{f}_1(t) \\ u_2' = & \lambda_2 u_2 + b_{23}u_2 + \cdots + b_{2n}u_n + \tilde{f}_2(t) \\ \cdots & \\ u_{n-1}' = & \lambda_{n-1}u_{n-1} + b_{n-1,n}u_n + \tilde{f}_{n-1}(t) \\ u_n' = & \lambda_n u_n + \tilde{f}_n(t) \end{cases} \tag{5.8}$$

The last equation in (5.8) is a scalar equation in u_n with almost periodic forcing term $\tilde{f}_n$. Therefore, if we show that $u_n \in AP^0(\mathbb{C})$ and we substitute it in the previous equation on u_{n-1}, we obtain for u_{n-1} an equation similar to the last equation in (5.8)

$$u'(t) = \lambda u(t) + g(t), \tag{5.9}$$

where $g(t) = b_{n-1,n}u_n(t) + \tilde{f}_{n-1}(t) \in AP^0(\mathbb{C})$.

We continue this process until we obtain that $u_1 \in AP^0(\mathbb{C})$, provided any bounded solution u of equation (5.9) is shown to belong to $AP^0(\mathbb{C})$, regardless of the value of $\lambda \in \mathbb{C}$ and $g \in AP^0(\mathbb{C})$.

Now, the solution of the equation (5.9) is given by

$$u(t) = e^{\lambda t}\left[c + \int_0^t e^{-\lambda x} g(s)dx\right], \tag{5.10}$$

where c is an arbitrary constant complex number.

We distinguish three different situations according to the value of the real part $\Re(\lambda)$ of λ:

Case 1. : $\Re(\lambda) < 0$.

Since $e^{\lambda t}$ tends to ∞ as t goes to $-\infty$, the only chance to obtain u from the equation (5.10) bounded on $\mathbb{R}$ is to require $c + \displaystyle\int_0^{-\infty} e^{-\lambda s} g(s)ds = 0$. The last integral exists, due to the boundedness of the function g on $\mathbb{R}$. Then, this choice for the constant c leads to the following solution for the scalar equation (5.9) :

$$u(t) = \int_{-\infty}^t e^{\lambda(t-s)} g(s)ds, \quad t \in \mathbb{R}. \tag{5.11}$$

The expression already given for the solution u defines an almost periodic function, since it is seen the convolution product where the first factor is an element of $L^1(\mathbb{R}, \mathbb{C})$.

But, since the homogeneous equation corresponding to equation (5.9), $u'(t) = \lambda u(t)$, has only the zero solution bounded on $\mathbb{R}$ when the real part of λ is such $\Re(\lambda) < 0$, the equation (5.11) represents the unique almost periodic solution of equation (5.9).

It can easily be seen that

$$\sup_{t\in\mathbb{R}} |u(t)| \leq \frac{1}{|\Re(\lambda)|} \sup_{t\in\mathbb{R}} |g(t)|.$$

Case 2. $\Re(\lambda) = 0$.

When the real part of λ is zero, for some $\omega \neq 0$, we can write $e^{\lambda t} = e^{i\omega t}$ and so that the solution pf (5.9) shall be

$$u(t) = ce^{i\omega t} + e^{i\omega t} \int_0^t e^{-i\omega s} g(s)ds.$$

We derive then that u is bounded on $\mathbb{R}$ if and only if the integral $\int_0^t e^{i\omega s} g(s)ds$ is bounded.

Since the function $s \mapsto e^{i\omega s} g(s)$ is in $AP^0(\mathbb{C})$, its integral is also in $AP^0(\mathbb{C})$.

As a consequence, all bounded solutions to (5.9) are almost periodic.

Case 3. $\Re(\lambda) > 0$.

This case can do treated in a similar way to Case 1., and then for $\Re(\lambda) > 0$, there exists only one bounded almost periodic solution to the equation (5.9) which is given by

$$u(t) = -\int_t^{\infty} e^{\lambda(t-s)} g(s)ds, \ t \in \mathbb{R}.$$

To conclude, we start with a bounded solution x of the equation (5.2), which by means of the linear transformation $x = Tu$ generates a bounded solution to (5.4). This solution of (5.4) is necessarily in $AP^0(\mathbb{C}^n$, as showed the above discussion.

Hence, x belongs also to $AP^0(\mathbb{C}^n)$ and this completes the proof of our Proposition. □

Remark 45. *Historically, the above result is also sometimes known as Bohr-Neugebauer result, [9].*

References

[1] L. Amerio, G. Prouse, Almost Periodic Functions and Functional Equations, van Nostrand Reinhold Company, 1971. MR0275061. Zbl 0215.15701.

[2] J. Bass, Cours de Mathématiques, tome 3, Masson, Paris, 1971. MR0360039.

[3] A. S. Besicovitch, Almost Periodic Functions, Cambridge University Press, Cambridge 1932 (Dover, 1955). MR0068029. Zbl 0004.25303. (Zbl 0065.07102).

[4] H. Bochner, Almost Periodic Functions, Julius Springer, Berlin, 1933 (Chelsea Publishing Company, N.Y., 1947).

[5] H. Bohr, Almost Periodic Functions, Julius Springer, Berlin, 1932 (Chelsea Publishing Company, N.Y., 1947). MR0020163. Zbl 0005.20303.

[6] H. Bohr, *Zur Theorie der fastperiodischen Funktionen, I. Eine Verallgemeinerung der Theorie der Fourierreihen*, Acta Math., 45 (1925), 29-127. MR1555192. JFM 50.0196.01.

[7] H. Bohr, *Zur Theorie der fastperiodischen Funktionen, II. Eine Verallgemeinerung der Theorie der Fourierreihen*, Acta Math., 46 (1925), 101-214. MR1555201. JFM 51.0212.02.

[8] H. Bohr, *Zur Theorie der fastperiodischen Funktionen, III. Eine Verallgemeinerung der Theorie der Fourierreihen*, Acta Math., 47 (1926), 237-281. MR1555216. JFM 52.0330.04.

[9] H. Bohr, O. Neugebauer, *Über lineare Differentialgleichungen mit konstanten Koeffizienten und fastperiodischer rechter Seite. Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen.* Mathematisch-Physikalische Klasse. 1926: 8–22. JFM 52.0464.01.

[10] J. -B Bost, Tores invariants des systèmes dynamiques Hamiltoniens, Séminaire Bourbaki, 37ème année, no 639, Vol. 1984/85, 1986. MR0837218. Zbl 0602.58021.

[11] H. Brezis, Analyse Fonctionnelle, Masson, Paris, 1993. Zbl 1147.46300.

[12] H. W. Broer, B. Huitemag, M. B. Sevryuk, Quasi-Periodic Motions in families of Dynamical Systems, Lecture Notes in Mathematics, no 1645, Springer, Berlin, 1996. MR1484969. Zbl 0870.58087.

[13] C. Corduneanu, *Almost Periodic Functions*, Chelsea, 1989. Zbl 0672.42008.

[14] J. Diestel, J. J. Uhl, Vector Measures, Mathematical Surveys no 15, AMS, 1977. MR0453964. Zbl 0369.46039.

[15] J. Favard, *Leçons sur les fonctions presque-périodiques*, Gauthiers-Villars, Paris, 1933. JFM 59.0996.01.

[16] A. M. Fink, *Almost Periodic Differential Equations*, Lectures Notes in Mathematics n^{o}377, Springer-Verlag, Berlin, 1974. MR0460799. Zbl 0325.34039.

[17] D. Lassoued, M. Feckan, *Boundedness and almost periodicity of solutions of linear differential systems*, Mathematica Slovaca, vol. 72, no. 5, 2022, pp. 1203-1214. MR4496036. Zbl 1508.34012.

[18] D. Lassoued, Z. Hui, *Some theorems on the almost-periodic solutions of discrete dynamical systems*, Journal of Differential Equations, Volume 297, 2021, pp. 469–507. MR4282663. Zbl 1468.42005.

[19] D. Lassoued, R. Shah and T. Li, *Almost periodic and asymptotically almost periodic functions: part I*, Advances in Difference Equations, (2018) 2018:47. MR3757665. Zbl 1445.42003.

[20] S. Lang, *Real and Functional Analysis*, Springer Verlag, N.Y., 1993. MR1216137. Zbl 0831.46001.

[21] B. M. Levitan, V. V. Zhikov, *Almost periodic functions and differential equations*, Cambridge University Press, Cambridge, 1982. MR0690064. Zbl 0499.43005.

[22] A. A. Pankov, *Bounded and Almost Periodic Solutions of Nonlinear Operators Differential Equations*, Kluwer Academic Publishers, 1990. MR1120781. Zbl 0712.34001.

[23] K. Vo-Khac, *Distributions, analyse de Fourier, opérateurs aux dérivées partielles*, tome 1, Vuibert, Paris, 1971.

[24] K. Vo-Khac, *Distributions, analyse de Fourier, opérateurs aux dérivées partielles*, tome 2, Vuibert, Paris, 1972.

[25] A. Walther, *Fastperiosche Folgen und ihre Fouriersche Analyse*, Atti del Congresso Internazionale dei Matematici, Bologna, 2 (1928), 289–298. JFM 56.0257.01

[26] T. Yoshizawa, *Stability Theory and the Existence of Periodic Solutions and Almost Periodic Solutions*, Springer-Verlag, N.Y., 1975. MR0466797. Zbl 0304.34051.

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