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ON SIMPLICITY OF CUNTZ ALGEBRAS AND ITS APPLICATIONS

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Abstract. Cuntz algebra $\mathcal{O}_2$ is the universal C^*-algebra generated by two isometries s_1, s_2 satisfying $s_1s_1^* + s_2s_2^* = 1$. This is separable, simple, infinite C^*-algebra containing a copy of any nuclear C^*-algebra. The C^*-algebra $\mathcal{O}_2$ plays a central role in the modern theory of C^*-algebras and appears in many fundamental statements, including a formulation of the celebrated Uniform Coefficient Theorem (UCT). There are several extensions of this notion, including Cuntz algebra $\mathcal{O}_n$, Cuntz-Krieger algebra $\mathcal{O}_A$ for a matrix A, Cuntz-Pimsner algebra $\mathcal{O}_X$ and its relaxation by Katsura for a C^*-correspondence X, and Cuntz-Nica-Pimsner algebra $\mathcal{NO}_X$, for a product system X. We give an overview of the construction of these classes of C^*-algebras with a focus on conditions ensuring their simplicity, which is needed in the Elliott Classification Program, as it stands now. The results we present are now part of the literature, but we hope to shed a light on recent developments in a fascinating area of modern operator algebras.

1 Introduction

The Cuntz algebra $\mathcal{O}_n$, introduced in the late 70's by Joachim Cuntz [4], is the universal C^*-algebra generated by n isometries $s_1, \cdots s_n$ of an infinite-dimensional Hilbert space satisfying $\sum_{i=1}^n s_is_i^* = 1$. These are the first concrete examples of a separable infinite simple C^*-algebra and any such algebra contains a subalgebra that has some $\mathcal{O}_n$ as a quotient. In the early 80's, Joachim Cuntz and Wolfgang Krieger extended the class of Cuntz algebras to a class of simple C^*-algebras generated by partial isometries, known as the Cuntz-Krieger algebras [5]. These are purely infinite, but not necessarily simple. For an $n \times n$ matrix A with entries 0 or 1 and non zero rows and columns, the Cuntz-Krieger algebra $\mathcal{O}_A$ is the universal C^*-algebra generated by n partial isometries $s_1, \cdots s_n$ of an infinite-dimensional Hilbert space satisfying $\sum_{i=1}^n s_is_i^* = 1$ and $s_i^*s_i = \sum_{j=1}^n A_{i,j}s_js_j^*$, for $i = 1, \cdots, n$. The Cuntz-Krieger algebra $\mathcal{O}_A$ is simple (and thereby independent of the choice of generators)

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if and only if A is irreducible (i.e., for any pre choice of a row and column indicies some power of A has a nonzero entry at that row and column) and not a permutation matrix (i.e., no power of A is the identity). Such results are reconfirmed via Renault theory by Kumjian et al [27].

In the late 90's, Mihai Pimsner [33] developed a far reaching generalization of previous constructions by associating to each C^*-correspondence X over a C^*-algebra $\mathcal{A}$ two C^*-algebras $\mathcal{T}_X$ and $\mathcal{O}_X$, respectively extending Toeplitz and Cuntz-Krieger algebras. The Cuntz-Pimsner algebras $\mathcal{O}_X$ also generalize crossed products by $\mathbb{Z}$. For a $\mathbb{Z}$-C^*-algebra, if we endow $X = \mathcal{A}$ with the canonical $\mathcal{A}$-valued inner product and right $\mathcal{A}$-module structure $a \cdot b := a(1 \cdot b)$, then the Cuntz-Pimsner algebra $\mathcal{O}_X$ is isomorphic to the full crossed product $\mathcal{A} \rtimes \mathbb{Z}$. Jürgen Schweizer [38, Theorem 3.9] observed that if $\mathcal{A}$ is unital and X is full $\mathcal{A}$-correspondence with injective left action, then $\mathcal{O}_X$ is simple if and only X is aperiodic (i.e., the n-fold tensor $X^{\otimes n}$ is unitarily equivalent to the identity correspondence over $\mathcal{A}$ only if $n = 0$) and $\mathcal{A}$ has no nontrivial invariant ideal. Later, Menevşe Eryüzlü and Mark Tomforde [9, Theorem 4.3] showed that for $\mathcal{O}_X$ being simple, one could replace the Schweizer's aperiodicity condition with the so called (S) condition, that is, for each positive element $a \in \mathcal{A}_+$, $\varepsilon > 0$, and $n \geq 1$, there is $m > n$ and a non returning unit vector $\zeta \in X^{\otimes m}$ (i.e., as elements in $\mathcal{O}_X$ via universal representation, $\zeta^* \xi \zeta = 0$, for $\xi \in X^{\otimes n}$) with $\|\langle a\zeta, \zeta \rangle\| > \|a\| - \varepsilon$. The general framework of Pimsner is used in Kajiwara et al [16, 17] to restate and reprove simplicity results of Cuntz-Krieger algebras.

In general, a right Hilbert $\mathcal{A}$-bimodules behaves like a generalized endomorphisms of $\mathcal{A}$ and thereby $\mathcal{T}_X$ and $\mathcal{O}_X$ looks respectively like $\mathcal{A} \rtimes \mathbb{N}$ and $\mathcal{A} \rtimes \mathbb{Z}$ [3]. The Inspired by the graph C^*-algebras, Takeshi Katsura [18] extended Pimsner approach to modules X with not necessarily isometric left actions, and defined the most general version of the C^*-algebra $\mathcal{O}_X$ for an A-correspondence X, and observed that the Cuntz-Pimsner representation of X generates an isomorphic copy of $\mathcal{O}_X$ when it is injective and respects the gauge action. In another direction, adapting a notion product systems over of Hilbert spaces on the interval $(0, \infty)$ due to William Arveson [1], Neal J. Fowler introduced notion of C^*-algebras associated to product systems of Hilbert C^*-bimodules [11, 12] (see also, [6, 12] for the same notion on a general semigroup). In particular, he introduced the Toeplitz algebras over product systems and the critical concept of compactly aligned systems. for A product system over a semigroup P of Hilbert $\mathcal{A}$–bimodules is nothing but an action of P on $\mathcal{A}$ by generalized endomorphisms. Alexandru Nica [30] had already introduced certain Toeplitz type algebras for semigroups P sitting inside a group G inducing a quasi-lattice order. It was Fowler who managed to associate a Toeplitz C^*-algebra $\mathcal{T}_X^{\text{cov}}$ and a Cuntz-Pimsner C^*-algebra $\mathcal{O}_X$ to a so called compactly aligned product system X over a quasi-lattice ordered group (G, P), with $\mathcal{O}_X$ no longer a quotient of $\mathcal{T}_X^{\text{cov}}$ and the canonical morphism from A to $\mathcal{O}_X$ no longer injective in general, and show that it resembles twisted crossed product $A \rtimes_\sigma P$ for twist σ coming from X [11, 12].

Aidan Sims and Trent Yeend [39] introduced the Cuntz-Nica-Pimsner algebra $\mathcal{NO}_X$ of a product system X over a quasi-lattice ordered group (G, P), which happens to be a quotient of $\mathcal{T}_X^{\rm cov}$ and the canonical representation of X on $\mathcal{NO}_X$ is isometric under certain mild conditions. Finally, Toke Carlsen, Nadia Larsen, Aidan Sims and Sean Vittadello [3] managed to describe the core of $\mathcal{NO}_X$ as well as the canonical coaction of G on $\mathcal{NO}_X$ and prove a gauge-invariance uniqueness theorem for $\mathcal{NO}_X$, where Katsura's gauge action of $\mathbb{T}$ (that is, a coaction of $\mathbb{Z}$) is replaced by a coaction of G. Moreover, for a large class of product systems X, they construct a reduced version $\mathcal{NO}_X^r$, generated by an injective Nica covariant Toeplitz representation of X, which admits a coaction of G compatible with that of the gauge group on $\mathcal{T}_X^{\rm cov}$, satisfying the expected couniversal property (like that of Exel [10] for Fell bundles and Katsura [18] for C^*-algebras of correspondences). This is the same as the full Cuntz-Nica-Pimsner algebra $\mathcal{NO}_X$ under certain amenability type condition.

As far as we know, there are few (if any) surveys on simplicity of these important classes of C^*-algebras. In particular, we could not trace any article directly discussing simplicity of Cuntz-Nica-Pimsner algebras, though as explained in the last section, necessary and sufficient conditions are essentially known.

2 From Cuntz algebra to higher rank graph C^*-algebra

The notion of a *universal* C^*-algebra of a set of generators and relations is needed to define most of the later constructions. Let us start by recalling this construction due to Bruce Blackadar [2]. First thing to note is that such a universal C^*-algebra does not exist for any choice of relations, as arbitrary set of relations may not realizable by operators on Hilbert spaces. This suggests that one should start with a notion of representation.

Let $\mathscr{G}$ be a set of generators and $\mathscr{R}$ be a set of relations between generators (which are meaningful for operators on Hilbert spaces). A representation $\pi : \mathscr{G} \to \mathbb{B}(\mathscr{H})$ of $(\mathscr{G}, \mathscr{R})$ on a Hilbert space $\mathscr{H}$ is a map such that $\pi(\mathscr{G})$ satisfies the same relations as $\mathscr{R}$ in $\mathbb{B}(\mathscr{H})$.

Let $\mathscr{A}$ be the free $*$-algebra on $\mathscr{G}$, then a representation π of $(\mathscr{G}, \mathscr{R})$ on $\mathscr{H}$ algebraically extends to a $*$-representation of $\mathscr{A}$ on $\mathscr{H}$. For $x \in \mathscr{A}$, let us define

$$\|x\| := \sup\{\|\pi(x)\| : \ \pi \text{ is a representation of } (\mathscr{G}, \mathscr{R})\}.$$

Now if this supremum is finite, for all $x \in \mathscr{G}$, it defines a C^*-seminorm on $\mathscr{A}$. The completion $C^*(\mathscr{G}, \mathscr{R})$ of $\mathscr{A}$ is then a C^*-algebra, called the universal C^*-algebra of $\mathscr{G}$ subjected to relations $\mathscr{R}$. A typical instance of relations leading to a finite seminorm is one of the form

$$\|p(x_{i_1}, \cdots, x_{i_n}, x_{i_1}^*, \cdots, x_{i_n}^*)\| \leq \varepsilon,$$

for a polynomial p of $2n$ noncommuting variables with complex coefficients.

For a finite set of generators, a finite set of such relations, plus a finite set of finite algebraic relations, the universal C^*-algebra is known to exist, though the case of infinitely many generators and relations is not ruled out. If the universal C^*-algebra $C^*(\mathscr{G},\mathscr{R})$ exists, it has a universal property: for any C^*-algebra $\mathcal{A}$ with a subset X in a one-to-one correspondence with $\mathscr{G}$, satisfying relations as those of $\mathscr{R}$, there is a surjective $*$-homomorphism from $C^*(\mathscr{G},\mathscr{R})$ onto the C^*-subalgebra of $\mathcal{A}$ generated by X.

Now we are ready to describe the classes of C^*-algebras considered in this survey. Generally, these are constructed as universal C^*-algebras for some given sets of generators and relations. We discuss the simplicity results in each case separately.

2.1 Cuntz Algebras

We start with Cuntz algebras, introduced in [4] by Joachim Cuntz, which are unital, simple, nuclear, and purely infinite.

Definition 1. *Let $\mathscr{H}$ be a separable Hilbert space, $n \geq 2$ and $s_1, \ldots s_n$ be isometries in $B(\mathscr{H})$, that is, $s_i^* s_i = id_{\mathscr{H}} =: 1$, for $i = 1, \cdots, n$. The universal C^*-algebra generated by $\{s_i\}_{i=1}^n$ subjected to a single relation $\sum_{i=1}^n s_i s_i^* = 1$, is called the Cuntz algebra $\mathcal{O}_n$. It follows from definition that $s_i^* s_j = 0$, if $i \neq j$, that is, the range projections of the distinct generators are orthogonal.*

For the Cuntz algebra $\mathcal{O}_n$ with generators $s_1, \cdots s_n$, let W be the set of all k-tuples $(i_1, \cdots, i_k)$, for $k \geq 1$, with $i_j \in \{1, \cdots, n\}$, for $j = 1, \cdots k$, let $\ell : W \to \mathbb{N}$ to the corresponding length function. To each $(i_1, \ldots i_k) =: \mu \in W$, associate an isometry $s_\mu := s_{1_1} s_{i_2} \ldots s_{i_k}$. The set $E_k := \{s_\mu s_\nu^* : \ell(\mu) = \ell(\nu)\}$ forms a matrix unit system for $\mathbb{M}_{n^k}(\mathbb{C})$, which then spans $F_k := \mathbb{M}_{n^k}(\mathbb{C})$. Since $\{E_k\}$ is an increasing sequence, $F = \bigcup_{k\in K} F_k$ is a UHF-algebra $\mathbb{M}_{n^\infty}$ of type n^∞. Now for the range projection $p_1 := s_1 s_1^*$, the cutdown ρ_1 by p_1 gives rise to an action of $\mathbb{N}$ on $\mathbb{M}_{n^\infty}$ by endomorphisms, and the Cuntz algebra $\mathcal{O}_n$ is isomorphic to the corresponding semigroup crossed product $\mathbb{M}_{n^\infty} \rtimes_\rho \mathbb{N}$ [4, section 2]. We have the following fundamental result due to Cuntz [4].

Theorem 2. *For $n \geq 2$ the Cuntz algebra $\mathcal{O}_n$ is simple and purely infinite.*

2.2 Cuntz-Krieger Algebras

A more general constructions along the same lines as that of the Cuntz algebras was suggested by Cuntz and Krieger in [5].

Definition 3. *Let A be a $n \times n$ matrix with entries either 0 or 1, and no zero columns or rows. Let $\{s_i\}_{i=1}^n$ be a set of nonzero partial isometries, with mutually orthogonal range projections, acting on a Hilbert space $\mathscr{H}$. The Cuntz-Krieger algebra $\mathcal{O}_A$ is then defined as the universal C^*-algebra with generators $\{s_i\}_{i=1}^n$ and additional relations $s_i^* s_i = \sum_{j=1}^n a_{i,j} s_j s_j^*$, for $i, j = 1, \cdots, n$.*

In the particular case where A is a matrix with 1 in all entries, the Cuntz-Krieger relations give back the Cuntz relation $1 = s_i^* s_i = \sum_{j=1}^n s_j s_j^*$, and we get the Cuntz algebra $\mathcal{O}_n$. Note that, in this case, the fact that the generators are not just partial isometries, but are isometries, is not changing anything, as by universality, $\mathcal{O}_A$ only depends on the choice of matrix A, and not on the choice of generating partial isometries.

Now let W be as in the case Cuntz algebra, and $M_A := \{\mu \in W : s_\mu \neq 0\}$, and let Σ be the set of indices $i \in \{1, \dots, n\}$ for which there are distinct elements $\mu = (\mu_1, \dots, \mu_r)$ and $\nu = (\nu_1, \dots, \nu_s)$ in M_A with $\mu_1 = \nu_1 = \mu_r = \mu_s = i$, while $\mu_k, \nu_l \neq i$, for $1 < k < r$ and $1 < l < s$. We say that an $n \times n$ matrix A satisfies condition (I) if for all $i \in \{1, \dots, n\}$, there exists an $r-$tuple $(i_1, \dots, i_n) \in M_A$ such that $i_1 = i$ and $i_r \in \Sigma$. Cuntz and Krieger observed that for distinct matrices A and B satisfying condition (I), the corresponding Cuntz-Krieger algebras $\mathcal{O}_A$ and $\mathcal{O}_B$ are non isomorphic [5]. It was already observed by Cuntz that the Cuntz algebras $\mathcal{O}_n$ are non isomorphic for distinct values of n, and non of these "uniqueness" results are trivial.

In order to state the simplicity theorem for Cuntz-Krieger algebras, let us recall that a $n \times n$ matrix A is called *irreducible* if for all $1 \leq i, j \leq n$, there is $m \in \mathbb{N}$ such that $(A^m)_{i,j} > 0$, that is the (i, j)-th entry of a power of A is non zero [5].

Theorem 4. *Let A be a $n \times n$ matrix with entries 0 or 1, satisfying condition (I), then the Cuntz-Krieger C^*-algebra $\mathcal{O}_A$ is simple if and only if A is irreducible.*

2.3 Graph Algebras

The C^*-algebras of (higher rank, topological) graphs are studied, among others, by Enomoto-Watatani [8], Kumjian et al [26, 27, 25], Katsura [19, 20, 21, 22], and Drinen-Tomford [7].

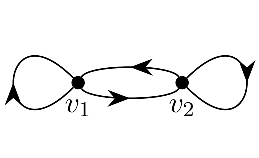
A directed graph $E = (E^0, E^1)$ consists of a vertex set E^0, an edge set E^1, and range and source maps $r, s : E^1 \to E^0$. One can define the graph C^*-algebra $C^*(E)$ of E, again as a universal C^*-algebra. A graph is called row-finite if every vertex receives only finitely many edges, in other words, for each $v \in E^0$, $|r^{-1}(v)| < \infty$. A vertex which receives no edge is called a source. A path is a (finite or infinite) sequence of edges $e_1, e_2, \dots$ with $s(e_{i+1}) = r(e_i)$, for $i = 1, \dots, n-1$. For a path $\mu = e_1, \dots, e_n$, set $s(\mu) = s(e_1)$ and $r(\mu) = r(e_n)$. By a cycle we mean a path μ with $s(\mu) = r(\mu)$.

Definition 5. *Let $E = (E^0, E^1)$ be a row-finite directed graph and $\mathscr{H}$ be a Hilbert space. A Cuntz-Krieger family $\{S, P\}$ consists of a set of partial isometries $S = \{s_e\}_{e \in E^1}$ with mutually orthogonal range projections and a set of mutually orthogonal projections $P = \{p_v\}_{v \in E^0}$, satisfying relations, $p_v = \sum_{e \in r^{-1}(v)} s_e s_e^*$, and $s_e^* s_e = p_{s(e)}$, provided that v is not source. The graph C^*-algebra $C^*(E)$ is the corresponding universal C^*-algebra with respect to a Cuntz-Krieger family $\{S, P\}$.*

For every row-finite graph E, there is a universal C^*-algebra $C^*(E)$ generated by a given Cuntz-Krieger family $\{S, P\}$, in the sense that for any C^*-algebra $\mathcal{B}$ containing a Cuntz-Krieger family $\{S', T'\}$, there exist a $*$-homomorphism $\pi : C^*(E) \to \mathcal{B}$, mapping S and P to S' and T', respectively [35, Proposition 1.21].

As for the "uniqueness", let E be a row-finite directed graph such that every cycle has an entry point, and $\{S, P\}$ be a Cuntz-Krieger family in a C^*-algebra $\mathcal{B}$ with all projections in P non-zero, then there is an isomorphism between $C^*(E)$ and the C^*-algebra $C^*(S, P)$ generated by $S \cup P$ in $\mathcal{B}$.

To see how this class of algebras extends the class of Cuntz-Kerieger algebras, take a natural number $n \geq 2$, and a matrix A with entries 0 and 1, and define $E^0_A = \{1, 2, \ldots, n\}$ and $E^1_A = \{\overline{ij} : a_{i,j} = 1\}$ with $s(\overline{ij}) = j$ and $r(\overline{ij}) = i$. Let $S = \{s_i\}_{i=1}^n$ be a set partial isometries satisfying $s_i^* s_i = \sum_{j=1}^n a_{i,j} s_j s_j^*$ and let $P = \{s_i s_i^*\}_{i=1}^n$, which is a set of orthogonal projections, then $\{S, P\}$ is a Cuntz-Krieger family of E_A and $\mathcal{O}_A$ is nothing but $C^*(E_A)$. In particular, one could realize a Cuntz algebra as a graph C^*-algebra with the above recipe, but let us give an alternative (but equivalent) way here: for a natural number $n \geq 2$, let $F = (F^0, F^1)$ be the graph with only one vertex $F^0 = \{v\}$ and n edges $F^1 = \{e_1, e_2, \ldots, e_n\}$, having trivial range and source maps. Then $\mathcal{O}_n$ is isomorphic to $C^*(F)$. To illustrate these constructions, let us look at the particular case of $\mathcal{O}_2$: the left graph below is the graph just described, whereas the right graph is the graph constructed by the Cuntz-Kriger matrix recipe.



Note that the right graph is nothing but the dual graph of the left one, and indeed the C^*-algebra of the dual graph is always isomorphic to that of the original graph. More precisely, let E be a row-finite graph with no sources and let $\hat{E}$ be the dual graph with $\hat{E}^0 := E^1$, and $\hat{E}^1$:= the paths of length 2 (with range $r(e_1e_2) = e_1$ and $s(e_1e_2) = e_2$). Then the dual graph is row-finite and gives an isomorphic graph C^*-algebra.

Before stating the simplicity result for graph algebras [35, Theorem 4.14], we need to recall the concept of cofinality. Let u and v be two vertices of a directed graph. By writing $u \leq v$ we mean that there is a path μ from v to u, that is, $s(\mu) = v$ and $r(\mu) = u$. We say that a graph E is *cofinal* if for every path μ and vertex $u \in E^0$, there exists a vertex v on μ with $u \leq v$.

Theorem 6. *Let E be a row-finite directed graph, then $C^*(E)$ is simple if and only if E is cofinal and every cycle has an entry.*

In the context of simple graph C^*-algebras, there is an important dichotomy: let E be a row-finite graph and $C^*(E)$ is simple, then either $C^*(E)$ is an AF-algebra (if E has no cycle) or it is purely infinite (if E has cycle). Here, being an AF-algebra means that $C^*(E)$ is a direct limit of finite dimensional C^*-algebras.

More generally, Katsura studied C^*-algebras of topological graphs in [19, 20, 21, 22], as follows. A topological graph $E = (E^0, E^1, d, r)$ consists of two locally compact spaces E^0 and E^1, and two maps $d, r : E^1 \to E^0$, where d is locally homeomorphic and r is continuous. For the open subset E^0_{rg} of E^0 consisting of those $v \in E^0$ for which there exists a neighborhood V of v with $r^{-1}(V) \subseteq E^1$ compact, and $r(r^{-1}(V)) = V$, and closed subset $E^0_{\text{sg}} = E^0 \backslash E^0_{\text{rg}}$. For $v \in E^0$, a negative orbit of v is either a finite path $\mu \in E^n$ with $r^n(\mu) = v$ and $d^n(\mu) \in E^0_{\text{sg}}$, or an infinite path $\mu \in E^\infty$ with $r^\infty(\mu) = v$. Set $E^* := \bigcup_{n=0}^\infty E^n$ and extend d^n and r^n to get a local homeomorphism $d^* : E^* \to E^0$ and a continuous map $r^* : E^* \to E^0$. The positive orbit space $\text{Orb}^+(v)$ of $v \in E^0$ is defined by

$$\text{Orb}^+(v) := \{r^*(\mu) \in E^0 : \mu \in E^*, d^*(\mu) = v\}.$$

For each $\mu = (e_1, e_2, \cdots, e_n) \in E^n$ of $v \in E^0$, with $n \in \mathbb{N} \cup \{\infty\}$, the negative orbit space $\text{Orb}^-(v, \mu)$ is the subset $\{v, d(e_1), d(e_2), \cdots, d(e_n)\}$ of E^0 and the orbit space $\text{Orb}(v, \mu)$ of $v \in E^0$ with respect to a negative orbit μ is the of orbits $\text{Orb}^+(w)$, where w runs over $\text{Orb}^-(v, \mu)$. A topological graph E is said to be *minimal* if the orbit space $\text{Orb}(v, \mu)$ is dense in E^0 for every $v \in E^0$ and every negative orbit μ of v, and *topologically free* if the set of base points of loops without entrances has an empty interior. Let n, m be positive integers, and $k := \min\{n, m\}$. For subsets $U \subseteq E^n$ and $V \subseteq E^m$, let $U \pitchfork V := (U|_k) \cap (V|_k)$, where $U|_k := \{(e_1, e_2, \cdots, e_k) \in E^k : (e_1, e_2, \cdots, e_n) \in U\}$, and the same for $V|_k$. A non-empty open, relatively compact subset V of E^0 is a contracting open set if its there exist non-empty open subsets $U_k \subseteq E^{n_k}$, for $k = 1, 2, \cdots, m$ with $n_k \geq 1$ satisfying:

$(i) r^{n_k}(U_k) \subseteq V$, for $k = 1, 2, \cdots, m$,

$(ii) U_k \pitchfork U_l = \emptyset$, for $k \neq l$,

$(iii) \bar{V} \subset \bigcup_{k=1}^m d^{n_k}(U_k)$.

We say that a topological graph E is contracting at $v_0 \in E^0$ if $\text{Orb}^+(v_0) = E_0$, and any neighborhood V_0 of v_0 contains a contracting open, relatively compact set $V \subseteq V_0$, and E is *contracting* if it is contracting at some $v_0 \in E^0$.

To each topological graph E, one can associate a C^*-correspondence $C_d(E^1)$ over $C_0(E^0)$ defined by

$$C_d(E^1) := \{\xi \in C_0(E^1) : \langle \xi, \xi \rangle \in C_0(E^0)\}$$

where the inner product is given by

$$\langle \xi, \eta \rangle(v) := \sum_{e \in d^{-1}(v)} \xi(e)\eta(e), \quad (\xi, \eta \in C_d(E^1), v \in E^0),$$

and the left and right actions are defined by $(f\xi g)(e) = f(r(e))\xi(e)g(d(e))$, for $f, g \in C_0(E^0)$, $\xi \in C_d(E^1)$ and $e \in E^1$. The left action defines a $*$-homomorphism

$$\pi_r : C_0(E^0) \to \mathscr{L}(C_d(E^1)); \quad \pi_r(f)\xi = f\xi, \quad (f \in C_0(E^0), \xi \in C_d(E^1)).$$

The C^*-algebra $\mathcal{O}(E)$ is the universal C^*-algebra generated by the images of a $*$-homomorphism $t_0 : C_0(E^0) \to \mathcal{O}(E)$ and a linear map $t_1 : C_d(E^1) \to \mathcal{O}(E)$ satisfying:

$(i) t_1(\xi)^* t_1(\eta) = t_0(\langle \xi, \eta \rangle)$,
$(ii) t_0(f) t_1(\xi) = t_1(\pi_r(f)\xi)$,
$(iii) t_0(g) = \phi(\pi_r(g))$,

for $\xi, \eta \in C_d(E^1)$, $f \in C_0(E^0)$, and $g \in C_0(E^0_{\mathrm{rg}})$, where

$$\phi : \mathscr{K}(C_d(E^1)) \to \mathcal{O}(E); \quad \theta_{\xi,\eta} \mapsto t_1(\xi)t_1(\eta)^*, \quad (\xi, \eta \in C_d(E^1)).$$

Katsura proved that for a minimal and contracting topological graph E, the topological graph C^*-algebra C^*-algebra $\mathcal{O}(E)$ is simple and purely infinite [22, Theorem A]. He also obtained the following simplicity characterization [21, Theorem 8.12].

Theorem 7. *For a topological graph E, the following conditions are equivalent:*
(i) The C^*-algebra $\mathcal{O}(E)$ is simple.
(ii) E is minimal and topologically free.
(iii) E is minimal and not generated by a loop

2.4 Cuntz-Pimsner algebras and Katsura construction

Let $\mathcal{A}$ be a C^*-algebra and X a right Hilbert C^*-module over $\mathcal{A}$ with right inner product $\langle \cdot, \cdot \rangle_X$. Let $\mathscr{L}(X)$ be the set of adjointable operators on X, then X together with a left action given by a $*$-homomorphism $\phi_X : \mathcal{A} \to \mathscr{L}(X)$ is called a C^*-correspondence over $\mathcal{A}$. Every C^*-algebra is a C^*-correspondence over itself by trivial product and inner product $\langle a, b \rangle_{\mathcal{A}} := a^*b$. A morphism of $\mathcal{A}$-correspondences is a bimodule map $T : X_1 \to X_2$ satisfying ${}_{X_2}\langle T(x), T(y) \rangle = {}_{X_1}\langle x, y \rangle$, for $a \in \mathcal{A}$ and $x, y \in X_1$, which is the isomorphism of the category of correspondences when T is also surjective, writing $X_1 \simeq X_2$. If T is adjointable, and $T^*T = id_{X_1}$, $TT^* = id_{X_2}$, then we have a unitary equivalence, writing $X_1 \approx X_2$.

We denote by $\mathscr{K}(X)$ the subalgebra of compact operators on X. This is the closed subalgebra generated by "finite rank" operators (with the warning that their ranges are merely finitely generated, and not necessarily finite dimensional; in particular, operators in $\mathscr{K}(X)$ are not necessarily compact as operators on the Banach space X). A C^*-correspondence X over $\mathcal{A}$ is called essential if $\{\phi_X(a)x : a \in \mathcal{A}, x \in X\}$ is a total set in X.

Let X be a C^*-correspondence over $\mathcal{A}$. By a representation (π, t) of X in a C^*-algebra $\mathcal{B}$ we mean a $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}$ and a linear map $t : X \to \mathcal{B}$

satisfying, $t(\xi)^*t(\eta) = \pi(\langle \xi, \eta \rangle)$ and $\pi(a)^*t(\xi) = t(a \cdot \xi)$, for $a \in \mathcal{A}$ and $\xi, \eta \in X$. The *-algebra $C^*(\pi, t)$ is then generated by the ranges of the maps π and t. For a representation (π, t) of X in $\mathcal{B}$, one can define $\psi_t : \mathscr{K}(X) \to \mathcal{B}$, mapping each rank one operator $\theta_{\xi,\eta}$ to $t(\xi)t(\eta)^*$. The linear maps $\psi_n^{(\pi,t)} : X^{\otimes n} \to \mathcal{B}$ are then mapping $x_1 \otimes x_2 \otimes \cdots \otimes x_n$ to $t(x_1)t(x_2) \ldots t(x_n)$ [18, section 4]. Now for the Katsura ideal,

$$J_X = \phi_X^{-1}(\mathscr{K}(X)) \cap (\ker \phi_X)^{\perp},$$

a Cuntz-Pimsner covariant representation is one for which $\pi(a) = \psi_t(\phi_X(a))$, for all $a \in J_X$.

Definition 8. *Let X be a C^*-correspondence over C^*-algebra $\mathcal{A}$. The Cuntz-Pimsner algebra $\mathcal{O}_X$ is defined as the universal C^*-algebra given by Cuntz-Pimsner covariant representations.*

Katsura in [18] proved a gauge invariance uniqueness theorem and built universal covariant representation (π_X, t_X), with $\mathcal{O}_X \cong C^*(\pi_X, t_X)$, in the sense that, for all covariant representations (π, t) of X, there exist a homomorphism $\rho : \mathcal{O}_X \to C^*(\pi, t)$ with $\pi = \rho \circ \pi_X$ and $t = \rho \circ t_X$.

The graph algebras (and so the Cuntz and Cuntz-Krieger algebras) are special cases of this construction. To see this, let $E = (E^0, E^1)$ be a row-finite directed graph. Let $a \in \mathcal{A} := c_0(E^0)$, $f, g \in c_c(E^1)$, $v \in E^0$ and $e \in E^1$ be given, and define,

$$(f \cdot a)(e) := f(e)a(s(e)) \qquad\qquad \langle f, g \rangle_{\mathcal{A}}(v) = \sum_{e \in s^{-1}(v)} \overline{f(e)}g(e)$$

The completion X_E of $c_c(E^1)$ is then a Hilbert $\mathcal{A}$-module. Defining homomorphism $\phi : \mathcal{A} \to \mathscr{L}(X_E)$ by $\phi(a)(f)(e) = (a \cdot f)(e) := a(r(e))f(e)$, for $a \in \mathcal{A}$, $f \in c_c(E^1)$ and $e \in E^1$, we may equip X_E with a C^*-correspondence structure, referred to as the graph correspondence of E. Then $\phi^{-1}(\mathscr{K}(X_E))$ is the span of point mass functions δ_v with $|r^{-1}(v)| < \infty$, and $\phi(\delta_v) = 0$, for source vertex v [14, Proposition 4.4].

Indeed, for a vertex v with $|r^{-1}(v)| < \infty$, $\phi(\delta_v) = \sum_{e \in r^{-1}(v)} \theta_{\delta_e, \delta_e}$. In this case, the Katsura ideal is nothing but the closed linear span of the set $\{\delta_v : \ 0 < |r^{-1}(v)| < \infty\}$. In particular, a representation (π, t) is covariant if and only if for every vertex v with $|r^{-1}(v)| < \infty$, we have,

$$\pi(\delta_v) = \psi_t(\phi(\delta_v)) = \sum_{e \in r^{-1}(v)} t(\delta_e)t(\delta_e)^*.$$

Now setting $S = \{t(\delta_e)\}_{e \in E^1}$ and $P = \{\pi(\delta_v)\}_{v \in E^0}$, the above computation shows that every covariant representation of X_E induces an Cuntz-Krieger family, and by universality, $C^*(E)$ is nothing but $\mathcal{O}_{X_E}$.

For stating simplicity result for Katsura algebras, we need some preparation. Let X be a C^*-correspondence over a C^*-algebra $\mathcal{A}$. Then, X is called *minimal* if $\langle X, JX\rangle \subseteq J$, for a closed ideal J of A, implies $J = 0$ or A. It is called full if $\{\langle x, y\rangle : x, y \in X\}$ is a total set in $\mathcal{A}$. When $\mathcal{A}$ is unital, X is called *aperiodic* if for every non zero integer n, $X^{\otimes n} \approx I_{\mathcal{A}}$ implies $n = 0$, where $X^{\otimes n}$ is the n-fold tensor product, and $I_{\mathcal{A}}$ is the identity C^*-correspondence, that is, $\mathcal{A}$ itself with its canonical inner product.

The first simplicity result [38, Theorem 3.9], is for Cuntz-Pimsner algebras, i.e., Katsura algebras of a full C^*-correspondence with injective left action.

Theorem 9. *Let X be a full C^*-correspondence X over a unital C^*-algebra $\mathcal{A}$ with an injective left $\mathcal{A}$-action. Then $\mathcal{O}_X$ is simple if and only if X is aperiodic and minimal.*

Eryüzlü and Tomforde in [9] defined the condition (S) (where S stands for simplicity) and obtained a simplicity result for Katsura algebras [9, Theorem 4.3]. Let X be a C^*-correspondence over a C^*-algebra $\mathcal{A}$. Let (π_X, t_X) be a universal covariant representation of X into $\mathcal{O}_X$. A vector $\xi \in x^{\otimes n}$ with $n \in \mathbb{N}$ is called non-returning if for $m < n$ and $\eta \in X^{\otimes m}$, $t_X^{\otimes n}(\xi)t_X^{\otimes m}(\eta)t_X^{\otimes n}(\xi) = 0$. We say X satisfies condition (S) if for each $a \in \mathcal{A}$ with $a \geq 0$, and each $n \in \mathbb{N}$ and $\epsilon > 0$, there exist $m > n$ and a non-returning unit vector $\xi \in X^{\otimes m}$ with $\|\langle \xi, a\xi\rangle\| > \|a\| - \epsilon$. An ideal I of $\mathcal{A}$ is called invariant if $IX \subseteq XI$, and for all $a \in J_X$, $aX \subseteq XI$ implies $a \in I$.

Theorem 10. *Let X be a C^*-correspondence over a C^*-algebra $\mathcal{A}$. If X satisfies condition (S) and $\mathcal{A}$ has no non trivial invariant ideal, then $\mathcal{O}_X$ is simple.*

2.5 Higher rank graph algebras

As a generalization of the graph algebras, Kumjian and Pask introduced the notion of Higher rank graphs in [25]. For a natural number k, a k-graph (or a graph of rank k) (Λ, d) consist of a countable small category Λ, with source and range maps s, r, and a functor $d : \Lambda \to \mathbb{N}^k$ satisfying the factorization property, that is, for all $\lambda \in \Lambda$, $m, n \in \mathbb{N}^k$ with $d(\lambda) = m + n$, there exist unique elements $\mu, \nu \in \Lambda$ such that $\lambda = \mu\nu$ with $d(\mu) = m$ and $d(\nu) = n$.

Often for $n \in \mathbb{N}^k$, the notation Λ^n is used for $d^{-1}(n)$. Similar to the directed graph case, a k-graph is called row-finite if $r^{-1}(v) \cap \Lambda^n$ is finite for every $v \in \Lambda^0$ and $n \in \mathbb{N}^k$.

As a concrete example, for a natural number k, define Ω_k to be the small category with object set $\mathbb{N}^k$ and morphism set consisting of $(m, n) \in \mathbb{N}^k \times \mathbb{N}^k$ with $m \leq n$, with the range and source maps defined by $r(m, n) = m$ and $s(m, n) = n$. For $d : \Omega_k \to \mathbb{N}^k$; $d(m, n) = n - m$, (Ω_k, d) is a k-graph.

In order to build a universal C^*-algebra like the previous cases, we need to define proper relations. For a row-finite k-graph (Λ, d), a Cuntz-Krieger Λ-family is a family $S = \{s_\lambda : \lambda \in \Lambda\}$ of partial isometries, with $\{s_v : v \in \Lambda^0\}$ being mutually orthogonal projections, satisfying, $s_\lambda s_\mu = s_{\lambda\mu}$ for $s(\lambda) = r(\mu)$, $s_\lambda^* s_\lambda = s_{s(\lambda)}$, and $s_v = \sum_{\lambda \in r^{-1}(v)\cap\Lambda^n} s_\lambda s_\lambda^*$, for $v \in \Lambda^0$ and $n \in \mathbb{N}^k$.

The higher rank graph algebra $C^*(\Lambda)$ is then the universal C^*-algebra generated by Cuntz-Krieger Λ-families. Every row-finite k-graph Λ with no source, is generated by a universal Cuntz-Krieger Λ-family $\{s_\lambda\}_{\lambda\in\Lambda}$, in the sense that, for any Cuntz-Krieger Λ-family $\{t_\lambda\}_{\lambda\in\Lambda}$ in a C^*-algebra $\mathcal{B}$, there exist a homomorphism $\pi : C^*(\Lambda) \to \mathcal{B}$ with $\pi(s_\lambda) = t_\lambda$.

To see that a row-finite directed graph is a special case, we need to build a small category $\mathscr{P}_E$ (path category) out of a directed graph $E = (E^0, E^1)$ with source and range maps s_E and r_E. Let the object set $\mathscr{P}_E^0$ of $\mathscr{P}_E$ consist of vertices in E^0 and the morphisms of $\mathscr{P}_E$ simply be the finite paths μ with $s(\mu) = s_E(\mu)$ and $r(\mu) = r_E(\mu)$. Consider the product $(\mu,\nu) \to \mu\nu$, defined when $s(\mu) = r(\nu)$, by pasting the paths. Define the functor of $d : \mathscr{P}_E \to \mathbb{N}$ to be the path length function. The directed graphs are then just 1-graphs in this context.

The simplicity of higher rank graph C^*-algebras was characterized by Kumjian and Pask [25, Proposition 4.8]. First, we need to recall some definitions. Let Λ be a k-graph, and define the infinite path space of Λ as the set Λ^∞ of all k-graph morphisms $x : \Omega_k \to \Lambda$. For $v \in \Lambda^0$ put $\Lambda^\infty(v) = \{x \in \Lambda^\infty : x(0) = v\}$, and for $p \in \mathbb{N}^k$, define $\sigma^p : \Lambda^\infty \to \lambda^\infty$ by $\sigma^p(x)(m,n) = x(m+p, n+p)$, for $x \in \Lambda^\infty$ and $(m,n) \in \Omega_k$. We say that Λ is *cofinal* if for every $x \in \Lambda^\infty$ and vertex $v \in \Lambda^0$, there is $\lambda \in \Lambda$ and $n \in \mathbb{N}^k$ with $s(\lambda) = x(n)$ and $r(\lambda) = v$. A path $x \in \Lambda^\infty$ is periodic if there exist $p \in \mathbb{Z}^k$ such that for $(m,n) \in \mathbb{N}^k$ with $m+p \geq 0$, $x(m+p, n+p) = x(m,n)$; and aperiodic if $\sigma^n x$ is not periodic for all $n \in \mathbb{N}^k$. We say that Λ satisfies the aperiodicity condition, if for every vertex $v \in \Lambda^0$ there is an aperiodic path $x \in \Lambda^\infty(v)$.

Theorem 11. *If Λ satisfies the aperiodicity condition, then $C^*(\Lambda)$ is simple if and only if Λ is cofinal.*

This result is improved by Robertson-Sims [37] who showed that if Λ is a row-finite k-graph with no sources, then $C^*(\Lambda)$ is simple if and only if Λ is cofinal and satisfies Kumjian-Pask aperiodicity condition, known also as Condition (A).

3 Product systems and Cuntz-Nica-Pimsner algebras

Let us review the notion of product system and construction of the full and reduced Cuntz-Nica-Pimsner algebras $\mathcal{NO}_X$ and $\mathcal{NO}_X^r$. The C^*-algebras of higher rank graphs and Cuntz-Pimsner algebras fit in this framework.

Let $\mathcal{A}$ be a C^*-algebra, P be a discrete monoid with identity e, and for each $p \in P$, X_p be a C^*-correspondence over $\mathcal{A}$. A product system of C^*-correspondences over P is a semigroup (under tensor product) $X = \{X_p\}_{p\in P}$ consisting of C^*-correspondences, with $X_e \simeq \mathcal{A}$, as C^*-correspondences, such that the multiplication in X by elements of X_e implements the left and right actions of $\mathcal{A}$ on each X_p, and for $p, q \in P\backslash\{e\}$, there is an isomorphism of $\mathcal{A}$-correspondences $M_{p,q} : X_p \otimes_{\mathcal{A}} X_q \to X_{pq}$, satisfying $M_{p,q}(x \otimes_A y) = xy$, for $x \in X_p$ and $y \in X_q$. When $p \neq e$, there is a morphism $\iota_p^{pq} : \mathscr{L}(X_p) \to \mathscr{L}(X_{pq})$ satisfying $\iota_p^{pq}(S)(xy) = (Sx)y$, for $x \in X_p, y \in X_q$, and $S \in \mathscr{L}(X_p)$. If we identify $\mathscr{K}(X_e)$ with $\mathcal{A}$, we also get $\iota_e^p : \mathscr{K}(X_e) \to \mathscr{L}(X_q)$, given by the left action ϕ_p of X_p. Here, for Banach right and left $\mathcal{A}$-modules X and Y, the balanced tensor product $X \otimes_{\mathcal{A}} Y$ is defined as the quotient of $X \otimes Y$ by the closed submodule generated by elements of the form $x \cdot a \otimes y - x \otimes a \cdot y$, for $x \in X, y \in Y$, and $a \in \mathcal{A}$.

In this review, we restrict ourselves to monoids arising from quasi-lattice ordered groups. Let G be a discrete group with identity e, $P \leq G$ be a sub-semigroup of G with $P \cap P^{-1} = \{e\}$. The pair (G, P) is called a quasi-lattice ordered group if for the partial order on G defined for $p, q \in G$ by $p \leq q$ if $p^{-1}q \in P$, any two elements $p, q \in P$ with a common upper bound have a least common upper bound $p \vee q \in P$. We write $p \vee q = \infty$ when p and q fail to have a common upper bound in P and $p \vee q < \infty$, otherwise. We denote the $\mathcal{A}$-valued inner product on X_p by $\langle \cdot, \cdot \rangle_p$.

A representation ψ of X in a C^*-algebra $\mathcal{B}$ is a map $\psi : X \to \mathcal{B}$ such that ψ_e is a $*$-homomorphism, and for $p \in P$, $\psi_p := \psi|_{X_p}$ is linear, satisfying $\psi_p(x)\psi_q(z) = \psi_{pq}(xz)$ and $\psi_e(\langle x, y \rangle_p) = \psi_p(x)^*\psi_p(y)$, for $p, q \in P$, $x, y \in X_p$ and $z \in X_q$. A representation ψ is called injective if ψ_e is injective. In this case, it follows that each ψ_p is an isometry. For each $p \in P$, there is a $*$-homomorphism $\psi^{(p)} : \mathscr{K}(X_p) \to \mathcal{B}$ satisfying, $\psi^{(p)}(\theta_{x,y}) = \psi_p(x)\psi_p(y)^*$, for $x, y \in X_p$ [33]. When the product system is a Hilbert C^*-bimodule, a representation of X is said to be a Hilbert C^*-bimodule representation of X if moreover, $\psi_e({}_p\langle x, y \rangle) = \psi_p(x)\psi_p(y)^*$, for $p \in P$ and $x, y \in X_p$. The system X is called compactly aligned if $\iota_p^{p\vee q}(S)\iota_q^{p\vee q}(T) \in \mathscr{K}(X_{p\vee q})$, for each $p \vee q < \infty$, $S \in \mathscr{K}(X_p)$, and $T \in \mathscr{K}(X_q)$. For the rest of the section, (G, P) is a quasi-lattice ordered group and X is a product system over P.

We denote the C^*-algebra generated by the image of a representation ψ in $\mathcal{B}$ by $C^*(\psi)$. If ι is the universal representation of product system X over P and $\mathcal{T}_X := C^*(\iota)$, then for each representation ψ of X, there is a $*$-homomorphism $\rho : \mathcal{T}_X \to C^*(\psi)$ with $\rho(i_p(x)) = \psi_p(x)$, $p \in P$ and $x \in X_p$.

Let X be a compactly aligned product system over P. A representation ψ of X is Nica coveariant if for each $p, q \in P$ and $S \in \mathscr{K}(X_p)$ and $T \in \mathscr{K}(X_q)$,

$$\psi^{(p)}(S)\psi^{(q)}(T) = \begin{cases} \psi^{(p\vee q)}(\iota_p^{(p\vee q)}(S)\iota_q^{(p\vee q)}(T)) & \text{if } p \vee q < \infty \\ 0 & \text{if } p \vee q = \infty \end{cases}$$

The C^*-algebra generated by the universal Nica covariant representation $\iota = \iota_X$ is denoted here by $\mathcal{T}_X^{\text{cov}}$ (or by $\mathcal{T}_{\text{cov}}(X)$ in some texts). It is known that the set $\{\iota(x)\iota(y)^* : x, y \in X\}$ is total in $\mathcal{T}_X^{\text{cov}}$ [11] (see also, [3, 2.2]).

3.1 Full and reduced Cuntz-Nica-Pimsner algebras

Let (G, P) be a quasi lattice ordered group and X is a compactly aligned product system over P. For the closed ideal ideals $I_e = \mathcal{A}$ and $I_q = \bigcap_{e<p\leq q} ker(\phi_p)$, for

$q \in P\backslash\{e\}$, consider the $\mathcal{A}$-correspondence $\tilde{X}_q = \bigoplus_{p\leq q} X_p.I_{p^{-1}q}$, with implementing left action $\tilde{\phi}_q$. We say that X is $\tilde{\phi}$-injective if all the left actions $\tilde{\phi}_q$ are injective. For $p, q \in P$, let $\tilde{\iota}_p^q : \mathscr{L}(X_p) \to \mathscr{L}(\tilde{X}_q)$ be defined by $\tilde{\iota}_p^q(T) = \bigoplus_{r\leq q} \iota_p^r(T)$.

A representation ψ of X is called Cuntz-Nica-Pimsner covariant (CNP-covariant) if for each finite subset $F \subseteq P$, each $p \in F$ and $T_p \in \mathscr{K}(X_p)$, if $\sum_{p\in F} \tilde{\iota}_p^q(T_p) = 0$, then $\sum_{p\in F} \psi^{(p)}(T_p) = 0$, for large q, that is, for each $s \in P$ there exists $r \geq s$ such that the latter sum is zero for $r \leq q$.

The C^* algebra $\mathcal{NO}_X$ generated by the range of universal CNP-covariant representation j_X is called the (full) Cuntz-Nica-Pimsner algebra of X. Since j_X is an injective representation, by universality of $\mathcal{T}_X^{\text{cov}}$, there exists a canonical surjective $*$-homomorphism $\rho_{CNP} : \mathcal{T}_X^{\text{cov}} \to \mathcal{NO}_X$. Let $\iota = \iota_X$ be the universal Nica covarinat representation generating $\mathcal{T}_X^{\text{cov}}$, then the core $\mathcal{F}_X$ of $\mathcal{T}_X^{\text{cov}}$ is the span closure $\{i_p(x)i_p(y)^* : x, y \in X_p\}$, which also coincides with span closure of the range of $i_X^{(p)}$.

A criterion for injectivity of induced representations is available as follows [3, Theorem 3.8]: assume that either the left actions on all fibers of X are injective or that P is directed and X is $\tilde{\phi}$-injective. For a CNP-covariant representation ψ of X on a C^*-algebra B where $\rho : \mathcal{NO}_X \to B$ is the induced homomorphism, ρ is in injective on $\rho_{CNP}(\mathcal{F}_X)$ if and only if ψ is injective as a representation.

Let $\delta_G : C^*(G) \to C^*(G) \otimes C^*(G)$ be the canonical homomorphism mapping $g \in G$ to $i_G(g) \otimes i_G(g)$, where i_G is the canonical inclusion of G in the full group C^*-algebra $C^*(G)$. A (full) coaction of G on a C^*-algebra $\mathcal{A}$ is a nondegenerate injective $*$-homomorphism $\delta : \mathcal{A} \to \mathcal{A} \otimes C^*(G)$ satisfying, $(\delta \otimes id_{C^*(G)}) \circ \delta = (id_{\mathcal{A}} \otimes \delta_G) \circ \delta$. The generalized fixed-point algebra of δ is $\mathcal{A}_e^\delta$, where $\mathcal{A}_g^\delta = \{a \in \mathcal{A} | \delta(a) = a \otimes i_G(g)\}$, for $g \in G$. The generalized fixed-point algebra associated with $\mathcal{T}_X^{\text{cov}}$ is nothing but the core of $\mathcal{T}_X^{\text{cov}}$ [3]. A representation ψ of X in $\mathcal{A}$ is said to be gauge compatible if there is a coaction δ of G on A with $\delta(\psi_p(x_p)) = \psi_p(x_p) \otimes i_G(p)$, for $p \in P$ and $x \in X_p$. The C^*-algebras $\mathcal{T}_X^{\text{cov}}$ and $\mathcal{NO}_X$ associated to a compactly aligned product system X over P have gauge compatible coactions with respect to their universal representation [3].

Again, let us suppose that either the left action on each fiber is injective or P is directed. Also assume that X is $\tilde{\phi}$-injective. Then the C^*-algebra $\mathcal{NO}_X^r$ generated by the couniversal gauge compatible injective Nica covariant representation of X is called the reduced Cuntz-Nica-Pimsner C^*-algebra of X, whose canonical coaction is denoted by ν^r (we refer the reader to [3, Theorem 4.1] for the notion of couniversal representation). The C^*-algebras $\mathcal{NO}_X$ and $\mathcal{NO}_X^r$ are known to be the same as the full and reduced cross sectional algebras of the Fell bundle $\{(\mathcal{NO}_X)_g^\nu \times \{g\}\}_{g\in G}$ [3]. When X is $\tilde{\phi}$-injective, we say that $\mathcal{NO}_X$ has the gauge invariant uniqueness property if for any C^*-algebra B, the injectivity of any surjective $*$-homomorphism $\phi : \mathcal{NO}_X \to \mathcal{B}$ is equivalent to the injectivity of $\phi|_{\iota_X(\mathcal{A})}$ plus the existence of a coaction β of G on B with $\beta \circ \phi = (\phi \otimes \text{id}_{C^*(G)}) \circ \nu$. This is known to be automatic whenever G is amenable [3, Corollary 4.12]. In general, the following guage invariance uniqueness property is verified in [3]: suppose that either the left action on each fiber is injective or P is directed, and X is $\tilde{\phi}$-injective, then the following conditions are equivalent:

(i) the canonical surjection of $\mathcal{NO}_X$ onto $\mathcal{NO}^r_X$ is an isomorphism,

(ii) $\mathcal{NO}_X$ has the gauge invariant uniqueness property,

(iii) given two injective gauge compatible CNP-covariant representations $\psi_i : X \to B_i$ of X whose image generates $\mathcal{B}_i$, $i = 1, 2$, there is a $*$-isomorphism $\phi : \mathcal{B}_1 \to \mathcal{B}_2$ with $\phi \circ \psi_1 = \psi_2$.

As promised, we now explain how Cuntz-Pimsner algebras and higher rank graph algebras could be considered as Cuntz-Nica-Pimsner algebras. Let X be a C^*-correspondence over a C^*-algebra $\mathcal{A}$, then the product system $X^\otimes$ over $\mathbb{N}$ could be defined as the monoid of n-fold tensor products $X^{\otimes n}$ with $X^{\otimes 0} := \mathcal{A}$, where the isomorphism $M_{m,n} : X^{\otimes m} \otimes X^{\otimes n} \to X^{\otimes m+n}$ gives the product of the monoid. Since $\mathbb{N}$ is totally ordered, the product system is compactly aligned. Now we could describe how Sims and Yeend proved that the Cuntz-Pimsner algebras are special cases of Cuntz-Nica-Pimsner algebras [39, Proposition 5.3]. Let X be a C^*-correspondence over a C^*-algebra $\mathcal{A}$, let (π_X, t_X) be the universal covatiant representation of X on $\mathcal{O}_X$ and j_X be the universal representation of $X^\otimes$ on $\mathcal{NO}_{X^\otimes}$, then there is an isomorphism $\theta : \mathcal{O}_X \to \mathcal{NO}_{X^\otimes}$, mapping $\pi_X(a)$ and $t_X(x)$ to $j_X(a)$ and $j_X(x)$, respectively. For a representation (π, t) of X the representation $\psi^{(\pi,t)}$ (defined as in the subsection 2.4) is CNP-covariant if and only if (π, t) is covariant.

As a concrete example, let (Λ, d) be a row-finite k-graph and $A = c_0(\Lambda_0)$. For $n \in \mathbb{N}^k$, let X_{Λ^n} be defined as X_E (see, subsection 2.4). For $\lambda \in \Lambda^n$ and $\mu \in \Lambda^m$, define $\delta_\lambda \delta_\mu = \delta_{\lambda\mu}$, if (λ, μ) is multiplicable, and 0 otherwise. Then $P_\Lambda = \{X_{\Lambda^n}\}_{n \in \mathbb{N}^k}$ is a product system over $\mathbb{N}^k$ [39, Section 5]). For λ, ν, consider the collection $MCE(\mu, \nu)$ consisting of those $\lambda \in \Lambda$ satisfying $d(\lambda) = d(\mu) \wedge d(\nu), \lambda = \mu\mu' = \nu\nu'$, for some $\mu'\nu' \in \Lambda\}$. The k-graph Λ is then called finitely aligned if $MCE(\mu, \nu)$ is finite, for every choice $\mu, \nu \in \Lambda$. Sims and Yeend [39, Proposition 5.4] observed that C^*-algebras of higher rank graphs could be described as Cuntz-Nica-Pimsner algebras as follows. Let Λ be a finitely aligned k-graph, and P_Λ be the associated product system. Let $\{s_\lambda\}_{\lambda \in \Lambda}$ be the universal Cuntz-Krieger-Λ family in $C^*(\Lambda)$, and let j_{P_Λ} be the universal CNP-covariant representation of X in $\mathcal{NO}_{P_\Lambda}$. Then there is an isomorphism $C^*(\Lambda) \to \mathcal{NO}_{P_\Lambda}$, mapping s_λ to $j_{P_\Lambda}(\delta_\lambda)$, for all $\lambda \in \Lambda$.

3.2 Kishimoto Condition and Simplicity

Throughout this subsection, following [3], we assume that either the left action on each fiber is injective or P is directed, and X is $\tilde{\phi}$-injective. This gurantees the existence of the reduced Cuntz-Nica-Pimsner C^*-algebra $\mathcal{NO}^r_X$ of X generated by the couniversal, gauge compatible injective Nica covariant representation of X. As a simple observation, note that for a closed ideal I of $\mathcal{NO}_X$, $I \cap j_e(\mathcal{A}) = \{0\}$ implies $I \cap (\mathcal{NO}_X)^\delta_e = \{0\}$.

Next let us define the notions of minimality and Kishimoto condition for a quasi-lattice ordered group (G, P) and product system X over P. This is related to the notion of aperiodicity for C^*-correspondences, introduced by Muhly and Solel [29, Definition 5.1], inspired by the results of Kishimoto [24, Lemma 1.1] and Olesen and Pedersen [31, Theorems 6.6 and 10.4] (see also, [15]). In a more general setting, aperiodicity of Fell bundles [28, Definition 4.1] and their intersection property have been studied in [28]. It is known that, aperiodicity implies the intersection property. Also, the connection between simplicity of crossed sectional algebras of a Fell bundle with intersection property is described in [28, Corollary 3.13]. In our context, aperiodicity of Fell bundles associated with a product system and the Kishimoto condition are basically the same (cf., [28, Lemma 4.2]). There is an alternative, more direct way to obtain a simplicity result for Cuntz-Nica-Pimsner algebras, and the rest of this subsection is devoted to a rather detailed description of such a direct approach, which is essentially an adaptation of the argument for the case of crossed products (cf., [32]).

Let us call a product system X over P *minimal* if $\langle X_p, JX_p\rangle \subseteq J$, for a closed ideal J of $\mathcal{A}$ implies $J = 0$ or $\mathcal{A}$, and *essential* if each X_p is essential, for $p \in P$. The system X is said to satisfy the *Kishimoto condition*, if for any norm one positive element $x \in (\mathcal{NO}^r_X)^\nu_e$, $\varepsilon > 0$, and finite subsets $F \subseteq G\backslash\{e\}$, $S \subseteq \bigcup_{g\in F}(\mathcal{NO}^r_X)^\nu_g$, there exists norm one positive element $c \in (\mathcal{NO}^r_X)^\nu_e$ with $\|cxc\| > 1-\varepsilon$, $\|cx_gc\| < \varepsilon$, for each $g \in F$ and $x_g \in S$.

Since $\mathcal{NO}^r_X$ is the reduced cross sectional algebra of the Fell bundle $\{(\mathcal{NO}^r_X)^\nu_g \times \{g\}\}_{g\in G}$, by the standard theory of Fell bundles, there is a faithful conditional expectation $\mathbb{E} : \mathcal{NO}^r_X \to (\mathcal{NO}^r_X)^\nu_e$, vanishing on all fibers except the one at e [10, Proposition 2.12]. Also, $\mathcal{NO}^r_X$ is a graded C^*-algebra and $\bigoplus_{g\in G}(\mathcal{NO}^r_X)^\nu_g$ is dense in $\mathcal{NO}^r_X$ [10, Proposition 3.2].

We have the following characterization of the Kishimoto condition in terms of finite subsets of $\mathcal{A}$ under the above assumptions: assume that there exist unitaries $u_g \in (\mathcal{NO}^r_X)^\nu_g \cap ((\mathcal{NO}^r_X)^\nu_e)'$, for $g \in G$, such that $u_{g^{-1}} = u_g^*$. When X is a product system consisting of $\mathcal{A}$-bimodules, the Kishimoto condition is equivalent to the condition that for each norm one positive element $a \in \mathcal{A}$, finite subsets $F \subset G\backslash\{1\}$, $S \subset \mathcal{A}$, and $\varepsilon > 0$, there exists a norm one positive element $c \in \mathcal{A}$ with $\|cxc\| > 1-\varepsilon$, $\|csc\| < \varepsilon$, for all $g \in F$ and $s \in S$.

Next, let us observe that in the particular case of product systems coming from actions and corresponding twisted crossed products, the above Kishimoto condition is nothing but the classical Kishimoto condition for actions: let α be an action of G on a unital C^*-algebra $\mathcal{A}$ with unit 1, ω be a $\mathbb{T}$-valued cocycle on G, and let $\bar{\omega}$ be the conjugate cocycle. Put $X_p = \mathcal{A}$, for each $p \in P$, and consider A as an $\mathcal{A}$-Hilbert bimodule with actions and $\mathcal{A}$-valued inner products,

$$\langle x, y\rangle := x^*y,\ {}_p\langle x, y\rangle_p := \alpha_{p^{-1}}(xy^*),\ a\cdot x = \alpha_p(a)x,\ x\cdot a = xa,$$

for $x, y \in X_p = \mathcal{A}$ and $a \in \mathcal{A}$. Define isomorphism $X_p \otimes_{\mathcal{A}} X_q \to X_{pq};\ x \otimes y \mapsto$

$\bar{\omega}(q,p)\alpha_q(x)y$, then $X = \{X_p\}_{p\in P}$ is a product system consisting of essential C^*-bimodules. If P be a directed and G generated by P as a group, then there is a canonical isomorphism between $\mathcal{A} \rtimes^r_{\alpha,\omega} G$ and $\mathcal{NO}^r_X$ [3, Lemma 5.1]. Let us recall the definition of the classical Kishimoto condition for actions. This notion first appeared in [24], where Kishimoto showed that if a discrete group G acts on a C^*-algebra $\mathcal{A}$, say by α, with no non trivial G-invariant closed ideal (i.e., α is minimal), then the reduced crossed product $\mathcal{A} \rtimes_r G$ is simple whenever the strong Connes spectrum [23] of automorphisms α_g is non trivial, for $g \in G\backslash\{e\}$. Kishimoto showed that the above two conditions together are stronger than the following formulation of the Kishimoto condition [24, Lemma 3.2], a name coined by Chris Phillips in [32], but it is implicit in his proof of simplicity of the reduced crossed product [24, Theorem 3.1], that minimality of α and the Kishimoto condition are good enough to imply simplicity. Let G be a discrete group, α be an action of G on a C^*-algebra $\mathcal{A}$. We say that α satisfies the Kishimoto condition if for all norm one positive elements $a \in \mathcal{A}$, finite subsets $F \subseteq G\backslash\{1\}$, $S \subseteq \mathcal{A}$, and $\varepsilon > 0$, there exists a norm one positive element $c \in \mathcal{A}_+$ with $\|cxc\| > 1-\varepsilon$, $\|cb\alpha_g(c)\| < \varepsilon$, for all $g \in F$ and $b \in S$.

Let P be directed and generate G, α be an action of G on a unital C^*-algebra $\mathcal{A}$ with unit 1, and ω be a $\mathbb{T}$-valued cocycle on G. Let X be the product system of the twisted crossed product $\mathcal{A} \rtimes^r_{\alpha,\omega} G$, then X satisfies the Kishimoto condition if and only if α satisfies the Kishimoto condition. It is easy to see that the general Kishimoto condition defined in this section coincides with the classical Kishimoto condition for the special case of twisted crossed products.

Finally we have the following simplicity result for the Cuntz-Nica-Pimsner algebras.

Theorem 12. *Let (G,P) be a quasi-lattice ordered group, X be a product system over P consisting of C^*-correspondences over unital C^*-algebra $\mathcal{A}$. If X is minimal or $\mathcal{A}$ is simple, X satisfies the Kishimoto condition and $j^r_e(\mathcal{A})$ contains an approximate identity of $\mathcal{NO}^r_X$, then $\mathcal{NO}^r_X$ is simple.*

This result follows from more general results on simplicity of cross sectional C^*-algebras of Fell bundles [28, Corollary 3.13].

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