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SHARP HARDY'S INEQUALITIES VIA CONFORMABLE FRACTIONAL INTEGRALS

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Abstract. The aim of this research is to present Hardy-type inequalities with a sharp constant that are related to the fractional conformable integral operator.

1 Introduction and Preliminaries

Fractional calculus has gained popularity in recent years due to its numerous applications in various fields of mathematics and science. This article deals with new definitions of fractional derivatives proposed in recent works by [1], [8] and [9]. Specifically, we focus on the conformable fractional derivative, which shares many properties with the traditional Newton's derivative. For a differentiable function f, the conformable derivative is defined by

$$D_\alpha(f)(t) = t^{1-\alpha}\frac{df}{dt}(t). \tag{1.1}$$

We explore the properties and applications of the conformable fractional integral associated to this derivative, with a particular focus on Hardy's inequalities with sharp constants.

First, we present here some basics concerning the conformable fractional integral. We define, for an order $\alpha \in (0,1]$, the set $L^1_\alpha([0,b])$ of α-fractional integrable functions and the conformable fractional integral operator of of a function $f \in L^1_\alpha([0,b])$. More detailed information about fractional calculus can be found in [11].

Definition 1. *Let $\alpha \in (0,1]$ and $0 < b$. A function $f : [0,+\infty) \to \mathbb{R}$ is α-fractional integrable on $[0,b]$ if the integral*

$$\int_0^b f(t)d_\alpha t := \int_0^b f(t)\, t^{\alpha-1}dt, \tag{1.2}$$

exists and is finite.

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Definition 2. ***(Conformable Fractional Operator)***
Let $\alpha \in (0,1]$ and $f \in L^1_\alpha([0,b])$. The conformable fractional integral of order α of f is defined by

$$I_\alpha f(x) = \int_0^x f(t) d_\alpha t := \int_0^x t^{\alpha-1} f(t)\, dt, \tag{1.3}$$

for all $x > 0, \alpha \in (0,1]$.

In recent years, conformable fractional calculus has been used to derive various inequalities, such as Opial inequalities presented in [16, 17], Hermite-Hadamard's inequality [10, 15], Tchebychev's inequality [2], and Steffensen's inequality [18]. In addition, Hardy's inequality has been proved in [7] for $p > 1$ as follows

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty \left(f(t)t^{\alpha-1}\right)^p dt, \tag{1.4}$$

and in [6], a weighted version of the inequality (1.4) was presented with some conditions on the exponent $r < p - 1$

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p x^r\, dx \le \left(\frac{p}{p-r-1}\right)^p \int_0^\infty \left(f(t)t^{\alpha-1}\right)^p t^r\, dt. \tag{1.5}$$

When $0 < p < 1$, these inequalities were established for the class of non-increasing and non-negative functions (see [5]).

The conformable fractional versions of inequalities (1.4) and (1.5) are presented in [12] and [13] as follow

Theorem 3. *([12, 13]) Let f be a nonnegative function on $(0;+\infty)$, $\alpha \in (0;1]$, $p > 1$ and assume that $\int_0^\infty \left(f(t)\ t^{\alpha-1}\right)^p\ d_\alpha t$ is convergent. Then the following inequality holds*

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\ d_\alpha t\right)^p\ d_\alpha x \le \left(\frac{p}{p-\alpha}\right)^p\ \int_0^\infty \left(f(t)\ t^{\alpha-1}\right)^p\ d_\alpha t. \tag{1.6}$$

Theorem 4. *([13]) Let $\alpha \in (0;1]$; f be a nonnegative function on $(0;+\infty)$, $p \ge 1$ and $r < p - \alpha$. Assume $\int_0^\infty \left(f(t)\ t^{\alpha-1}\right)^p\ t^r\ d_\alpha t$ is convergent. Then the following inequality holds*

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\ d_\alpha t\right)^p x^r\ d_\alpha x \le \left(\frac{p}{p-r-\alpha}\right)^p\ \int_0^\infty \left(f(t)\ t^{\alpha-1}\right)^p\ t^r\ d_\alpha t. \tag{1.7}$$

The goal of this study is to establish the sharpness of the constants in the inequalities (1.7), (1.6) and to establish a new Hardy's type inequality for conformable fractional integral with sharp constants for $0 < p < 1$.

2 Main results

We now reformulate Theorem 4 with a sharp constant result.

Theorem 5. *Let $\alpha \in (0;1]$; f be a nonnegative function on $(0;+\infty)$, $p \geq 1$ and $r < p-\alpha$. Assume $\int_0^\infty \left(f(t)\, t^{\alpha-1}\right)^p\, t^r\, d_\alpha t < \infty$. Then the following inequality holds*

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\, d_\alpha t\right)^p x^r\, d_\alpha x \leq \left(\frac{p}{p-r-\alpha}\right)^p \int_0^\infty \left(f(t)\, t^{\alpha-1}\right)^p\, t^r\, d_\alpha t. \tag{2.1}$$

In addition, if $-\alpha < r < p-\alpha$ then the constant $\left(\frac{p}{p-r-\alpha}\right)^p$ is sharp.

Proof. Suppose that there exists $C \leq \left(\frac{p}{p-r-\alpha}\right)^p$ verifying inequality (2.1) and denote by N and D the integrals in the left and the right hand side of the inequality (2.1) respectively.
Since $-\alpha < r < p-\alpha$ then $r+\alpha > 0$ and $p-r-\alpha > 0$. Let be $0 < \varepsilon < p-r-\alpha$ and define the function

$$f_\varepsilon(x) = \begin{cases} x^{1-\alpha} & \text{if } x \in (0,1) \\ x^{1-\alpha} \times x^{-\frac{r}{p}-\frac{\alpha+\varepsilon}{p}} & \text{if } x \in [1,\infty). \end{cases} \tag{2.2}$$

Replacing (2.2) in N and D, we get

$$\begin{aligned} N &= \int_0^1 \left(\frac{1}{x}\int_0^x t^{1-\alpha}\, d_\alpha t\right)^p x^r\, d_\alpha x + \int_1^\infty \left(\frac{1}{x}\int_0^x t^{1-\alpha} \times t^{-\frac{r}{p}-\frac{\alpha+\varepsilon}{p}}\, d_\alpha t\right)^p x^r\, d_\alpha x \\ &= \int_0^1 x^{r+\alpha-1}\, dx + \frac{1}{\left(-\frac{r}{p}-\frac{\alpha+\varepsilon}{p}+1\right)^p}\int_1^\infty x^{-\varepsilon-1}\, dx \\ &= \frac{1}{r+\alpha} + \frac{1}{\varepsilon\left(-\frac{r}{p}-\frac{\alpha+\varepsilon}{p}+1\right)^p}\ , \end{aligned}$$

and

$$\begin{aligned} D &= \int_0^1 t^{r+\alpha-1}\, dt\ +\ \int_1^\infty \left(t^{-\frac{r}{p}-\frac{\alpha+\varepsilon}{p}}\right)^p\ t^{r+\alpha-1}\, dt \\ &= \frac{1}{r+\alpha}\ +\ \frac{1}{\varepsilon}\ . \end{aligned}$$

We deduce from (2.1)

$$
\begin{aligned}
C &\geq \frac{\dfrac{1}{r+\alpha}+\dfrac{1}{\varepsilon\left(-\frac{r}{p}-\frac{\alpha+\varepsilon}{p}+1\right)^{p}}}{\dfrac{1}{r+\alpha}\ +\ \dfrac{1}{\varepsilon}} \\
&= \frac{\dfrac{\varepsilon}{r+\alpha}+\dfrac{1}{\left(-\frac{r}{p}-\frac{\alpha+\varepsilon}{p}+1\right)^{p}}}{\dfrac{\varepsilon}{r+\alpha}\ +\ 1},
\end{aligned}
$$

taking $\varepsilon \to 0$, we get

$$C \geq \left(\frac{p}{p-r-\alpha}\right)^{p}.$$

this means that $\left(\frac{p}{p-r-\alpha}\right)^{p}$ is the best possible.

□

The following Corollary is obtained by setting $r = 0$.

Corollary 6. *Under the hypothesis of Theorem 3, the inequality (1.6) holds and* $\left(\dfrac{p}{p-\alpha}\right)^{p}$ *is a sharp constant.*

Hardy type inequality for $0 < p < 1$

The following lemma is required to prove the next theorem.

Lemma 7. *Let $\alpha \in (0,1]$, $0 < p < 1$ and $f \in L^{1}_{\alpha}(0,+\infty)$ be a function verifying the condition*

$$f(x) \leq \frac{M}{x^{\alpha}}\left(\int_{0}^{x} f^{p}(t)t^{\alpha\,(p-1)}\,d_{\alpha}t\right)^{\frac{1}{p}}, \quad a.e,\ x>0, \tag{2.3}$$

then

$$\left(\int_{0}^{x} f(t)d_{\alpha}t\right)^{p} \leq p^{p}M^{p(1-p)}\int_{0}^{x} f^{p}(t)t^{\alpha\,(p-1)}\,d_{\alpha}t. \tag{2.4}$$

Proof. Let $x > 0$ and f verifying the inequality (2.3) almost everywhere in $(0, x)$, since

$$f(t) = (f(t)t)^{1-p}(f^{p}(t)t^{p-1}),$$

thus

$$f(t) \ \ \leq \left[\frac{M}{t^{\alpha}}\left(\int_{0}^{t} f^{p}(\mu)\mu^{\alpha\,(p-1)}\,d_{\alpha}\mu\right)^{\frac{1}{p}} t\right]^{1-p}(f^{p}(t)t^{p-1}),$$

which yields

$$f(t)t^{\alpha-1} \leq M^{1-p}\left(\int_0^t f^p(\mu)\mu^{\alpha\,(p-1)}\,d_\alpha\mu\right)^{\frac{1}{p}-1} f^p(t)t^{p\alpha-1}.$$

Taking $\phi(t)=\int_0^t f^p(\mu)\mu^{\alpha\,(p-1)}\,d_\alpha\mu=\int_0^t f^p(\mu)\mu^{p\alpha-1}d\mu$ and integrating the above inequality on $(0,x)$, we get

$$\begin{aligned}\int_0^x f(t)\,d_\alpha t \; &\leq M^{1-p}\int_0^x (\phi(t))^{\frac{1}{p}-1}\,\phi'(t)dt\\ &= pM^{1-p}\,(\phi(x))^{\frac{1}{p}}\\ &= pM^{1-p}\left(\int_0^x f^p(\mu)\mu^{\alpha\,(p-1)}\,d_\alpha\mu\right)^{\frac{1}{p}}.\end{aligned}$$

It achieves the desired inequality (2.4).

□

Theorem 8. *Let $\alpha\in(0,1]$, $M>0$, $r<p-\alpha$ and $0<p<1$. If $f\in L^1_\alpha(0,+\infty)$ verifies the condition (2.3), then the following inequality holds*

$$\int_0^\infty\left(\frac{1}{x}\int_0^x f(t)\,d_\alpha t\right)^p x^r\,d_\alpha x \leq \frac{p^pM^{p(1-p)}}{p-r-\alpha}\int_0^\infty\left(f(t)\,t^{\alpha-1}\right)^p\,t^r\,d_\alpha t. \tag{2.5}$$

In addition, if $-\alpha<r<p-\alpha$, then the constant $\frac{p^pM^{p(1-p)}}{p-r-\alpha}$ is sharp.

Proof. Let $x>0$ and f verifying the inequality (2.3) almost everywhere in $(0,x)$, we denote by Lhs the integral in left-hand side of inequality (2.5). By applying Lemma 7 and Fubini's Theorem, we get

$$\begin{aligned}Lhs\; &= \int_0^\infty\left(\frac{1}{x}\int_0^x f(t)\,d_\alpha t\right)^p x^r\,d_\alpha x\\ &\leq \int_0^\infty p^pM^{p(1-p)}\int_0^x f^p(t)t^{\alpha\,(p-1)}\,d_\alpha t\;x^{r-p+\alpha-1}\,dx\\ &= p^pM^{p(1-p)}\int_0^\infty\left(\int_t^\infty x^{r-p+\alpha-1}\,dx\right) f^p(t)\,t^{p\alpha-1}\,dt\\ &= \frac{p^p\,M^{p(1-p)}}{p-r-\alpha}\int_0^\infty t^{r-p+\alpha}\,f^p(t)\,t^{p\alpha-1}\,dt\\ &= \frac{p^p\,M^{p(1-p)}}{p-r-\alpha}\int_0^\infty\left(f(t)\,t^{\alpha-1}\right)^p\,t^r\,d_\alpha t\,,\end{aligned}$$

which gives the required inequality.
Proof of sharpness of the constant

Suppose that there is $C \leq \dfrac{p^p M^{p(1-p)}}{p-r-\alpha}$ which verifies the inequality (2.5), then

$$C \geq \frac{\displaystyle\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\, d_\alpha t\right)^p x^r \, d_\alpha x}{\displaystyle\int_0^\infty \left(f(t)\, t^{\alpha-1}\right)^p \, t^r \, d_\alpha t}. \tag{2.6}$$

i) If $M^p > p-r-\alpha$, lets define $f(t) = t^{\frac{M^p}{p}-\alpha}\chi_{(0,1)}(t)$. So, in the denominator of (2.6) we get

$$\begin{aligned} I_D &:= \int_0^\infty \left(f(t)\ t^{\alpha-1}\right)^p t^r \, d_\alpha t \\ &= \int_0^1 t^{M^p-p+r+\alpha-1}\, dt \\ &= \frac{1}{M^p-p+r+\alpha}, \end{aligned}$$

and in the numerator of (2.6) we have

$$\begin{aligned} I_N &:= \int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\, d_\alpha t\right)^p x^r \, d_\alpha x \\ &= \int_0^1 \left(\frac{1}{x}\int_0^x t^{\frac{M^p}{p}-1}\, dt\right)^p x^{r+\alpha-1}\, dx + \int_1^\infty \left(\frac{1}{x}\int_0^1 t^{\frac{M^p}{p}-1}\, dt\right)^p x^{r+\alpha-1}\, dx \\ &= \left(\frac{p}{M^p}\right)^p \left(\int_0^1 x^{M^p-p+r+\alpha-1}\, dx + \int_1^\infty x^{-p+r+\alpha-1}\, dx\right) \\ &= \frac{p^p\ M^{p(1-p)}}{(M^p-p+r+\alpha)\ (p-r-\alpha)}. \end{aligned}$$

Therefore, from (2.6), we get

$$C \ \geq \frac{p^p\ M^{p(1-p)}\,(M^p-p+r+\alpha)}{(M^p-p+r+\alpha)\,(p-r-\alpha)} = \frac{p^p\ M^{p(1-p)}}{(p-r-\alpha)},$$

we conclude that $C = \dfrac{p^p\ M^{p(1-p)}}{(p-r-\alpha)}$ is the sharp constant.

ii) If $M^p < p - r - \alpha$, let us define $f(t) = t^{\frac{M^p}{p} - \alpha} \chi_{(1,\infty)}(t)$.
We deduce that

$$\begin{aligned} I_D &:= \int_0^\infty \left(f(t)\, t^{\alpha-1}\right)^p t^r \, d_\alpha t \\ &= \int_1^\infty t^{M^p - p + r + \alpha - 1} \, dt \\ &= \frac{1}{-M^p + p - r - \alpha}\,, \end{aligned}$$

and

$$\begin{aligned} I_N &:= \int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\; d_\alpha t\right)^p \; x^r \; d_\alpha x \\ &= \int_1^\infty \left(\frac{1}{x}\int_1^x t^{\frac{M^p}{p}-1}\, dt\right)^p \; x^{r+\alpha-1}\, dx \\ &= \left(\frac{p}{M^p}\right)^p \int_1^\infty \left(x^{\frac{M^p}{p}} - 1\right)^p x^{-p+r+\alpha-1} \; dx. \end{aligned}$$

For $0 < p < 1$ and $0 < B < A$ we have $(A-B)^p \geq A^p - B^p$ [3], so for $x > 1$ we deduce that

$$\left(x^{\frac{M^p}{p}} - 1\right)^p \geq \left(x^{M^p} - 1\right),$$

hence

$$\begin{aligned} I_N &\geq \left(\frac{p}{M^p}\right)^p \int_1^\infty \left(x^{M^p} - 1\right) x^{-p+r+\alpha-1}\, dx \\ &= \frac{p^p \; M^{p(1-p)}}{(-M^p + p - r - \alpha)\; (p - r - \alpha)}, \end{aligned}$$

this yields

$$C \geq \frac{I_N}{I_D} \geq \frac{p^p \; M^{p(1-p)} \; (-M^p + p - r - \alpha)}{(-M^p + p - r - \alpha)\; (p - r - \alpha)} = \frac{p^p \; M^{p(1-p)}}{(p - r - \alpha)},$$

then $C = \dfrac{p^p \; M^{p(1-p)}}{(p - r - \alpha)}$ is a sharp constant.

iii) If $M^p = p - r - \alpha$, let be $\tau \in (1, \infty)$ and define $f(t) = t^{\frac{M^p}{p} - \alpha} \chi_{(1,\tau)}(t)$, then

$$\begin{aligned} I_D &:= \int_0^\infty \left(f(t)\, t^{\alpha-1} \right)^p t^r \, d_\alpha t \\ &= \int_1^\tau t^{M^p - p + r + \alpha - 1} \, dt \\ &= \ln(\tau), \end{aligned}$$

and

$$\begin{aligned} I_N &:= \int_0^\infty \left(\frac{1}{x} \int_0^x f(t) \, d_\alpha t \right)^p x^r \, d_\alpha x \\ &= \left(\frac{p}{M^p} \right)^p \left(\int_1^\tau \left(x^{\frac{M^p}{p}} - 1 \right)^p x^{-p+r+\alpha-1} \, dx \, + \, \int_\tau^\infty \left(\tau^{\frac{M^p}{p}} - 1 \right)^p x^{-p+r+\alpha-1} \, dx \right) \\ &= \left(\frac{p}{M^p} \right)^p (I_1 + I_2) \, . \end{aligned}$$

We have

$$\begin{aligned} I_1 &\geq \int_1^\tau \left(x^{M^p} - 1 \right) \, x^{-p+r+\alpha-1} \, dx \\ &= \left[\int_1^\tau x^{-1} \, dx - \int_1^\tau x^{-p+r+\alpha-1} \, dx \right] \\ &= \left[\ln(\tau) + \frac{\tau^{-p+r+\alpha} - 1}{p - r - \alpha} \right] \, , \end{aligned}$$

and

$$\begin{aligned} I_2 &= \left(\tau^{\frac{M^p}{p}} - 1 \right)^p \, \int_\tau^\infty x^{-p+r+\alpha-1} \, dx \\ &= \left(\tau^{\frac{M^p}{p}} - 1 \right)^p \, \frac{\tau^{-p+r+\alpha}}{p - r - \alpha} \\ &\geq \left(\tau^{M^p} - 1 \right) \frac{\tau^{-p+r+\alpha}}{p - r - \alpha} \\ &= \frac{1 - \tau^{-p+r+\alpha}}{p - r - \alpha}, \end{aligned}$$

consequently

$$I_1 + I_2 \, \geq \, \ln(\tau),$$

then
$$I_N \geq \left(\frac{p}{M^p}\right)^p \ \ln(\tau).$$

So
$$C \geq \frac{I_N}{I_D} \geq \left(\frac{p}{M^p}\right)^p = \frac{p^p \ M^{p(1-p)}}{(p-r-\alpha)},$$

which gives the sharpness of $C = \dfrac{p^p \ M^{p(1-p)}}{(p-r-\alpha)}$.

□

3 Example of functions satisfying condition (2.3)

3.1 Cone of non-increasing functions

Proposition 9. *Let $\alpha \in (0,1]$ and $p > 0$. Non-negative measurable and non-increasing functions on $(0,+\infty)$ satisfy the condition (2.3).*

Proof. Let f be a non-increasing function on $(0,x)$, hence
$$\begin{aligned}\left(\int_0^x f^p(t) t^{\alpha(p-1)} d_\alpha t\right)^{\frac{1}{p}} &\geq f(x)\left(\int_0^x t^{p\alpha-1} dt\right)^{\frac{1}{p}} \\ &= \frac{x^\alpha}{(p\alpha)^{\frac{1}{p}}} \ f(x) = \frac{x^\alpha}{M} \ f(x),\end{aligned}$$

which gives the condition (2.3) with $M = (p\alpha)^{\frac{1}{p}}$. □

3.2 Slowly varying functions

Notation 10. *For two non-negative functions f and g defined on $(0,\infty)$, we write $f \lesssim g$ if there is a constant $k > 0$ such that $f(t) \leq k\, g(t)$ for all $t \in (0,\infty)$. Analogously, we define $f \gtrsim g$. If $f \lesssim g$ and $f \gtrsim g$, we say that f and g are equivalent and write $f \sim g$.*

Definition 11. *[14] A non-negative measurable function f on $(0,\infty)$ is said to be slowly varying if for each $\varepsilon > 0$, there are non-negative measurable functions g_ε and $g_{-\varepsilon}$ such that*

- *g_ε is non-decreasing and $g_\varepsilon(t) \sim t^\varepsilon f(t)$,*
- *$g_{-\varepsilon}$ is non-increasing and $g_{-\varepsilon}(t) \sim t^{-\varepsilon} f(t)$.*

Proposition 12. *Let $\alpha \in (0,1]$ and $p > 0$. Slowly varying functions satisfy the condition (2.3).*

Proof. Let f be a slowly varying function, then for any $\varepsilon > 0$ there exist g_ε and $g_{-\varepsilon}$ as described in definition (11) such that

$$g_\varepsilon(t) \sim t^\varepsilon f(t) \quad and \quad g_{-\varepsilon}(t) \sim t^{-\varepsilon} f(t),$$

hence

$$g_\varepsilon(t) \lesssim t^\varepsilon f(t) \lesssim g_\varepsilon(t) \quad and \quad g_{-\varepsilon}(t) \lesssim t^{-\varepsilon} f(t) \lesssim g_{-\varepsilon}(t). \tag{3.1}$$

step i : To proof f^p is slowly varying for $p > 0$, we replace ε by $\frac{\varepsilon}{p}$ in (3.1), then

$$g^p_{\frac{\varepsilon}{p}}(t) \lesssim t^\varepsilon f^p(t) \lesssim g^p_{\frac{\varepsilon}{p}}(t) \quad and \quad g^p_{-\frac{\varepsilon}{p}}(t) \lesssim t^{-\varepsilon} f^p(t) \lesssim g^p_{-\frac{\varepsilon}{p}}(t)$$

where $g^p_{\frac{\varepsilon}{p}}$ is non-decreasing and $g^p_{-\frac{\varepsilon}{p}}$ is non-increasing .

step ii : Since f^p is slowly varying, there exist a non-increasing function G_{-1} such that

$$G_{-1}(t) \gtrsim t^{-1} f^p(t) \gtrsim G_{-1}(t).$$

We have

$$\begin{aligned}
\int_0^x t^{\alpha p-1} f^p(t)\, dt &= \int_0^x t^{\alpha p} t^{-1} f^p(t)\, dt \\
&\gtrsim \int_0^x t^{\alpha p} G_{-1}(t)\, dt \\
&\geq G_{-1}(x) \int_0^x t^{\alpha p}\, dt \\
&\gtrsim x^{-1} f^p(x) \int_0^x t^{\alpha p}\, dt \\
&= \frac{k}{\alpha p + 1} x^{\alpha p} f^p(x),
\end{aligned}$$

which gives inequality (2.3).

□

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