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ADVANCE IN THE ADOMIAN DECOMPOSITION METHOD FOR SOLVING NEW EMDEN-FOWLER TYPE EQUATIONS

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Abstract. In this manuscript, we developed the technique of the Adomian method for solving many Lane-Emden-Fowler type equations. The use of the Adomian Decomposition method to effectively solve these newly developed singular 4^{th} and 5^{th} order equations. The efficacy of our approach is further validated by analyzing various fourth and fifth orders Emden-Fowler type examples encompassing nonlinear cases.

1 Introduction

In the latter part of the 19^{th} century, astrophysicist J. Homer Lane derived and studied an equation that represents the equilibrium of stellar configurations [20]. Thereafter, this work was then extended [14], leading to what is now referred to as the Lane-Emden equation, which can be expressed as follows:

$$u'' + \frac{k}{t}u' + g(t)h(u) = 0, \qquad u(0) = a, u'(0) = 0, \tag{1.1}$$

where k is the shape factor, $g(t)$ and $h(u)$ are functions of t and u, respectively. The Emden-Fowler equation (1.1) can be used to model a variety of physical phenomena, including the structure of stars, the propagation of waves in plasmas, the diffusion of particles in gases, relativistic mechanics, pattern formation, population evolution, and chemical reactor systems. For $g(t) = 1$, equation (1.1) becomes the standard Lane-Emden equation [20, 14]:

$$u'' + \frac{k}{t}u' + h(u) = 0, \qquad u(0) = a, u'(0) = 0, \tag{1.2}$$

The Lane-Emden equation (1.2) describes the temperature variation in a spherical gas cloud influenced by mutual molecular attraction and subject to thermodynamic

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laws. It is utilized in astrophysics to model the interiors of polytropic stars, radiative cooling, self-gravitating gas clouds, and galaxy cluster modeling [20, 14, 15, 27, 10, 7]. The singularity at $t = 0$ poses a significant challenge for equation (1.1) and equation (1.2).

The function $h(u)$ in the Lane-Emden equation takes on various forms. The most popular are $g(t) = 1$ and $h(u) = u^m$ with m as a constant parameter. The Lane-Emden equation in this form is known as the Lane-Emden equation of the first kind, or index m. It is used to model the thermal behavior of a spherical gas under mutual attraction, following classical thermodynamic laws. Notably, for $m = 0$ and $m = 1$, the Lane-Emden equation is linear, while it is nonlinear otherwise. On the other hand, the Lane-Emden equation of the second kind employs $h(u)$ in the form $h(u) = e^u$. It models the non-dimensional density distribution $y(x)$ in an isothermal gas sphere, a common approach to studying stellar structures. There are also other nonlinear forms for $h(u)$, such as $cos(u)$, $sin(u)$, $cosh(u)$, $sinh(u)$, and more. An interesting variant is $h(u) = (u^2 - C)^{\frac{3}{2}}$. For this value of $h(u)$, the Lane-Emden equation becomes the white-dwarf equation introduced by Chandrasekhar [7] to study the gravitational potential of degenerate white-dwarf stars. When $C = 0$, it reduces to the Lane-Emden equation with index $m = 3$. White dwarfs, or degenerate dwarfs, are stellar remnants primarily composed of electron-degenerate matter.

The Lane-Emden equation (1.1) was derived by employing the equation:

$$\frac{1}{t^k}\frac{d}{dt}(t^k\frac{d}{dt})u + g(t)h(u) = 0, \qquad u(0) = a, u'(0) = 0, \tag{1.3}$$

where k represents the shape factor. Upon setting $g(t) = 1$, the Lane-Emden equation emerges from the same derivation. The Lane-Emden and the Emden-Fowler equations have undergone extensive scrutiny, with researchers employing a diverse array of numerical and analytic methods to derive both exact and approximate solutions. The literature encompasses the Adomian decomposition method ADM [4, 28, 29, 5, 30, 31, 21, 17], variational iteration method [18], homotopy perturbation method [34], rational Legendre pseudo-spectral approach [25], collocation method [24], galerkin method [6], finite element method, and other techniques [22, 19, 23].

This paper extends the work of Wazwaz et al.[33] by introducing a novel differential operator to solve three new types of Emden-Fowler equations: a fourth-order and two fifth-order. The operator's versatility allows it to handle a broader range of Emden-Fowler equations, including those of higher orders (detailed in Section 2). Solutions for these new equation types are presented in Section 3.

This paper introduces a novel differential operator to solve new types of Emden-Fowler equations. Section 2 details the systematic derivation of these equations for various dimensions, employing a uniform method for first-to-fifth-order cases (with cases 8, case 12 and case 13 for both orders being novel). Section 3 presents solutions for these new equation types, while Section 4 discusses the implications and concludes the paper.

2 Deriving the Emden - Fowler Type Equations for Integral Order

Consider the general for the Emden-Fowler type equations:

$$t^{-r}\frac{d}{dt}t^{r-k}\frac{d^m}{dt^m}(t^k\frac{d^n}{dt^n})u+g(t)h(u)=0, \tag{2.1}$$

where $r \geq 1$, $k \geq 0$ and $m, n \in N$. When $m, n = 0$ to obtain the Emden- Fowler equation of the first order [16], to get the Emden- Fowler equations of the 2nd, 3rd, 4th, and 5th order, we should take

$$m+n=1,$$

$$m+n=2,$$

$$m+n=3,$$

and

$$m+n=4, \tag{2.2}$$

where m, $n \geq 0$. There are two probabilities for 2nd order (debated in [29, 8]), three possibilities for 3rd order (discussed in [8, 32, 9]), four possibilities for 4th order (three cases debated in [33, 8, 9]), and five possible choices for 5th order (three cases discussed in [33, 8, 9]) of the Emden-Fowler type equations. To give the following type equations, respectively

First-Order Equation [16]

When $m = 0$, $n = 0$ in (2.1) gives:

$$t^{-r}\frac{d}{dt}t^r u+g(t)h(u)=0, \tag{2.3}$$

simplifying (2.3), we obtain:

$$u^{'}+\frac{r}{t}y+g(t)h(u)=0. \tag{2.4}$$

Second-Order Equations [29, 8]

Case 1. *at $m = 0$, $n = 1$ in equation (2.1) will be:*

$$t^{-r}\frac{d}{dt}(t^r\frac{d}{dt})u+g(t)h(u)=0, \tag{2.5}$$

simplifying equation (2.5), we get:

$$u^{''}+\frac{r}{t}u^{'}+g(t)h(u)=0. \tag{2.6}$$

Case 2. *When $m = 1$, $n = 0$ in equation (2.1) to get*

$$t^{-r}\frac{d}{dt}t^{r-k}(\frac{d}{dt}t^{k})u + g(t)h(u) = 0, \tag{2.7}$$

equaling equation (2.7), we have

$$u^{''} + \frac{r+k}{t}u^{'} + g(t)h(u) = 0. \tag{2.8}$$

Equations of the Third Order [8, 32, 9]

Case 3. *at $m = 0$, $n = 2$ in equation (2.1) to get*

$$t^{-r}\frac{d}{dt}(t^{r}\frac{d^2}{dt^2})u + g(t)h(u) = 0, \tag{2.9}$$

clarifying equation (2.9), we have

$$u^{'''} + \frac{r}{t}u^{''} + g(t)h(u) = 0. \tag{2.10}$$

Case 4. *at $m = 1$, $n = 1$ in equation (2.1) gives*

$$t^{-r}\frac{d}{dt}t^{r-k}\frac{d}{dt}(t^{k}\frac{d}{dt})u + g(t)h(u) = 0, \tag{2.11}$$

elucidating equation (2.11), we obtain

$$u^{'''} + \frac{r+k}{t}u^{''} + \frac{k(r-1)}{t^2}u' + g(t)h(u) = 0. \tag{2.12}$$

Case 5. *when $m = 2$, $n = 0$ in equation (2.1) will be*

$$t^{-r}\frac{d}{dt}t^{r-k}(\frac{d^2}{dt^2}t^{k})u + g(t)h(u) = 0, \tag{2.13}$$

Fourth-Order Equations

Case 6. *at $m = 0$, $n = 3$ in equation (2.1) will be*

$$t^{-r}\frac{d}{dt}(t^{r}\frac{d^3}{dt^3})u + g(t)h(u) = 0, \tag{2.14}$$

when simplifying equation (2.14), we have [33]

$$u^{''''} + \frac{r}{t}u^{'''} + g(t)h(u) = 0. \tag{2.15}$$

Case 7. *at $m = 1$, $n = 2$ in equation (2.1) will be*

$$t^{-r}\frac{d}{dt}t^{r-k}\frac{d}{dt}(t^k\frac{d^2}{dt^2})u + g(t)h(u) = 0, \tag{2.16}$$

after simplifying equation (2.16) gives [8]

$$u^{''''} + \frac{r+k}{t}u^{'''} + \frac{k(r-1)}{t^2}u^{''} + g(t)h(u) = 0. \tag{2.17}$$

Case 8. *when $m = 2$, $n = 1$ in equation (2.1) to get*

$$t^{-r}\frac{d}{dt}t^{r-k}\frac{d^2}{dt^2}(t^k\frac{d}{dt})u + g(t)h(u) = 0. \tag{2.18}$$

In this case, the provision of a new differential equation from the differential opertor above, after simplifying equation (2.18), gives

$$u^{''''} + \frac{r+2k}{t}u^{'''} + \frac{2k(r-1)+k(k-1)}{t^2}u^{''} + \frac{k(k-1)(r-2)}{t^3}u^{'} + g(t)h(u) = 0, \tag{2.19}$$

initial conditions

$$u(0) = a, u'(0) = u''(0) = u'''(0) = 0. \tag{2.20}$$

The initial conditions above are assumed to be similar to the initial conditions of the standard Emden-Fowler equation. It is worth noting that the singular point $t = 0$ appears three times in the equation as t, t^2, and t^3 with shape factors $r + 2k$, $2k(r-1) + k(k-1)$, and $k(k-1)(r-2)$, respectively. Additionally, the third and fourth terms vanish when $k = 1$ and $r = 1$, and the shape factor in this case reduces to 3. However, the fourth also vanishes, in addition to the previous case, when $k = 2$ and $r = 2$, and in this case the shape factors for the second and third terms are both 6, respectively. When $g(t) = 1$ and the initial conditions equation (2.20), the Lane-Emden type equation of the fourth order, as

$$u^{''''} + \frac{r+2k}{t}u^{'''} + \frac{2k(r-1)+k(k-1)}{t^2}u^{''} + \frac{k(k-1)(r-2)}{t^3}u^{'} + h(u) = 0.$$

Case 9. *at $m = 3$, $n = 0$ in equation (2.1) given [9]*

$$t^{-r}\frac{d}{dt}t^{r-k}\frac{d^3}{dt^3}(t^k u) + g(t)h(u) = 0. \tag{2.21}$$

Equations of the Fifth Order

Case 10. *at $m = 0$, $n = 4$ in equation (2.1) given [33]*

$$t^{-r}\frac{d}{dt}(t^r\frac{d^4}{dt^4})u + g(t)h(u) = 0. \tag{2.22}$$

Case 11. *when $m = 1$, $n = 3$ in equation (2.1) given [8]*

$$t^{-r}\frac{d}{dt}t^{r-k}\frac{d}{dt}(t^k\frac{d^3}{dt^3})u + g(t)h(u) = 0. \tag{2.23}$$

Case 12. *for $m = 2$, $n = 2$ in equation (2.1) will be*

$$t^{-r}\frac{d}{dt}t^{r-k}\frac{d^2}{dt^2}(t^k\frac{d^2}{dt^2})u + g(t)h(u) = 0, \tag{2.24}$$

equaling equation (2.24), we have:

$$u^{(v)} + \frac{r+2k}{t}u^{(iv)} + \frac{2k(r-1)+k(k-1)}{t^2}u''' + \frac{k(k-1)(r-2)}{t^3}u'' + g(t)h(u) = 0, \tag{2.25}$$

initial conditions

$$u(0) = a, u'(0) = b, u''(0) = u'''(0) = u''''(0) = 0,$$

and

Case 13. *for $m = 3$, $n = 1$ in equation (2.1) to get*

$$t^{-r}\frac{d}{dt}t^{r-k}\frac{d^3}{dt^3}(t^k\frac{d}{dt})u + g(t)h(u) = 0, \tag{2.26}$$

summarize equation (2.26), we get:

$$u^{(5)} + \frac{r+3k}{t}u^{(4)} + \frac{3k(r-1)+3k(k-1)}{t^2}u''' + \frac{3k(k-1)(r-2)+k(k-1)}{t^3}u''$$
$$+\frac{k(k-1)(k-2)(r-3)}{t^4}u' + g(t)h(u) = 0, \tag{2.27}$$

initial conditions

$$u(0) = a, u'(0) = u''(0) = u'''(0) = u''''(0) = 0.$$

Case 14. *at $m = 4$, $n = 0$ in equation (2.1) given [9]*

$$t^{-r}\frac{d}{dt}t^{r-k}\frac{d^4}{dt^4}(t^ku) + g(t)h(u) = 0. \tag{2.28}$$

2.1 Derivation of Generalized Emden-Fowler Equations

Building upon the established Emden–Fowler type equations, we introduce generalized Emden–Fowler-type equations applicable for arbitrary orders m and n.

$$t^{-r}\frac{d}{dt}t^{r-k}\frac{d^m}{dt^m}(t^k\frac{d^n}{dt^n})u+g(t)h(u)=0. \tag{2.29}$$

Therefore, this formula yields generalized Emden-Fowler-type equations of higher order in the form:

$$u^{(1+m+n)}+\frac{r+mk}{t}u^{(m+n)}+\sum_{p=1}^{m}\binom{m}{p}\frac{(r-p)}{t^{p+1}}\left(\prod_{j=1}^{p}(k-j+1)\right)u^{(m+n-p)}$$

$$+\sum_{p=2}^{m}\binom{m}{p}\frac{1}{t^p}\left(\prod_{j=1}^{p}(k-j+1)\right)u^{(1+m+n-p)}+g(t)h(u)=0, \tag{2.30}$$

with ICs

$$u(0)=a_0, u'(0)=a_1, ..., u^{(n-1)}=a_n,$$
$$u^{(n)}(0)=u^{(n+1)}(0)=...=u^{(m+n)}=0.$$

The singular point at $t=0$ appears m times in the equation as $t, t^2, \cdots, t^m$, each with a distinct shape factor. When $g(t)=1$, equation (2.30) reduces to the generalized Lane-Emden equations of higher orders.

2.2 Generalization of Higher Order Verification

Theorem 15. *For $m\in N$, the generalized of higher order derivative operator in the Emden-Fowler type equations is given by:*

$$t^{-r}\frac{d}{dt}t^{r-k}\frac{d^m}{dt^m}(t^k\frac{d^n}{dt^n}u)=u^{(1+m+n)}+\frac{r+mk}{t}u^{(m+n)}+$$
$$\sum_{p=1}^{m}\binom{m}{p}\frac{(r-p)}{t^{p+1}}\left(\prod_{j=1}^{p}(k-j+1)\right)u^{(m+n-p)}+\sum_{p=2}^{m}\binom{m}{p}\frac{1}{t^p}\left(\prod_{j=1}^{p}(k-j+1)\right)u^{(1+m+n-p)} \tag{2.31}$$

Proof. By mathematical induction for $m=0$, we get

$$L.H.S=t^{-r}\frac{d}{dt}t^r\frac{d^n}{dt^n}u=u^{(n+1)}(t)+\frac{r}{t}u^{(n)}(t),$$

and

$$R.H.S = u^{(1+n)}(t) + \frac{r}{t}u^{(n)}(t).$$

Or

$m = 1$, we have

$$L.H.S = t^{-r}\frac{d}{dt}t^{r-k}\frac{d}{dt}t^{k}\frac{d^n}{dt^n}u$$

$$= u^{(n+2)}(t) + \frac{r+k}{t}u^{(n+1)}(t)\frac{k(r-1)}{t^2}u^{(n)}(t),$$

$$R.H.S = u^{(2+n)} + \frac{r+k}{t}u^{(1+n)}\sum_{p=1}^{1}\binom{1}{1}\frac{(r-1)}{t^2}\prod_{j=1}^{1}(k-j+1)u^{(n)}$$

$$= u^{(2+n)}(t) + \frac{r+k}{t}u^{(1+n)}(t) + \frac{k(r-1)}{t^2}u^{(n)}(t).$$

Suppose that equation (2.31) is true for m=s, this means that we get

$$t^{-r}\frac{d}{dt}t^{r-k}\frac{d^s}{dt^s}(t^k\frac{d^n}{dt^n}u) = u^{(1+s+n)} + \frac{r+sk}{t}u^{(s+n)} +$$

$$\sum_{p=1}^{s}\binom{s}{p}\frac{(r-p)}{t^{p+1}}\left(\prod_{j=1}^{p}(k-j+1)\right)u^{(s+n-p)} + \sum_{p=2}^{s}\binom{s}{p}\frac{1}{t^p}\left(\prod_{j=1}^{p}(k-j+1)\right)u^{(1+s+n-p)} \tag{2.32}$$

To prove the claim for $m = s+1$, we apply the direct proof

$$L.H.S = t^{-r}\frac{d}{dt}t^{r-k}\frac{d^{s+1}}{dt^{s+1}}(t^k\frac{d^n}{dt^n}u)$$

$$= t^{-r}\frac{d}{dt}t^{r-k}\frac{d^s}{dt^s}(\frac{d}{dt}t^k\frac{d^n}{dt^n}u) = t^{-r}\frac{d}{dt}t^{r-k}\frac{d^s}{dt^s}(t^k u^{(n+1)} + kt^{k-1}u^{(n)})$$

$$= t^{-r}\frac{d}{dt}t^{r-k}\frac{d^s}{dt^s}t^k u^{(n+1)} + t^{-r}\frac{d}{dt}t^{r-k}\frac{d^s}{dt^s}kt^{k-1}u^{(n)}$$

$$= u^{(1+(s+1)+n)} + \frac{r+sk}{t}u^{((s+1)+n)} + \sum_{p=1}^{s}\binom{s}{p}\frac{(r-p)}{t^{p+1}}\left(\prod_{j=1}^{p}(k-j+1)\right)u^{((s+1)+n-p)}$$

$$+\sum_{p=2}^{s}\binom{s}{p}\frac{1}{t^p}\left(\prod_{j=1}^{p}(k-j+1)\right)u^{(1+(s+1)+n-p)} + \frac{k}{t}u^{(1+s+n)} +$$

$$\frac{k(r-1)+k(k-1)}{t^2}u^{(s+n)} + \sum_{p=2}^{s}\binom{s}{p}\frac{(r-1-p)}{t^{p+2}}\left(\prod_{j=1}^{p+1}(k-j+1)\right)u^{(s+1+n-p)} +$$

$$\sum_{p=3}^{s}\binom{s}{p}\frac{1}{t^{p+1}}\left(\prod_{j=1}^{p+1}(k-j+1)\right)u^{(1+s+1+n-p)},$$

then

$$u^{(1+(s+1)+n)}+\frac{r+(s+1)k}{t}u^{((s+1)+n)}+\frac{k(r-1)+k(k-1)}{t^2}u^{(s+n)}+$$

$$\sum_{p=1}^{s}\left[\binom{s}{p}+\binom{s}{p-1}\right]\frac{(r-p)}{t^{p+1}}\left(\prod_{j=1}^{p}(k-j+1)\right)u^{((s+1)+n-p)}$$

$$+\sum_{p=2}^{s}\left[\binom{s}{p}+\binom{s}{p-1}\right]\frac{(r-p)}{t^{p}}\left(\prod_{j=1}^{p}(k-j+1)\right)u^{(1+(s+1)+n-p)},$$

therefore

$$u^{(1+(s+1)+n)}+\frac{r+(s+1)k}{t}u^{((s+1)+n)}+$$

$$\frac{k(r-1)}{t^2}u^{(s+n)}+\sum_{p=1}^{s}\left[\binom{s}{p}+\binom{s}{p-1}\right]\frac{(r-p)}{t^{p+1}}\left(\prod_{j=1}^{p}(k-j+1)\right)u^{((s+1)+n-p)}$$

$$\frac{k(k-1)}{t^2}u^{(s+n)}+\sum_{p=2}^{s}\left[\binom{s}{p}+\binom{s}{p-1}\right]\frac{(r-p)}{t^{p}}\left(\prod_{j=1}^{p}(k-j+1)\right)u^{(1+(s+1)+n-p)},$$

$$u^{(1+(s+1)+n)}+\frac{r+(s+1)k}{t}u^{((s+1)+n)}+\sum_{p=1}^{s+1}\binom{s+1}{p}\frac{(r-p)}{t^{p+1}}\left(\prod_{j=1}^{p}(k-j+1)\right)u^{((s+1)+n-p)}$$

$$+\sum_{p=2}^{s+1}\frac{1}{t^p}\left(\prod_{j=1}^{p}(k-j+1)\right)u^{(1+(s+1)+n-p)}=R.H.S$$

Therefore equation (2.31) is valid for all $m\in$ N. □

3 The Method and Applications

The Adomian Decomposition Method ADM [2, 1, 3, 26, 11, 12, 13] is the application of the ADM for the analytic treatment of the Emden-Fowler-type equations. This method's specifics are now well-established and widely utilized in the literature [2, 1]. The ADM allows for the utilization of an infinite decomposition series.

$$u(t) = \sum_{n=0}^{\infty} u_n(t), \tag{3.1}$$

the solution $u(x)$, and polynomials of the infinite series

$$h(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, ..., u_n), \tag{3.2}$$

where $h(y)$ the nonlinear, $u_n(x)$ the components of the solution $u(x)$ will be determined recurrently, and A_n are the Adomian polynomials which are obtained from the definitional formula which given in [2]

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[h\left(\sum_{i=0}^{n} \lambda^i u_i\right)\right]_{\lambda=0}, n = 0, 1, 2, ...$$

where $Nu = h(u(t))$ is the nonlinearity, the formulas of Adomian polynomials from A_0 to A_4 as

$$A_0 = h(u_0),$$

$$A_1 = u_1 h'(u_0),$$

$$A_2 = u_2 h'(u_0) + \frac{1}{2!} u_1^2 h''(u_0),$$

$$A_3 = u_3 h'(u_0) + u_1 u_2 h''(u_0) + \frac{1}{3!} u_1^3 h'''(u_0),$$

$$A_4 = u_4 h'(u_0) + (u_1 u_3 + \frac{1}{2!} u_2^2) h''(u_0) + \frac{1}{2!} u_1^2 u_2 h'''(u_0) + \frac{1}{4!} u_1^4 h''''(u_0). \tag{3.3}$$

3.1 The 3^{rd} Type of Emden–Fowler Equation of Fourth Order

The 3^{rd} type of Emden–Fowler equation of fourth order is:

$$u^{''''} + \frac{r+2k}{t} u^{'''} + \frac{2k(r-1) + k(k-1)}{t^2} u^{''} + \frac{k(k-1)(r-2)}{t^3} u^{'} + g(t)h(u) = 0, \tag{3.4}$$

initial conditions

$$u(0) = a, u'(0) = u''(0) = u'''(0) = 0.$$

Re-written equation (3.4) in proposed linear operator

$$L(u) = -g(t)h(u), \tag{3.5}$$

the proposed operator L,

$$L(.) = t^{-r}\frac{d}{dt}t^{r-k}\frac{d^2}{dt^2}(t^k\frac{d}{dt}(.)). \tag{3.6}$$

To overcome the singular behavior at $t = 0$, the optimal definition of L^{-1} as proposed in equation (3.6)

$$L^{-1}(.) = \int_0^t t^{-k}\int_0^t\int_0^t t^{k-r}\int_0^t t^r(.)dtdtdtdt, \tag{3.7}$$

when $u'(0) = u''(0) = u'''(0) = 0$, we obtain

$$\begin{aligned} L^{-1}(Lu) &= \int_0^t t^{-k}\int_0^t\int_0^t t^{k-r}\int_0^t \frac{d}{dt}t^{r-k}\frac{d^2}{dt^2}(t^k\frac{d}{dt})dtdtdtdt \\ &= \int_0^t t^{-k}\int_0^t\int_0^t \frac{d^2}{dt^2}(t^k\frac{d}{dt})dtdtdtdt \\ &= \int_0^t t^{-k}\int_0^t \frac{d}{dt}(t^k\frac{d}{dt})dtdt \\ &= u(t) - u(0). \end{aligned}$$

Integral four-times and applying the L^{-1} in equation (3.5), we have

$$u(t) = a - L^{-1}(g(t)h(u)).$$

The solution $u(t)$ and nonlinearity $h(u)$ into two parts as in equation (3.1) and equation (3.2). The recursive relation for solution components is the derived

$$\begin{aligned} u_0 &= a, \\ u_{i+1} &= -L^{-1}(g(t)A_i), \quad i \geq 0. \end{aligned} \tag{3.8}$$

We will analyze two examples for different values of the parameters r and k when $r < k$ and $r > k$ for the specific functions, respectively.

Example 16. Consider the Emden-Fowler equation:

$$u'''' + \frac{5}{t}u''' + \frac{2}{t^2}u'' - \frac{2}{t^3}u' - (40 - 174t^4 + 15t^8 - 2t^{12})u^{-15} = 0, \tag{3.9}$$

under initial conditions

$$u(0) = 1, u'(0) = u''(0) = u'''(0) = 0,$$

substituting $r = 1, k = 2$ in equation (3.4) and $g(t)h(u) = (40 - 174t^4 + 15t^8 - 2t^{12})u^{-15}$.

Polynomials of the Adomian for u^{-15}, given

$$A_0 = u_0^{-15},$$
$$A_1 = -15u_1u_0^{-16},$$
$$A_2 = -15u_2u_0^{-16} + 120u_1^2u_0^{-17},$$
$$A_3 = -15u_3u_0^{-16} + 240u_1u_2u_0^{-17} + 680u_1^3u_0^{-18},$$

employing equation (3.8) the relation of repetitive

$$u_0 = 1,$$
$$u_1 = \frac{1}{4}t^4 - \frac{29}{576}t^8 + \frac{1}{1248}t^{12} - \frac{1}{30464}t^{16},$$
$$u_2 = \frac{-25}{576}t^8 + \frac{3277}{89856}t^{12} - \frac{78265}{25346048}t^{16} + ...,$$
$$u_3 = \frac{1565}{89856}t^{12} - \frac{417455}{16293888}t^{16} + ...,$$
$$u_4 = \frac{1162075}{228114432}t^{16} - \frac{1812362453}{184772689920}t^{20} + ...,$$
$$\cdots$$

By expansion of Taylar series, the solution is

$$u(t) = 1 + \frac{1}{4}t^4 - \frac{3}{32}t^8 + \frac{7}{128}t^{12} - \frac{77}{2048}t^{16} + ...,$$

which converges to the exact solution $u(t) = (1 + t^4)^{\frac{1}{4}}$.

Example 17. The Emden-Fowler equation:

$$u'''' + \frac{4}{t}u''' + \frac{2}{t^2}u'' - (4t^6 - 54t^4 + 45t^2 - 2)\frac{u^9}{t^2} = 0, \tag{3.10}$$

with initial conditions

$$u(0) = 1, u'(0) = 0, u''(0) = -1, u'''(0) = 0,$$

at $r = 2, k = 1$ in equation (3.4) and $g(t)h(u) = (4t^6 - 54t^4 + 45t^2 - 2)\frac{u^9}{t^2}$.

By the Adomian of polynomials for u^9, given

$$A_0 = u_0^9,$$

$$A_1 = 9u_1u_0^8,$$

$$A_2 = 9u_2u_0^8 + \frac{72}{2}u_1^2u_0^7,$$

$$A_3 = 9u_3u_0^8 + 72u_1u_2u_0^7 + 84u_1^3u_0^6,$$

using equation (3.8) in equation (3.3) the repetitive relation

$$u_0 = 1,$$

$$u_1 = -\frac{1}{2}t^2 + \frac{5}{16}t^4 - \frac{3}{50}t^6 + \frac{1}{784}t^8,$$

$$u_2 = \frac{1}{16}t^4 - \frac{37}{160}t^6 + \frac{148257}{1254400}t^8 - \frac{10573}{441000}t^{10} + ...,$$

$$u_3 = -\frac{17}{800}t^6 + \frac{18279}{125440}t^8 - \frac{78731713}{564480000}t^{10} + ...,$$

$$u_4 = \frac{1479}{179200}t^8 - \frac{4486687}{56448000}t^{10} + ...,$$

$$...$$

By Taylar expansion series, the $u(t)$ is

$$u(t) = 1 - \frac{1}{2}t^2 + \frac{3}{8}t^4 - \frac{5}{16}t^6 + \frac{35}{128}t^8 - \frac{63}{256}t^{10} + ...,$$

obtain to the exact solution $u(t) = \frac{1}{\sqrt{1+x^2}}$.

3.2 The Emden–Fowler Equation of 5^{th} order

In this part, we consider two new cases of Emden-Fowler with initial conditions (ICs). In the study of the third type of equation of 5^{th} order when $m = 2$, $n = 2$ in equation (2.1), we get equation (2.25) = equation (3.11), and in the study of the fourth type of Emden-Fowler equation of 5^{th} order when $m = 3$, $n = 1$ in equation (2.1), we obtain equation (2.27) = equation (3.16), respectively.

3.2.1 The Third Type of Emden–Fowler Equation

The 3^{rd} type of Emden–Fowler equation of fifth order is:

$$u^{'''''} + \frac{r+2k}{t}u^{''''} + \frac{2k(r-1)+k(k-1)}{t^2}u^{'''} + \frac{k(k-1)(r-2)}{t^3}u^{''} + g(t)h(u) = 0, \tag{3.11}$$

under ICs

$$u(0) = a, u'(0) = b, u''(0) = u'''(0) = u''''(0) = 0.$$

Re-written equation (3.11) in operator

$$L(u) = -g(t)h(u), \tag{3.12}$$

the operator L,

$$L(.) = t^{-r}\frac{d}{dt}t^{r-k}\frac{d^2}{dt^2}(t^k\frac{d^2}{dt^2}(.)). \tag{3.13}$$

To overcome the singular behavior at $t = 0$, the optimal definition of L^{-1} as proposed in equation (3.13)

$$L^{-1}(.) = \int_0^t \int_0^t t^{-k} \int_0^t \int_0^t t^{k-r} \int_0^t t^r (.)dtdtdtdtdt, \tag{3.14}$$

when $u''(0) = u'''(0) = u''''(0) = 0$, we get

$$L^{-1}(Lu) = \int_0^t \int_0^t t^{-k} \int_0^t \int_0^t t^{k-r} \int_0^t \frac{d}{dt}t^{r-k}\frac{d^2}{dt^2}(t^k\frac{d^2}{dt^2})dtdtdtdtdt$$

$$= \int_0^t \int_0^t t^{-k} \int_0^t \int_0^t \frac{d^2}{dt^2}(t^k\frac{d^2}{dt^2})dtdtdtdt$$

$$= \int_0^t \int_0^t t^{-k} \int_0^t \frac{d}{dt}(t^k\frac{d^2}{dt^2})dtdtdt$$

$$= \int_0^t \int_0^t t^{-k}(t^k\frac{d^2}{dt^2})dtdt$$

$$= u(t) - u(0) - tu'(0).$$

Integral five-times and applying the L^{-1} in equation (3.12), we get

$$u(t) = a + bt - L^{-1}(g(t)h(u)).$$

The solution $u(t)$ and nonlinearity $h(u)$ into two parts as in equation (3.1) and equation (3.2). The solution components is the derived

$$u_0 = a + bt,$$

$$u_{i+1} = -L^{-1}(g(t)A_i), \quad i \geq 0. \tag{3.15}$$

3.2.2 The Fourth Type of Emden–Fowler Equation

The 4^{th} type of Emden–Fowler equation of 5^{th} order is:

$$u^{(5)} + \frac{r+3k}{t}u^{(4)} + \frac{3k(r-1)+3k(k-1)}{t^2}u^{'''} +$$
$$\frac{3k(k-1)(r-2)+k(k-1)(k-2)}{t^3}u^{''} + \frac{k(k-1)(k-2)(r-3)}{t^4}u^{'} + g(t)h(u) = 0, \tag{3.16}$$

with ICs

$$u(0) = a, u'(0) = u''(0) = u'''(0) = u''''(0) = 0.$$

Re-written equation (3.16) in linear operator

$$L(u) = -g(t)h(u). \tag{3.17}$$

The linear operator L,

$$L(.) = t^{-r}\frac{d}{dt}t^{r-k}\frac{d^3}{dt^3}(t^k\frac{d}{dt}(.)). \tag{3.18}$$

To overcome the singular behavior at $t = 0$, the optimal definition of L^{-1} as proposed in equation (3.18)

$$L^{-1}(.) = \int_0^t t^{-k}\int_0^t\int_0^t\int_0^t t^{k-r}\int_0^t t^r(.)dtdtdtdtdt, \tag{3.19}$$

at $u'(0) = u''(0) = u'''(0) = u''''(0) = 0$, we have

$$\begin{aligned}
L^{-1}(Lu) &= \int_0^t t^{-k}\int_0^t\int_0^t\int_0^t t^{k-r}\int_0^t \frac{d}{dt}t^{r-k}\frac{d^3}{dt^3}(t^k\frac{d}{dt})dtdtdtdtdt \\
&= \int_0^t t^{-k}\int_0^t\int_0^t\int_0^t \frac{d^3}{dt^3}(t^k\frac{d}{dt})dtdtdtdt \\
&= \int_0^t t^{-k}\int_0^t\int_0^t \frac{d^2}{dt^2}(t^k\frac{d}{dt})dtdtdt \\
&= \int_0^t t^{-k}\int_0^t \frac{d}{dt}(t^k\frac{d}{dt})dtdt \\
&= \int_0^t t^{-k}(t^k\frac{d}{dt})dt \\
&= u(t) - u(0),
\end{aligned}$$

and applying the L^{-1} in equation (3.17), we have

$$u(t) = a - L^{-1}(g(t)h(u)).$$

The solution $u(t)$ and nonlinearity $h(u)$ into two parts as in equation (3.1) and equation (3.2). The recursive relation for solution components is the derived

$$u_0 = a,$$
$$u_{i+1} = -L^{-1}(g(t)A_i), \qquad i \geq 0. \tag{3.20}$$

We will study two examples for different values of the parameters k and r, respectively.

Example 18. Consider the 5^{th} order Emden-Fowler equation:

$$u''''' + \frac{4}{t}u'''' + \frac{2}{t^2}u''' - \frac{16}{t}(t^{16} - 220t^{12} + 910t^8 - 396t^4 + 9)e^{-5u} = 0, \tag{3.21}$$

with ICs

$$u(0) = 0, u'(0) = u''(0) = u'''(0) = u''''(0) = 0,$$

putting $r = 2, k = 1$ in (3.11) and $g(t)h(u) = \frac{16}{t}(t^{16} - 220t^{12} + 910t^8 - 396t^4 + 9)e^{-5u}$.

The polynomials A_n for e^{-5u}, given

$$A_0 = e^{-5u_0},$$
$$A_1 = -5u_1e^{-5u_0},$$
$$A_2 = (-5u_2 + \frac{25}{2}u_1^2)e^{-5u_0},$$
$$A_3 = (-5u_3 + 25u_1u_2 - \frac{125}{6}u_1^3)e^{-5u_0},$$

putting equation (3.15) in equation (3.3), obtain

$$u_0 = 0,$$
$$u_1 = t^4 - \frac{22}{49}t^8 + \frac{182}{1815}t^{12} - \frac{11}{2205}t^{16} + \frac{1}{146205}t^{20},$$
$$u_2 = \frac{-5}{98}t^8 + \frac{54}{245}t^{12} - \frac{8068616}{65367225}t^{16} + \frac{600379}{26268165}t^{20} + ...,$$
$$u_3 = \frac{75}{5929}t^{12} - \frac{7026}{60025}t^{16} + \frac{24024987946}{212378114025}t^{20} + ...,$$
$$u_4 = \frac{-15775}{3486252}t^{16} + \frac{21065582}{338721075}t^{20} + ...,$$
$$...$$

the solution in series is

$$u(t) = t^4 - \frac{1}{2}t^8 + \frac{1}{3}t^{12} - \frac{1}{4}t^{16} + \frac{1}{5}t^{20} + ...,$$

which converges to the exact solution $u(t) = \ln(1 + t^4)$.

Example 19. Consider the 4^{th} type of Emden-Fowler equation:

$$u^{(5)} + \frac{3}{t}u^{(4)} + \frac{\frac{1}{3}}{t^2}u''' + \frac{\frac{10}{27}}{t^3}u'' + \frac{\frac{-10}{27}}{t^4}u' -$$
$$\frac{25}{27}(3375t^{20} + 29025t^{15} + 58365t^{10} + 24094t^5 + 546)e^{-t^5}u^2 = 0, \tag{3.22}$$

under ICs

$$u(0) = 1, u'(0) = u''(0) = u'''(0) = u''''(0) = 0,$$

substituting $r = 2, k = \frac{1}{3}$ in equation (3.16) and $g(t)h(u) = \frac{25}{27}(3375t^{20} + 29025t^{15} + 58365t^{10} + 24094t^5 + 546)e^{-t^5}u^2$.

Polynomials of the Adomian for u^2, given

$$A_0 = u_0^2,$$
$$A_1 = 2u_1u_0,$$
$$A_2 = 2u_2u_0 + u_1^2,$$
$$A_3 = 2u_3u_0 + 2u_1u_2,$$

by equation (3.20) the relation of recursive

$$u_0 = 1,$$
$$u_1 = t^5 + \frac{841}{1760}t^{10} + \frac{4318}{62049}t^{15} - \frac{2173}{298584}t^{20} - \frac{1105}{18898824}t^{25} + ...,$$
$$u_2 = \frac{39}{1760}t^{10} + \frac{20951833}{218412480}t^{15} + \frac{10695256379}{278694324480}t^{20} + \frac{99927736691}{519550484152700}t^{25} + ...,$$
$$u_3 = \frac{6433}{5600320}t^{15} + \frac{1217497981}{115321789440}t^{20} + \frac{176524171155407573}{21945812450610048000}t^{25} + ...,$$
$$u_4 = \frac{913717}{85752099840}t^{20} + \frac{323563029679}{2078989432608000}t^{25} + ...,$$
$$...$$

the solution is

$$u(t) = 1 + t^5 + \frac{1}{2}t^{10} + \frac{1}{6}t^{15} + \frac{1}{24}t^{20} + \frac{1}{120}t^{25} + ...,$$

which closed to the solution $u(t) = e^{t^5}$.

3.3 Examples of the Type of Lane–Emden Equations of Fourth and Fifth Order

In this section, we will test two examples of the 4^{th} and 5^{th} orders Lane-Emden equations with $k = 1$ and different value r in equation (3.4) and equation (3.11) . These examples belong to the third types, respectively. The nonlinear term in each equation is u^m.

Example 20. We first consider the Lane-Emden equation of 4^{th} order:

$$u'''' + \frac{r+2}{t}u''' + \frac{2(r-1)}{t^2}u'' + u^m = 1 - \frac{3(1+r)}{10}, \tag{3.23}$$

with ICs

$$u(0) = 1, u'(0) = u''(0) = u'''(0) = 0,$$

substituting $k = 1$ in equation (3.4), $h(u) = u^m$ and $g(t) = 1$.

Polynomials of the Adomian for u^m

$$A_0 = u_0^m,$$

$$A_1 = m\, u_1 u_0^{m-1},$$

$$\ldots$$

using equation (3.8) in equation (3.3), the recursive relation

$$u_0 = 1 + \frac{7-3r}{480(1+r)}t^4,$$

$$u_1 = -L^{-1}(A_0) = \frac{(-9600-9600\,r)}{460800\,(1+r)^2}t^4 + \frac{m\,(-7+3\,r)}{215040\,(1+r)\,(5+r)}t^8 - \frac{(-1+m)\,m\,(7-3\,r)^2}{729907200\,(1+r)^2\,(9+r)}t^{12} + \ldots,$$

$$u_2 = -L^{-1}(A_1) = \frac{m\,\left(-2138400-2376000\,r-237600\,r^2\right)}{5109350400\,(1+r)^2\,(5+r)\,(9+r)}t^8 + \frac{m\,\left(4900-5005\,m-1120\,r+1060\,m\,r-420\,r^2+465\,m\,r^2\right)}{5109350400\,(1+r)^2\,(5+r)\,(9+r)}t^{12} - \frac{(-1+m)\,m\,(7-3\,r)^2\,(-924\,(5+r)\,(9+r)+m\,(21751+r\,(7542+575\,r)))}{39239811072000\,(1+r)^3\,(5+r)\,(9+r)\,(13+r)}t^{16} + \ldots,$$

$$\ldots$$

the solution in series form is given by

$$u(t) = 1 - \frac{1}{160}\,t^4 + \frac{m\,(-17+3\,r)}{215040\,(5+6\,r+r^2)}\,t^8 -$$

$$\frac{m\,(-7+3\,r)\,\left(-7\,\left(-45-4\,r+r^2\right)+m\,\left(-320-33\,r+7\,r^2\right)\right)}{1703116800\,(1+r)^2\,(45+14\,r+r^2)}\,t^{12} +$$

$$\frac{(1-m)\,m\,(7-3\,r)^2\,(-924\,(5+r)\,(9+r)+m\,(21751+r\,(7542+575\,r)))}{39239811072000\,(1+r)^3\,(5+r)\,(9+r)\,(13+r)}\,t^{16} + \ldots,$$

when $r=0$, note that

$$u(t) = 1 - \frac{1}{160}\,t^4 - \frac{17\,m}{1075200}\,t^8 + \left(\frac{(4900-5005\,m)\,m}{229920768000} - \frac{49\,(-1+m)\,m}{6569164800}\right)\,t^{12} -$$

$$\frac{7\,(-1+m)\,m\,(-41580+21751\,m)}{3279327068160000}\,t^{16} + \ldots,$$

and when $m=0$, note that the exact solution is given as $u(t) = 1 - \frac{1}{160}\,t^4$.

Example 21. We next consider the Lane-Emden equation of 5^{th} order:

$$u^{'''''} + \frac{4}{t}u^{''''} + \frac{2}{t^2}u^{'''} + u^m = 0, \tag{3.24}$$

under initial conditions

$$u(0) = 1, u'(0) = u''(0) = u'''(0) = u''''(0) = 0,$$

when $r = 2\,k = 1$ in (3.11) and $g(t) = 1$, polynomials of Adomian for u^m, given

$$\begin{aligned} A_0 &= u_0^m, \\ A_1 &= m u_1 u_0^{m-1}, \\ A_2 &= m u_2 u_0^{m-1} + m(m-1)\frac{u_1^2}{2!}u_0^{m-2}, \\ A_3 &= m u_3 u_0^{m-1} + m(m-1)u_1 u_2 u_0^{m-2} + m(m-1)(m-2)\frac{u_1^3}{3!}u_0^{m-3}, \\ &\ldots \end{aligned} \tag{3.25}$$

Using equation (3.20) into equation (3.3), we get

$$u_0 = 1,$$

$$u_1 = -L^{-1}(A_0) = -\frac{1}{720}t^5,$$

$$u_2 = -L^{-1}(A_1) = \frac{m}{37324800}t^{10},$$
$$u_3 = -L^{-1}(A_2) = -\frac{m(37m-36)}{18545200128000}t^{15},$$
$$u_4 = -L^{-1}(A_3) = \frac{m(108097m^2-306829m+198744)}{5205889890651340800000}t^{20},$$
$$u_5 = -\frac{m(1534478153m^3-8447787017m^2+14661940128m-7748631072)}{6345021892964104588492800000}t^{25},$$

the solution in series form is given by

$$u(t) = 1 - \frac{1}{720}t^5 + \frac{m}{37324800}t^{10} - \frac{m(37m-36)}{18545200128000}t^{15} +$$
$$\frac{m(108097m^2-306829m+198744)}{5205889890651340800000}t^{20} -$$
$$-\frac{m(1534478153m^3-8447787017m^2+14661940128m-7748631072)}{6345021892964104588492800000}t^{25} + ...,$$

when $m = 0$, note that the exact solution is given as $u(t) = 1 - \frac{1}{720}t^5$.

4 Conclusion

In this study, we introduced a new differential operator that can solve a variety of different types of Emden-Fowler and Lane-Emden equations, unlike classical limited differential operators. The presentation of a 3^{rd} type of Emden-Fowler equation of 4^{th} order and the provision of two examples for different values that are described by the accurate approximate solution of the exact solution. In addition, it displays two cases of the 5^{th} order, each with an example. These cases were updated to show our strong analysis, where we used the ADM for nonlinear equations. We obtained convergence of the exact solution as shown in the examples for each of the cases mentioned earlier.

We propose using the ADM to solve these equations, highlighting its effectiveness in handling both linear and nonlinear problems. demonstrate the application of this method to specific examples with initial conditions ICs emphasizing the handling of singular points and nonlinearities.

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