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HADAMARD AND QUASI-HADAMARD PROPERTIES FOR CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS

Shashi Kant, Neelmani Gupta and Sanjay Issar

Abstract. In the present paper two subclasses $\mathcal{S}_q^*(Ne)$ and $\mathcal{K}_q(Ne)$ of analytic functions are introduced by using q-derivative operator. Several properties including Hadamard product, quasi-Hadamard product, the necessary and sufficient condition and coefficient estimates for the functions belonging to these subclasses are obtained.

1 Introduction and definitions

Let $\mathcal{A}$ denote the class of analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S}$ be class of all functions in $\mathcal{A}$, which are univalent in $\mathbb{U}$. Let Ω be the family of analytic function w satisfying the conditions $w(0) = 0,\ \ |w(z)| < 1$ for all $z \in \mathbb{U}$.

Let f and g be two analytic functions in $\mathbb{U}$, then function f is said to be subordinate to g if there exists an analytic function $w \in \Omega$ such that $f(z) = g(w(z))\ \ (z \in \mathbb{U})$. We denote this subordination by $f \prec g$. In particular, if the function g is univalent in $\mathbb{U}$ the above subordination is equivalent to $f(0) = g(0)$ *and* $f(\mathbb{U}) \subseteq g(\mathbb{U})$.

Quantum calculus or q-calculus is an ordinary classical calculus without the notion of limit. It has many applications in various branches of Mathematics and Physics. Recently the area of q-calculus has attracted the serious attention of researchers. The application of q-calculus was initiated by Jackson [8, 10] to begin with investigated q-calculus applications, efficiently creating q-derivative and q-integral. Recently in the field of Geometric Function Theory many function classes have been introduced with the help of q-derivative and q-integral operators and investigated

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by a number of researchers including [[1], [24], [27], [33], [36], [37], [38], [39]]. The purpose of this article is to introduce and study two subclasses of univalent functions by applying q-derivative operator in conjunction with the principle of subordination.

For $0 < q < 1$, the q-derivative of a function f is defined as(see [8, 10, 11])

$$D_q f(z) = \begin{cases} \frac{f(z)-f(qz)}{(1-q)z} & (z \neq 0) \\ f'(0) & (z = 0), \end{cases}$$

provided that $f'(0)$ exists. For f given by (1.1), it can be easily obtained that

$$D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1},$$

where

$$[k]_q = \frac{1-q^k}{1-q} = 1 + q + q^2 + ... + q^{k-1}.$$

As q$\rightarrow 1^-$, $[k]_q \rightarrow k$ and $\lim_{q\rightarrow 1^-} D_q f(z) = f^{'}(z)$.

Also the $q - integral$ of the function f is defined as (see [8])

$$\int_0^z f(t) d_q t = (1-q) z \sum_{k=0}^{\infty} q^k f(zq^k) \quad , \tag{1.2}$$

provided that the series converges. Here we observe that

$$\int_0^z t^k d_q t = \frac{z^{k+1}}{[k+1]_q} \quad (k \neq -1)$$

and

$$\lim_{q\rightarrow 1^-} \int_0^z t^k d_q t = \frac{z^{k+1}}{k+1} \quad (k \neq -1) \ .$$

A function $f \in \mathcal{A}$ is starlike if $f(\mathbb{U})$ is starlike with respect to origin and function f is convex if $f(\mathbb{U})$ is convex. Analytically in term of subordination, a function $f \in \mathcal{A}$ is starlike and convex if and only if the subordination relations $\frac{zf^{'}(z)}{f(z)} \prec \frac{1+z}{1-z}$ and $\frac{zf^{''}(z)}{f^{'}(z)} \prec \frac{2z}{1-z}$ for $z \in \mathbb{U}$ respectively hold. Ma and Minda[18] introduced and studied the following two subclasses $\mathcal{S}^*(\phi)$ and $\mathcal{K}(\phi)$ of starlike and convex functions respectively in term of subordination relations:

$$\mathcal{S}^*(\phi) = \{f \in \mathcal{A} : \frac{zf^{'}(z)}{f(z)} \prec \phi(z)\}$$

and

$$\mathcal{K}(\phi) = \{f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \phi(z)\},$$

where ϕ is analytic and normalized by $\phi(0) = 1$ and $\phi'(0) > 0$ with $\Re\phi(z) > 0$ in $\mathbb{U}$. Seoudy and Aouf [27] further generalized these classes by using the q-derivative operator in the following manner:

$$\mathcal{S}^*_q(\phi) = \{f \in \mathcal{A} : \frac{zD_qf(z)}{f(z)} \prec \phi(z)\}$$

and

$$\mathcal{K}_q(\phi) = \{f \in \mathcal{A} : \frac{D_q(zD_qf(z))}{D_qf(z)} \prec \phi(z)\}.$$

Recently, several Ma - Minda type subclasses of starlike and convex functions have been introduced and studied by considering different image domains $\phi(\mathbb{U})$. For some examples, we can see[[4],[12],[14],[28]-[31],[40]]. Motivated essentially due to the work of Kumar and Cetinkaya [14], in the present paper we consider two subclasses $\mathcal{S}^*_q(Ne)$ and $\mathcal{K}_q(Ne)$ of starlike and convex functions respectively, which are associated with the analytic function $Ne(z) = 1 + z - \frac{z^3}{3}$. The function Ne maps an open unit disc onto the Nephroid shaped bounded symmetric region with respect to the real axis in the right - half of the complex plane. Analytically these classes are defined as:

$$\mathcal{S}^*_q(Ne) = \{f \in \mathcal{A} : \frac{zD_qf(z)}{f(z)} \prec 1 + z - \frac{z^3}{3}\} \tag{1.3}$$

and

$$\mathcal{K}_q(Ne) = \{f \in \mathcal{A} : \frac{D_q(zD_qf(z))}{D_qf(z)} \prec 1 + z - \frac{z^3}{3}\}. \tag{1.4}$$

From (1.3) and (1.4), we find that

$$f(z) \in \mathcal{S}^*_q(Ne) \Longleftrightarrow \int_0^z \frac{f(t)}{t} d_qt \in \mathcal{K}_q(Ne). \tag{1.5}$$

Let f and g be two analytic functions of the form

$$f(z) = a_1z + \sum_{k=2}^{\infty} a_kz^k$$

and

$$g(z) = b_1z + \sum_{k=2}^{\infty} b_kz^k \quad ,$$

then the Hadamard product (or convolution) of f and g is defined by

$$f(z) * g(z) = (f * g)(z) = a_1 b_1 z + \sum_{k=2}^{\infty} a_k b_k z^k.$$

In view of the definition of Hadamard product of two analytic functions , it is easy to verify that

$$f(z) = f(z) * \frac{z}{1-z}, \tag{1.6}$$

$$zD_q f(z) = zD_q f(z) * \frac{z}{(1-z)} = f(z) * \frac{z}{(1-z)(1-qz)} \tag{1.7}$$

and

$$zD_q(zD_q f(z)) = zD_q f(z) * \frac{z}{(1-z)(1-qz)}. \tag{1.8}$$

Further, let $\mathcal{T}$ be the subclass of analytic functions with negative coefficients of the form

$$f(z) = a_1 z - \sum_{k=2}^{\infty} a_k z^k \quad (a_1 > 0, a_k \geq 0) \tag{1.9}$$

defined in $\mathbb{U}$. For the function f defined by (1.9) and $g(z) = b_1 z - \sum_{k=2}^{\infty} b_k z^k \quad (b_1 > 0, b_k \geq 0)$, the quasi -Hadamard product of f and g is given by

$$f(z) *' g(z) = (f *' g)(z) = a_1 b_1 z - \sum_{k=2}^{\infty} a_k b_k z^k. \tag{1.10}$$

In a similar manner, we can define a quasi-Hadamard product of more than two functions. The quasi-Hadamard product of two or more functions has recently been defined and used by Owa ([20]-[22]), Kumar ([15]- [17]), Sekine [26], Aouf [2], Frasin and Aouf [7], Hossen [9], Darwish [5] and El-Ashwah et al. [6].

In the next section we give characterizations for defined subclasses of q-starlike and q-convex functions with the help of Hadamard products. For each of these subclasses first we find a function g depending on each concerning class $\mathcal{S}_q^*(Ne)$ and $\mathcal{K}_q(Ne)$, such that $\frac{1}{z}(f * g) \neq 0$ is both necessary and sufficient for f to be in $\mathcal{S}_q^*(Ne)$ and $\mathcal{K}_q(Ne)$. Further we use these findings to determine the coefficients estimates for a function belonging to $\mathcal{S}_q^*(Ne)$ and $\mathcal{K}_q(Ne)$. For some recent similar studies on various classes of analytic functions, one can find in [[3],[13],[19],[23],[25], [27], [32] , [38], [39]] and the references cited therein.

In section 3, we establish certain results concerning the quasi-Hadamard product of functions belonging in the classes $\mathcal{S}_q^*(Ne)$ and $\mathcal{K}_q(Ne)$ analogous to the results due to Kumar ([16] and [17]) and Sekine [26].

2 Hadamard product properties

Unless otherwise mentioned, we assume throughout this paper that $0 < q < 1$, and $\theta \in [0, 2\pi)$.

Theorem 1. *The function $f \in \mathcal{T}$ defined by (1.9) is in the class $\mathcal{K}_q(Ne)$ if and only if*

$$\frac{1}{z}\left[f(z) * \frac{z + [1-(1+q)L]qz^2}{(1-z)(1-qz)(1-q^2z)}\right] \neq 0 \tag{2.1}$$

for all $L = \frac{3+3e^{i\theta}-e^{i3\theta}}{3e^{i\theta}-e^{i3\theta}}$, where $\theta \in [0, 2\pi)$ and also $L = 1$.

Proof. Suppose that the function $f \in \mathcal{K}_q(Ne)$, then we have

$$\frac{D_q(zD_qf(z))}{D_qf(z)} \prec 1 + z - \frac{z^3}{3}. \tag{2.2}$$

Due to analyticity of the function $\frac{D_q(zD_qf(z))}{D_qf(z)}$ in $\mathbb{U}$, we have $D_qf(z) \neq 0$ which is equivalent to the fact that (2.1) holds for $L = 1$. In view of (2.2)

$$\frac{D_q(zD_qf(z))}{D_qf(z)} = 1 + \omega(z) - \frac{\omega^3(z)}{3}$$

where $\omega \in \Omega$ is a Schwarz function, hence

$$\frac{1}{z}\left[3zD_q(zD_qf(z)) - (3 + 3e^{\iota\theta} - e^{3\iota\theta})zD_qf(z)\right] \neq 0. \tag{2.3}$$

Using (1.7)and (1.8), we obtain

$$\frac{1}{z}\left[zD_qf(z) * \frac{3z}{(1-z)(1-qz)} - (3 + 3e^{\iota\theta} - e^{3\iota\theta})zD_qf(z) * \frac{z}{1-z}\right] \neq 0.$$

or

$$\frac{e^{3\iota\theta} - 3e^{\iota\theta}}{z}[zD_qf(z) * \frac{z - Lqz^2}{(1-z)(1-qz)}] \neq 0,$$

where L is given with (2.1). Using the identity $zD_qf(z) * g(z) = f(z) * zD_qg(z)$, we get

$$\frac{e^{3\iota\theta} - 3e^{\iota\theta}}{z}[f(z) * \frac{z + (1-(q+1)L)qz^2}{(1-z)(1-qz)(1-q^2z)}] \neq 0,$$

which leads to (2.1).

For only if part of the theorem, suppose that f satisfies the condition (2.1). Since it is shown here that assumption (2.1) is equivalent to (2.3), so we have

$$\frac{D_q(zD_qf(z))}{D_qf(z)} \neq 1 + e^{\iota\theta} - \frac{e^{3\iota\theta}}{3}. \tag{2.4}$$

Suppose that $\psi(z) = \frac{D_q(zD_qf(z))}{D_qf(z)}$ and $\varphi(z) = 1 + z - \frac{z^3}{3}$. The relation (2.4) means that

$$\psi(\mathbb{U}) \cap \varphi(\partial\mathbb{U}) = \emptyset.$$

Thus, the simply connected domain $\psi(\mathbb{U})$ is included in a connected component of $\mathbb{C}\backslash\varphi(\partial\mathbb{U})$. Therefore, using the fact that $\psi(0) = \varphi(0)$ together with the univalence of the function $\varphi(z) = 1 + z - \frac{z^3}{3}$, it follows that $\psi(z) \prec \varphi(z)$. Hence $f \in \mathcal{K}_q(Ne)$, which complete the proof of Theorem 1.

□

By using the technique as given in Srivastava and Zayed [38], we prove the following convolution condition for the subclass $\mathcal{S}_q^*(Ne)$.

Theorem 2. *The function $f \in \mathcal{T}$ defined by (1.9) is in the class $\mathcal{S}_q^*(Ne)$ if and only if*

$$\frac{1}{z}\left[f(z) * \frac{z - Lqz^2}{(1-z)(1-qz)}\right] \neq 0 \tag{2.5}$$

for all $L = \frac{3+3e^{i\theta}-e^{i3\theta}}{3e^{i\theta}-e^{i3\theta}}$, where $\theta \in [0, 2\pi)$ and also $L = 1$.

Proof. It follows from (1.5) that

$$f(z) \in \mathcal{S}_q^*(Ne) \text{ if and only if } \int_0^z \frac{f(t)}{t} d_qt \in \mathcal{K}_q(Ne).$$

Then according to the Theorem 1, the function f belongs to $\mathcal{S}_q^*(Ne)$ if and only if

$$\frac{1}{z}\left[\left(\int_0^z \frac{f(t)}{t} d_qt\right) * g(z)\right] \neq 0, \tag{2.6}$$

where

$$g(z) = \frac{z + [1-(1+q)L]qz^2}{(1-z)(1-qz)(1-q^2z)}.$$

From (1.2), we have

$$\int_0^z \frac{g(t)}{t} d_qt = \int_0^z \frac{1 + [1-(1+q)L]qt}{(1-t)(1-qt)(1-q^2t)} d_qt$$

$$= z(1-q)\sum_{k=0}^{\infty} \frac{q^k + [1-(1+q)L]zq^{2k+1}}{(1-zq^k)(1-zq^{k+1})(1-zq^{k+2})},$$

and therefore

$$\int_0^z \frac{g(t)}{t} d_q t = \frac{z - Lqz^2}{(1-z)(1-qz)}. \tag{2.7}$$

Using the identity

$$\left(\int_0^z \frac{f(t)}{t} d_q t\right) * g(z) = f(z) * \left(\int_0^z \frac{g(t)}{t} d_q t\right)$$

and (2.7) in relation (2.6), we get the desired result (2.5). □

Theorem 3. *The function $f \in \mathcal{T}$ defined by (1.9) is in the class $\mathcal{K}_q(Ne)$ if and only if*

$$a_1 - \sum_{k=2}^{\infty} [k]_q \left(1 - \frac{3q[k-1]_q}{3e^{i\theta} - e^{i3\theta}}\right) a_k z^{k-1} \neq 0 \tag{2.8}$$

Proof. If $f \in \mathcal{T}$ given by (1.9), then from Theorem 1, we have $f \in \mathcal{K}_q(Ne)$ if and only if (2.1) holds. Since

$$\frac{1}{(1-z)(1-qz)(1-q^2z)} = 1+(1+q+q^2)z+(1+q+2q^2+q^3+q^4)z^2+(1+q+2q^2+2q^3+2q^4+q^5+q^6)z^3+\ldots,$$

it follows that

$$\frac{z + [1-(1+q)L]qz^2}{(1-z)(1-qz)(1-q^2z)} = z + \sum_{k=2}^{\infty} [k]_q \left(1 - q(L-1)[k-1]_q\right) z^k,$$

where $L = \frac{3+3e^{i\theta}-e^{i3\theta}}{3e^{i\theta}-e^{i3\theta}}$ and so (2.1) may be written as

$$a_1 - \sum_{k=2}^{\infty} [k]_q \left(1 - \frac{3q[k-1]_q}{3e^{i\theta} - e^{i3\theta}}\right) a_k z^{k-1} \neq 0.$$

which completes the proof. □

Theorem 4. *The function $f \in \mathcal{T}$ defined by (1.9) is in the class $\mathcal{S}_q^*(Ne)$ if and only if*

$$a_1 - \sum_{k=2}^{\infty} (1 - \frac{3q[k-1]_q}{3e^{i\theta} - e^{i3\theta}}) a_k z^{k-1} \neq 0 \tag{2.9}$$

Proof. Since

$$\frac{z}{(1-z)(1-qz)} = z + \sum_{k=2}^{\infty} [k]_q z^k$$

so

$$\frac{z - Lqz^2}{(1-z)(1-qz)} = z + \sum_{k=2}^{\infty} [1 - q(L-1)[k-1]_q] z^k \tag{2.10}$$

In view of (2.5) and (2.10), a simple calculation provides the desired result. □

Theorem 5. *If the function $f \in \mathcal{T}$ defined by (1.9) satisfies the inequality*

$$\sum_{k=2}^{\infty}[k]_q\big(3[k]_q-1\big)a_k \leq 2a_1 \qquad (a_k \geq 0, a_1 > 0), \tag{2.11}$$

then $f \in \mathcal{K}_q(Ne)$.

Proof. Since

$$|a_1 - \textstyle\sum_{k=2}^{\infty}[k]_q\big(1-\frac{3q[k-1]_q}{3e^{i\theta}-e^{i3\theta}}\big)a_k z^{k-1}|$$

$$\geq a_1 - \textstyle\sum_{k=2}^{\infty}[k]_q\big|\big(1-\frac{3q[k-1]_q}{3e^{i\theta}-e^{i3\theta}}\big)\big|a_k$$

$$\geq a_1 - \textstyle\sum_{k=2}^{\infty}[k]_q\big|\big(1+\frac{3q[k-1]_q}{2}\big)\big|a_k > 0$$

Thus, the inequality (2.11) holds and our result follows from Theorem 3. □

Using similar arguments to those in the proof of Theorem 5, we may also prove the Theorem 6.

Theorem 6. *If the function $f \in \mathcal{T}$ defined by (1.9) satisfies the inequality*

$$\sum_{k=2}^{\infty}\big(3[k]_q-1\big)a_k \leq 2a_1 \qquad (a_k \geq 0, a_1 > 0), \tag{2.12}$$

then $f \in \mathcal{S}_q^*(Ne)$.

3 Quasi-Hadamard product properties

The quasi-Hadamard product properties of the classes $\mathcal{S}_q^*(Ne)$ and $\mathcal{K}_q(Ne)$ are obtained in this section. For that purpose we need to define the following subclass of analytic functions:

A function f of the form (1.9) belongs to the class $\mathcal{S}_q^{(c)}(Ne)$ if and only if

$$\sum_{k=2}^{\infty}([k]_q)^c\big(3[k]_q-1\big)a_k \leq 2a_1, \tag{3.1}$$

where c is an non-negative real number. We note that for a non-negative real number c the class $\mathcal{S}_q^{(c)}(Ne)$ is non-empty as the function of the form

$$f_1(z) = a_1 z - \sum_{k=2}^{\infty}\frac{2a_1}{([k]_q)^c\big(3[k]_q-1\big)}\lambda_k z^k,$$

where $a_1 > 0$, $\lambda_k > 0$ and $\sum_{k=2}^{\infty} \lambda_k \leq 1$ satisfies the inequality (3.1). Here we note that $\mathcal{S}_q^{(1)}(Ne) \equiv \mathcal{K}_q(Ne)$ and for $c = 0$, $\mathcal{S}_q^{(0)}(Ne) \equiv \mathcal{S}_q^{*}(Ne)$. Further, $\mathcal{S}_q^{(c)}(Ne) \subset \mathcal{S}_q^{(k)}(Ne)$ if $c > k \geq 0$, the containment being proper.

Let the functions of the form

$$f_i(z) = a_{1,i} - \sum_{k=2}^{\infty} a_{k,i} z^k \quad (a_{1,i} > 0, a_{k,i} \geq 0) \tag{3.2}$$

and

$$g_i(z) = b_{1,i} - \sum_{k=2}^{\infty} b_{k,i} z^k \quad (b_{1,i} > 0, b_{k,i} \geq 0) \tag{3.3}$$

be analytic in the unit disc $\mathbb{U}$.

Theorem 7. *Let the functions $f_i (i = 1, 2, \cdots, m)$ given by (3.2), belong to the class $\mathcal{S}_q^{*}(Ne)$. Then, the quasi-Hadamard product $f_1 *' f_2 *' \cdots *' f_m$ belongs to the class $\mathcal{S}_q^{(m-1)}(Ne)$.*

Proof. Here, we need to show that

$$\sum_{k=2}^{\infty} \left[\left([k]_q \right)^{m-1} \left(3[k]_q - 1 \right) \prod_{i=1}^{m} a_{k,i} \right] \leq 2 \prod_{i=1}^{m} a_{1,i}. \tag{3.4}$$

Since $f_i(z) \in \mathcal{S}_q^{*}(Ne)$, we have

$$\sum_{k=2}^{\infty} \left(3[k]_q - 1 \right) a_{k,i} \leq 2 a_{1,i} \qquad (a_{1,i} > 0, \ \ a_{k,i} \geq 0). \tag{3.5}$$

Therefore

$$a_{k,i} \leq \frac{2}{3[k]_q - 1} a_{1,i},$$

which implies

$$a_{k,i} \leq \frac{1}{[k]_q} a_{1,i}, \tag{3.6}$$

Using (3.6) for $i = 1, 2, \cdots, m-1$ and (3.5) for $i = m$, we get

$$\begin{aligned} \sum_{k=2}^{\infty} \left[\left([k]_q \right)^{m-1} \left(3[k]_q - 1 \right) \prod_{i=1}^{m} a_{k,i} \right] &= \sum_{k=2}^{\infty} \left[\left(\prod_{i=1}^{m-1} \left([k]_q a_{k,i} \right) \left(3[k]_q - 1 \right) a_{k,m} \right] \\ &\leq \left(\prod_{i=1}^{m-1} a_{1,i} \right) \sum_{k=2}^{\infty} \left(3[k]_q - 1 \right) a_{k,m} \end{aligned}$$

$$\leq 2\prod_{i=1}^{m} a_{1,i}.$$

Which completes the proof. □

Theorem 8. *Let the functions $f_i (i = 1, 2, \cdots, m)$ given by (3.2), belong to the the class $\mathcal{K}_q(Ne)$. Then, the quasi-Hadamard product $f_1 *' f_2 *' \cdots *' f_m$ belongs to the class $\mathcal{S}_q^{(2m-1)}(Ne)$.*

Proof. Since $f_i(z) \in \mathcal{K}_q(Ne)$, we have

$$\sum_{k=2}^{\infty}[k]_q\big(3[k]_q - 1\big)a_{k,i} \leq 2a_{1,i} \qquad (a_{1,i} > 0, \;\; a_{k,i} \geq 0). \tag{3.7}$$

Therefore

$$a_{k,i} \leq \frac{2}{[k]_q(3[k]_q - 1)}a_{1,i},$$

which implies

$$a_{k,i} \leq \frac{1}{([k]_q)^2}a_{1,i}, \tag{3.8}$$

Using (3.8) for $i = 1, 2, \cdots, m-1$ and (3.7) for $i = m$ in the following:

$$\begin{aligned}\sum_{k=2}^{\infty}\Big[\big([k]_q\big)^{2m-1}\big(3[k]_q - 1\big)\prod_{i=1}^{m} a_{k,i}\Big] &= \sum_{k=2}^{\infty}\Big[\{\prod_{i=1}^{m-1}([k]_q)^2 a_{k,i}\}[k]_q\big(3[k]_q - 1\big)a_{k,m}\Big] \\ &\leq \Big(\prod_{i=1}^{m-1} a_{1,i}\Big)\sum_{k=2}^{\infty}[k]_q\big(3[k]_q - 1\big)a_{k,m} \\ &\leq 2\prod_{i=1}^{m} a_{1,i}.\end{aligned}$$

which is the required condition for $f_1 *' f_2 *' \cdots *' f_m$ to be in the class $\mathcal{S}_q^{(2m-1)}(Ne)$. □

On using the similar arguments as used in the Theorem 7 and Theorem 8 we may obtain the following result:

Theorem 9. *Let the functions $f_i (i = 1, 2, \cdots, m)$ given by (3.2), belong to the the class $\mathcal{K}_q(Ne)$ and the functions $g_j (j = 1, 2, \cdots, s)$ given by (3.3) belong to the class $\mathcal{S}_q^*(Ne)$. Then, the quasi-Hadamard product $f_1 *' f_2 *' \cdots *' f_m *' g_1 *' g_2 *' \cdots *' g_s$ belongs to the class $\mathcal{S}_q^{(2m+s-1)}(Ne)$.*

4 Conclusions

In this paper, we have used q-calculus to introduce two subclasses $\mathcal{S}_q^*(Ne)$ and $\mathcal{K}_q(Ne)$ which are q-analogue of the classes studied by Kumar and Çetinkaya [14]. These classes are associated with the analytic function $Ne(z) = 1 + z - \frac{z^3}{3}$, which maps an open unit disc onto the Nephroid shaped bounded symmetric region. We studied some key issues, such as Hadamard products, the necessary and sufficient conditions and coefficient estimates of the functions belonging to the classes $\mathcal{S}_q^*(Ne)$ and $\mathcal{K}_q(Ne)$. We have also established certain results concerning the quasi-Hadamard products for the newly defined classes.

We now want to indicate, as pointed out in survey-cum-expository review paper by Srivastava ([33], p-340), the current trend of attempting to produce the so called (p,q)- derivative analogue of the known q-derivative results are nothing new but trivial and inconsequential as the additional parameter p is obviously redundant or superfluous. Also see([34] and [35]).

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Shashi Kant
Department of Mathematics, Government Dungar College,
Bikaner-334001, INDIA.
e-mail: drskant.2007@yahoo.com

Neelmani Gupta
Department of Mathematics, Government Dungar College,
Bikaner-334001, INDIA.

e-mail: neelmanigupta04@gmail.com

Sanjay Issar
Department of Mathematics, Government Dungar College,
Bikaner-334001, INDIA.
e-mail: sanjayissar@gmail.com

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