Surveys in Mathematics and its Applications

ISSN 1842-6298 (electronic), 1843-7265 (print)
Volume 20 (2025), 341 – 354

LEFSCHETZ FIXED POINT THEOREMS FOR EXTENSION TYPE SPACES WITH RESPECT TO A SELECTION MAP

Donal O'Regan

Abstract. In this paper we present fixed point theorems for a general class of maps defined on extension type spaces with respect to a map.

1 Introduction

In this paper we consider general classes of maps (which include KLU [13], $HLPY$ [14] and Scalzo [20] maps) which have a selection property and we present fixed point theorems for these maps. In particular we establish Lefschetz type fixed point theorems for extension type spaces with respect to a selection map. Our theorems improve and complement results in the literature; see [7, 9, 10, 16–19] and the references therein. We note also that our results include as special cases results for NES(compact) and $SANES$(compact) spaces.

First we describe the maps considered in this paper. Let H be the Čech homology functor with compact carriers and coefficients in the field of rational numbers K from the category of Hausdorff topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus $H(X) = \{H_q(X)\}$ (here X is a Hausdorff topological space) is a graded vector space, $H_q(X)$ being the q–dimensional Čech homology group with compact carriers of X. For a continuous map $f : X \to X$, $H(f)$ is the induced linear map $f_\star = \{f_{\star q}\}$ where $f_{\star q} : H_q(X) \to H_q(X)$. A space X is acyclic if X is nonempty, $H_q(X) = 0$ for every $q \geq 1$, and $H_0(X) \approx K$.

Let X, Y and Γ be Hausdorff topological spaces. A continuous single valued map $p : \Gamma \to X$ is called a Vietoris map (written $p : \Gamma \Rightarrow X$) if the following two conditions are satisfied:
(i). for each $x \in X$, the set $p^{-1}(x)$ is acyclic

2020 Mathematics Subject Classification: 47H10; 54H25.
Keywords: fixed points; acyclic maps; extension spaces.

(ii). p is a perfect map i.e. p is closed and for every $x \in X$ the set $p^{-1}(x)$ is nonempty and compact.

Let $\phi : X \to Y$ be a multivalued map (note for each $x \in X$ we assume $\phi(x)$ is a nonempty subset of Y). A pair (p, q) of single valued continuous maps of the form $X \overset{p}{\leftarrow} \Gamma \overset{q}{\to} Y$ is called a selected pair of ϕ (written $(p, q) \subset \phi$) if the following two conditions hold:
(i). p is a Vietoris map
and
(ii). $q\left(p^{-1}(x)\right) \subset \phi(x)$ for any $x \in X$.

Now we define the admissible maps of Gorniewicz [10]. An upper semicontinuous map $\phi : X \to 2^Y$ (nonempty subsets of Y) with compact values is said to be admissible (and we write $\phi \in Ad(X, Y)$) provided there exists a selected pair (p, q) of ϕ. An example of an admissible map is a Kakutani map. An upper semicontinuous map $\phi : X \to CK(Y)$ is said to be Kakutani (and we write $\phi \in Kak(X, Y)$); here Y is a Hausdorff topological vector space and $CK(Y)$ denotes the family of nonempty, convex, compact subsets of Y. Another example is an acyclic map which we now describe. Let X and Z be subsets of Hausdorff topological spaces and let $F : X \to K(Z)$ i.e. F has nonempty compact values. Recall a nonempty topological space is said to be a acyclic if all its reduced Čech homology groups over the rationals are trivial. Now we consider maps $F : X \to Ac(Z)$ i.e. $F : X \to K(Z)$ with acyclic values (i.e. F has nonempty acyclic compact values). We say $F \in AC(X, Z)$ (i.e. F is an acyclic map) if $F : X \to Ac(Z)$ is upper semicontinuous.

For a subset K of a topological space X, we denote by $Cov_X(K)$ the directed set of all coverings of K by open sets of X (usually we write $Cov(K) = Cov_X(K)$). Given a map $F : X \to 2^X$ and $\alpha \in Cov(X)$, a point $x \in X$ is said to be an α–fixed point of F if there exists a member $U \in \alpha$ such that $x \in U$ and $F(x) \cap U \neq \emptyset$.

Given two maps $F, G : X \to 2^Y$ and $\alpha \in Cov(Y)$, F and G are said to be α–close if for any $x \in X$ there exists $U_x \in \alpha$, $y \in F(x) \cap U_x$ and $w \in G(x) \cap U_x$. Of course, given two single valued maps $f, g : X \to Y$ and $\alpha \in Cov(Y)$, then f and g are α–close if for any $x \in X$ there exists $U_x \in \alpha$ containing both $f(x)$ and $g(x)$. We say f and g are homotopic if there is a homotopy $h_t : X \to Y$ $(t \in [0, 1])$ joining f and g.

The following fixed point result can be found in [2, 4].

Theorem 1. *Let X be a regular topological space, let $F : X \to 2^X$ be an upper semicontinuous map with closed values and suppose there exists a cofinal covering $\theta \subseteq Cov_X\left(\overline{F(X)}\right)$ such that F has an α–fixed point for every $\alpha \in \theta$. Then F has a fixed point.*

Remark 2. *From Theorem 1 in proving the existence of fixed points in uniform spaces for upper semicontinuous compact maps with closed values it suffices [3, page*

298] to prove the existence of approximate fixed points (since open covers of a compact set A admit refinements of the form $\{U[x] : x \in A\}$ where U is a member of the uniformity [12, page 199], so such refinements form a cofinal family of open covers). Note also that uniform spaces are regular (in fact completely regular [6]). Also note in Theorem 1 if F is compact valued, then the assumption that X is regular can be removed. We note here that when we apply Theorem 1 we will assume the space is uniform. Of course one could consider other appropriate spaces (like regular (Hausdorff) spaces) as well.

Let Q be a class of topological spaces. A space Y is an extension space for Q (written $Y \in ES(Q)$) if for any pair (X, K) in Q with $K \subseteq X$ closed, any continuous function $f_0 : K \to Y$ extends to a continuous function $f : X \to Y$. A space Y is an approximate extension space for Q (written $Y \in AES(Q)$) if for any $\alpha \in Cov\,(Y)$ and any pair (X, K) in Q with $K \subseteq X$ closed, and any continuous function $f_0 : K \to Y$ there exists a continuous function $f : X \to Y$ such that $f|_K$ is α–close to f_0. A space Y is a neighborhood extension space for Q (written $Y \in NES(Q)$) if $\forall X \in Q$, $\forall K \subseteq X$ closed in X, and any continuous function $f_0 : K \to Y$ there exists a continuous extension $f : U \to Y$ of f_0 over a neighborhood U of K in X. A space Y is an approximate neighborhood extension space for Q (written $Y \in ANES(Q)$) if $\forall \alpha \in Cov\,(Y)$, $\forall X \in Q$, $\forall K \subseteq X$ closed in X, and any continuous function $f_0 : K \to Y$ there exists a neighborhood U_α of K in X and a continuous function $f_\alpha : U_\alpha \to Y$ such that $f_\alpha|_K$ and f_0 are α–close. A space Y is a strongly approximate neighborhood extension space for Q (written $Y \in SANES(Q)$) if $\forall \alpha \in Cov\,(Y)$, $\forall X \in Q$, $\forall K \subseteq X$ closed in X, and any continuous function $f_0 : K \to Y$ there exists a neighborhood U_α of K in X and a continuous function $f_\alpha : U_\alpha \to Y$ such that $f_\alpha|_K$ and f_0 are α–close and homotopic.

Next we describe the maps due to Wu [21]. Let X and Y be subsets lying in Hausdorff topological vector spaces and we say $\Phi \in W(X, Y)$ if $\Phi : X \to 2^Y$ and there exists a lower semicontinuous map $\theta : X \to 2^Y$ with $\overline{co}\,(\theta(x)) \subseteq \Phi(x)$ for $x \in X$. Next we recall a selection theorem [1] (see the proof in Theorem 1.1 there) for Wu maps.

Theorem 3. *Let X be a paracompact subset of a Hausdorff topological vector space and Y a metrizable complete subset of a Hausdorff locally convex linear topological space. Suppose $\Phi \in W(X, Y)$ and let $\theta : X \to 2^Y$ be a lower semicontinuous map with $\overline{co}\,(\theta(x)) \subseteq \Phi(x)$ for $x \in X$. Then there exists a upper semicontinuous map $\Psi : X \to CK(Y)$ (collection of nonempty convex compact subsets of Y) with $\Psi(x) \subseteq \overline{co}\,(\theta(x)) \subseteq \Phi(x)$ for $x \in X$.*

Remark 4. *Let X be paracompact and Y a metrizable subset of a complete Hausdorff locally convex linear topological space E and $\Phi \in W(X, Y)$ with $\theta : X \to 2^Y$ a lower semicontinuous map and $\overline{co}\,(\theta(x)) \subseteq \Phi(x)$ for $x \in X$. Note [15] that $\overline{co}\,\theta : X \to 2^Y$*

(since $\overline{co}\,(\theta(x)) \subseteq \Phi(x) \subseteq Y$ *for* $x \in X$*) is lower semicontinuous, so from Michael's selection theorem there exists a continuous (single valued) map* $f : X \to Y$ *with* $f(x) \in \overline{co}\,(\theta(x))$ *for* $x \in X$*, so consequently* $f(x) \in \overline{co}\,(\theta(x)) \subseteq \Phi(x)$ *for* $x \in X$*.*

Let Z be a subset of a Hausdorff topological space Y_1 and W a subset of a Hausdorff topological vector space Y_2 and G a multifunction. We say $F \in HLPY(Z, W)$ [14] if W is convex and there exists a map $S : Z \to W$ (i.e. $S : Z \to P(W)$ (collection of subsets of W)) with $co\,(S(x)) \subseteq F(x)$ for $x \in Z$, $S(x) \neq \emptyset$ for each $x \in Z$ and $Z = \bigcup\{\,int\, S^{-1}(w) :\ w \in W\}$; here $S^{-1}(w) = \{z \in Z : w \in S(z)\}$ and note $S(x) \neq \emptyset$ for each $x \in Z$ is redundant since if $z \in Z$ then there exists a $w \in W$ with $z \in int\, S^{-1}(w) \subseteq S^{-1}(w)$ so $w \in S(z)$ i.e. $S(z) \neq \emptyset$. For the selection theorem below see [14].

Theorem 5. *Let* X *be a paracompact subset of a Hausdorff topological space,* Y *a convex subset of a Hausdorff topological vector space and* $F \in HLPY(X, Y)$ *(let* $S : X \to 2^Y$ *with* $co\,(S(x)) \subseteq F(x)$ *for* $x \in X$ *and* $X = \bigcup\{\,int\, S^{-1}(w) :\ w \in Y\}$*). Then there exists a continuous (single–valued) map* $f : X \to Y$ *with* $f(x) \in co\, S(x) \subseteq F(x)$ *for all* $x \in X$*.*

Remark 6. *These maps are related to the* DKT *maps in the literature and* $F \in DKT(Z, W)$ *[5] if* W *is convex and there exists a map* $S : Z \to W$ *with* $co\,(S(x)) \subseteq F(x)$ *for* $x \in Z$*,* $S(x) \neq \emptyset$ *for each* $x \in Z$ *and the fibre* $S^{-1}(w)$ *is open (in* Z*) for each* $w \in W$*. Note these maps were motivated from the Fan maps.*

Let X be a subset of a Hausdorff topological space and Y a subset of a Hausdorff topological vector space. We say $T : X \to 2^Y$ has the strong continuous inclusion property (SCIP) [13] at $x \in X$ if there exists an open set $U(x)$ in X containing x and a $F^x : U(x) \to 2^Y$ such that $F^x(w) \subseteq T(w)$ for all $w \in U(x)$ and $co\, F^x : U(x) \to 2^Y$ is compact valued and upper semicontinuous. We write $T \in KLU(X, Y)$ if T has the SCIP at every $x \in X$.

In this paper our map T will be a compact map so T has the SCIP is equivalent to T has the CIP [11].

Remark 7. *These maps contain as a special case the Scalzo maps [20] in the literature (see [13, pg 12]).*

Next we recall a selection theorem [13].

Theorem 8. *Let* X *be a paracompact subset of a Hausdorff topological space,* Y *a subset of a Hausdorff topological vector space and* $T \in KLU(X, Y)$*. Then there exists an upper semicontinuous map* $G : X \to CK(Y)$ *with* $G(w) \subseteq co\, T(w)$ *for all* $w \in X$*.*

Finally we present some preliminaries on the Lefschetz set for Ad maps needed in Section 2. Let $D(X,Y)$ be the set of all pairs $X \stackrel{p}{\Leftarrow} \Gamma \stackrel{q}{\to} Y$ where p is a Vietoris map and q is continuous. We will denote every such diagram by (p,q). Given two diagrams (p,q) and (p',q'), where $X \stackrel{p'}{\Leftarrow} \Gamma' \stackrel{q'}{\to} Y$, we write $(p,q) \sim (p',q')$ if there are maps $f : \Gamma \to \Gamma'$ and $g : \Gamma' \to \Gamma$ such that $q' \circ f = q$, $p' \circ f = p$, $q \circ g = q'$ and $p \circ g = p'$. The equivalence class of a diagram $(p,q) \in D(X,Y)$ with respect to $\sim$ is denoted by

$$\phi = \{X \stackrel{p}{\Leftarrow} \Gamma \stackrel{q}{\to} Y\} : X \to Y$$

or $\phi = [(p,q)]$ and is called a morphism from X to Y. We let $M(X,Y)$ be the set of all such morphisms. For any $\phi \in M(X,Y)$ a set $\phi(x) = q\,p^{-1}(x)$ where $\phi = [(p,q)]$ is called an image of x under a morphism ϕ.

Consider vector spaces over a field K. Let E be a vector space and $f : E \to E$ an endomorphism. Now let $N(f) = \{x \in E : \ f^{(n)}(x) = 0 \ \text{ for some } \ n\}$ where $f^{(n)}$ is the n^{th} iterate of f, and let $\tilde{E} = E \setminus N(f)$. Since $f(N(f)) \subseteq N(f)$ we have the induced endomorphism $\tilde{f} : \tilde{E} \to \tilde{E}$. We call f admissible if $dim\,\tilde{E} < \infty$; for such f we define the generalized trace $Tr(f)$ of f by putting $Tr(f) = tr(\tilde{f})$ where tr stands for the ordinary trace.

Let $f = \{f_q\} : E \to E$ be an endomorphism of degree zero of a graded vector space $E = \{E_q\}$. We call f a Leray endomorphism if (i). all f_q are admissible and (ii). almost all $\tilde{E}_q$ are trivial. For such f we define the generalized Lefschetz number $\Lambda(f)$ by

$$\Lambda(f) = \sum_q (-1)^q \, Tr\,(f_q).$$

With Čech homology functor extended to a category of morphisms we have the following well known result [10] (note the homology functor H extends over this category i.e. for a morphism

$$\phi = \{X \stackrel{p}{\Leftarrow} \Gamma \stackrel{q}{\to} Y\} : X \to Y$$

we define the induced map

$$H\,(\phi) = \phi_\star : H(X) \to H(Y)$$

by putting $\phi_\star = q_\star \circ p_\star^{-1}$). If $\phi : X \to Y$ and $\psi : Y \to Z$ are two morphisms (here X, Y and Z are Hausdorff topological spaces) then $(\psi \circ \phi)_\star = \psi_\star \circ \phi_\star$. Two morphisms $\phi, \psi \in M(X,Y)$ are homotopic (written $\phi \sim \psi$) provided there is a morphism $\chi \in M(X \times [0,1], Y)$ such that $\chi(x,0) = \phi(x)$, $\chi(x,1) = \psi(x)$ for every $x \in X$ (i.e. $\phi = \chi \circ i_0$ and $\psi = \chi \circ i_1$, where $i_0, i_1 : X \to X \times [0,1]$ are defined by $i_0(x) = (x,0)$, $i_1(x) = (x,1)$). Recall the following result [10]: If $\phi \sim \psi$ then $\phi_\star = \psi_\star$.

A map $\phi \in Ad(X,X)$ is said to be a Lefschetz map if for each selected pair $(p,q) \subset \phi$ the linear map $q_\star\, p_\star^{-1} : H(X) \to H(X)$ (the existence of $p_\star^{-1}$ follows from the Vietoris Theorem) is a Leray endomorphism. If $\phi : X \to X$ is a Lefschetz map, we define the Lefschetz set $\mathbf{\Lambda}(\phi)$ (or $\mathbf{\Lambda}_X(\phi)$) by

$$\mathbf{\Lambda}(\phi) = \left\{ \Lambda(q_\star\, p_\star^{-1}) : \ (p,q) \subset \phi \right\}.$$

A Hausdorff topological space X is said to be a Lefschetz space (for the class Ad) provided every compact $\phi \in Ad(X,X)$ is a Lefschetz map and $\mathbf{\Lambda}(\phi) \neq \{0\}$ implies ϕ has a fixed point. We will use the following result from [7, Theorem 7.2] in Section 2 (recall the Tychonoff cube T is the cartesian product of copies of the unit interval and T lies in an appropriate locally convex topological vector space E [7, 8]).

Theorem 9. *Every open subset U of T is a Lefschetz space i.e. every compact map $\Phi \in Ad(U,U)$ is a Lefschetz map and if $\mathbf{\Lambda}(\Phi) \neq \{0\}$ then Φ has a fixed point.*

Suppose U is an open subset of T. If $\Phi \in AC(U,U)$ is a compact map then from Theorem 9 we note that $\mathbf{\Lambda}(\Phi)$ is well defined and in fact here note (see [10, Section 32 and Section 40]) that $\mathbf{\Lambda}(\Phi)$ is a singleton and if (p,q) is a selected pair of Φ then $\Phi_\star = q_\star p_\star^{-1}$. Thus for acyclic maps we talk about the Lefschetz number instead of the Lefschetz set. From Theorem 9 we have the following.

Theorem 10. *Let U be an open subset of T and $\Phi \in AC(U,U)$ a compact map. Then the Lefschetz number $\mathbf{\Lambda}(\Phi)$ is well defined and if $\mathbf{\Lambda}(\Phi) \neq 0$ then Φ has a fixed point.*

We now present a result which will be needed in Section 2.

Theorem 11. *Let X, Y and Z be subsets of Hausdorff topological spaces.*

(i). Suppose $F \in AC(X,Z)$, $G \in AC(Y,X)$ and $F\,G \in Ad(Y,Z)$. Then there exists a selected pair (p,q) of $F\,G$ with $q_\star p_\star^{-1} = F_\star G_\star$.

(ii). Suppose $F \in AC(X,Z)$, $G \in AC(Y,X)$ and $F\,G \in AC(Y,Z)$. Then $(F\,G)_\star = F_\star G_\star$.

(iii). Suppose $F \in AC(X,Z)$ and $g \in C(Y,X)$ (i.e. $g : Y \to X$ is a continuous single valued map). Then $(F\,g)_\star = F_\star g_\star$.

Proof. (i). Let (p_1,q_1) be a selected pair of F (note from [10, Proposition 40.4], that $F_\star = (q_1)_\star (p_1)_\star^{-1}$) and (p_2,q_2) be a selected pair of G (note $G_\star = (q_2)_\star (p_2)_\star^{-1}$). Now [10, Proposition 40.5] guarantees that there exists a selected pair (p,q) of $F\,G$ with

$$q_\star p_\star^{-1} = (q_1)_\star (p_1)_\star^{-1} (q_2)_\star (p_2)_\star^{-1} = F_\star G_\star.$$

(ii). This is immediate from part (i) and [10, Proposition 40.4] since $q_\star p_\star^{-1} = (F\,G)_\star$ (note $F\,G \in AC(Y,Z)$ is assumed here).

(iii). This is immediate from part (ii) since $F\,g \in AC(Y,Z)$. □

Remark 12. *Theorem 11 (ii) was proved (with a different argument) in [19, Theorem 3.8] while Theorem 11 (iii) was proved in [9, Theorem 6.5].*

In Section 2 we consider two general classes of maps. Let X be a subset of a Hausdorff topological space and Y a subset of a Hausdorff topological vector space. We say $F \in HYAC(X,Y)$ if $F : X \to 2^Y$ and there exists a $\Phi \in AC(X,Y)$ with $\Phi(x) \subseteq co\,(F(x))$ for $x \in X$.

Let X be a subset of a Hausdorff topological space and Y a subset of a Hausdorff topological space. We say $F \in HYACC(X,Y)$ if $F : X \to 2^Y$ and there exists a $\Phi \in AC(X,Y)$ with $\Phi(x) \subseteq F(x)$ for $x \in X$.

Usually in the literature Lefschetz fixed point results are established for Kakutani type maps in neighborhood extension spaces. In this paper Lefschetz fixed point results are considered in new extension spaces in Hausdorff topological spaces (which include all known extension spaces in the literature [10]); in particular our extension type spaces include absolute retracts (AR's), absolute neighborhood retracts (ANR's), ES(compact) spaces, NES(compact) spaces (see Example 16), AES(compact) spaces and $ANES$(compact) spaces (see Example 23). Moreover the class of maps considered is new and this general class includes for example KLU, DKT, $HLPY$, W, Scalzo, Kakutani and acyclic maps. The proof of our results rely on (i). the fact that a compact space is homeomorphic to a closed subset of the Tychonoff cube T, (ii). the appropriate map is a Leray endomorphism, and (iii). the fact that every open subset of T is a Lefschetz space. To illustrate the above we present here two special cases of the main results in Section 2.

Theorem 13. *Let X be a compact subset of a Hausdorff topological space and $F \in AC(X,X)$. Also assume $X \in GNES$(compact) (w.r.t. F). Then the Lefschetz number $\mathbf{\Lambda}(F)$ is well defined and if $\mathbf{\Lambda}(F) \neq 0$ then F has a fixed point.*

Theorem 14. *Let X be a compact subset of a Hausdorff topological space, let X be a uniform space and let $F \in AC(X,X)$. Also assume $X \in GANES$(compact) (w.r.t. F). Then the Lefschetz number $\mathbf{\Lambda}(F)$ is well defined and if $\mathbf{\Lambda}(F) \neq 0$ then F has a fixed point.*

To conclude this section we show how the theory in this paper can be applied to generalized games (or abstract economies). A generalized game is given by $\Gamma = (X_i, A_i, P_i)_{i\in I}$ where I is the set of players (agents), X_i is a choice (or strategy) set which is a subset of a Hausdorff topological space, $A_i : X \equiv \prod_{i\in I} X_i \to 2^{X_i}$ is a constraint correspondence and $P_i : X \to 2^{X_i}$ is a preference correspondence. One is interested in finding an equilibrium point for Γ i.e. a point $x \in X$ with $x_i \in A_i(x)$ for all $i \in I$ and $A_i(x) \cap P_i(x) = \emptyset$ for some (or all) $i \in I$; here x_i is the projection of x on X_i. One way to consider this is to let $A(x) = \prod_{i\in I} A_i(x)$ and $P(x) = \prod_{i\in I} P_i(x)$ for $x \in X$.

We now consider the one person game $\Gamma = (X, A, P)$. Suppose $A : X \to Ac(X)$ is upper semicontinuous, $P : X \to 2^X$, $U = \{x \in X : A(x) \cap P(x) \neq \emptyset\}$ is open in X, $A \cap P : U \to Ac(X)$ is upper semicontinuous, and $x \notin A(x) \cap P(x)$ for $x \in U$. Let

$$F(x) = \begin{cases} A(x) \cap P(x), & x \in U \\ A(x), & x \notin U, \end{cases}$$

and we see (note $A(x) \cap P(x) \subseteq A(x)$ for $x \in U$) that $F : X \to Ac(X)$ is upper semicontinuous i.e. $F \in AC(X, X)$. Now with the assumptions in Section 2 on X we see that if $\mathbf{\Lambda}(F) \neq 0$ then there exists a $x \in X$ with $x \in F(x)$ and note if $x \in U$ then $x \in A(x) \cap P(x)$ which is a contradiction, so $x \notin U$ (i.e. $A(x) \cap P(x) = \emptyset$) and $x \in F(x) = A(x)$ i.e. x is an equilibrium of Γ.

2 Lefschetz Point Theory

Now we describe one of the spaces considered in this paper. Let X be a subset of a Hausdorff topological space and $\Phi \in AC(X, X)$.

Definition 15. *We say $X \in GNES(compact)$ (w.r.t. Φ) if for any compact subset Z of a Hausdorff topological space and $A \subseteq Z$ closed in Z, and any homeomorphism $g : X \to A$, there exists a neighborhood U of A in Z and a map $\Psi \in AC(U, X)$ with $\Psi(x) = \Phi\, g^{-1}(x)$ for $x \in A$.*

Example 16. *Let X be a subset of a Hausdorff topological space, $\Phi \in AC(X, X)$ and $X \in NES(compact)$. Then $X \in GNES(compact)$ (w.r.t. Φ).*

To see this let Z be a compact subset of a Hausdorff topological space and $A \subseteq Z$ closed in Z and let $g : X \to A$ be a homeomorphism. Then $g^{-1} : A \to X$ is continuous. Since $X \in NES(compact)$ then there exists a continuous extension $h : U \to X$ of g^{-1} over a neighborhood U of A in Z. Let $\Psi = \Phi\, h$. Note $h \in C(U, X)$ (continuous single valued) and $\Phi \in AC(X, X)$ guarantees that $\Psi = \Phi\, h \in AC(U, X)$. Also since $h|_A = g^{-1}$, for $x \in A$ we have $\Psi(x) = \Phi\, h(x) = \Phi\, g^{-1}(x)$.

Theorem 17. *Let X be a compact subset of a Hausdorff topological vector space and $F \in HYAC(X, X)$ (so in particular there exists a map $\Phi \in AC(X, X)$ with $\Phi(x) \subseteq co\,(F(x))$ for $x \in X$). Also assume $X \in GNES(compact)$ (w.r.t. Φ). Then the Lefschetz number $\mathbf{\Lambda}(\Phi)$ is well defined and if $\mathbf{\Lambda}(\Phi) \neq 0$ then Φ (so consequently $co\,F$) has a fixed point.*

Proof. Let Φ be as in the statement of Theorem 17. Let (p, q) be a selected pair of $\Phi \in AC(X, X)$ and note $\Phi_\star = q_\star p_\star^{-1}$. Note [8] every compact space is homeomorphic to a closed subset of the Tychonoff cube T, so as a result X can be embedded as a closed subset $K^\star$ of T and let $s : X \to K^\star$ be a homeomorphism. Since $X \in GNES(\text{compact})$ (w.r.t. Φ) then there exists a neighborhood U of $K^\star$ in T and a map $\Psi \in AC(U, X)$ with $\Psi(x) = \Phi\, s^{-1}(x)$ for $x \in K^\star$. Let (p', q') be a selected pair of Ψ and note $\Psi_\star = (q')_\star (p')_\star^{-1}$. Let $j_U : K^\star \hookrightarrow U$ be the natural embedding and $G = j_U\, s\, \Psi$. Note $G \in Ad(U,U)$ is a compact map and $(p', j_U\, s\, q')$ is a selected pair of G (as an aside note $G \in AC(U,U)$ since the homeomorphic image of an acyclic set is acyclic). Now Theorem 9 guarantees that $(j_U\, s\, q')_\star (p')_\star^{-1}$ is a Leray endomorphism. Also note since $\Psi\, j_U = \Phi s^{-1}$ then from Theorem 11 we have $\Psi_\star (j_U)_\star = \Phi_\star (s^{-1})_\star$ so

$$(q')_\star (p')_\star^{-1} (j_U)_\star s_\star = \Psi_\star (j_U)_\star s_\star = \Phi_\star (s^{-1})_\star s_\star = \Phi_\star = q_\star p_\star^{-1}$$

and

$$(j_U)_\star s_\star (q')_\star (p')_\star^{-1} = (j_U\, s\, q')_\star (p')_\star^{-1}.$$

Now [8, pp 214 (see (1.3))] (here we take $E' = H(U)$, $E'' = H(X)$, $u = (q')_\star (p')_\star^{-1}$, $v = (j_U)_\star s_\star$, $f' = (j_U\, s\, q')_\star (p')_\star^{-1}$, $f'' = q_\star p_\star^{-1}$ and note

$$\begin{aligned} u\, f' &= (q')_\star (p')_\star^{-1} (j_U\, s\, q')_\star (p')_\star^{-1} \\ &= (q')_\star (p')_\star^{-1} (j_U)_\star s_\star (q')_\star (p')_\star^{-1} = q_\star p_\star^{-1} (q')_\star (p')_\star^{-1} = f''\, u) \end{aligned}$$

guarantees that $q_\star p_\star^{-1} (= \Phi_\star)$ is a Leray endomorphism (so $\mathbf{\Lambda}(\Phi)$ is well defined) and $\mathbf{\Lambda}(q_\star p_\star^{-1}) = \mathbf{\Lambda}((j_U\, s\, q')_\star (p')_\star^{-1})$.

Now suppose $\mathbf{\Lambda}(\Phi) \neq 0$. Let (p,q) be a selected pair of Φ and let s, Ψ, p', q', j_U and G be as described above and note $\mathbf{\Lambda}((j_U\, s\, q')_\star (p')_\star^{-1}) = \mathbf{\Lambda}(q_\star p_\star^{-1}) = \mathbf{\Lambda}(\Phi) \neq 0$. Thus $\mathbf{\Lambda}(G) \neq \{0\}$ (as an aside note $\mathbf{\Lambda}(G) \neq 0$). Now Theorem 9 guarantees that there exists a $x \in U$ with $x \in G(x) = j_U\, s\, \Psi(x)$. Then there exists a $y \in \Psi(x)$ with $x = j_U\, s(y)$. Since $s(y) \in K^\star$ then $\Psi(x) = \Phi\, s^{-1}(x) = \Phi(y)$. As a result $y \in \Psi(x) = \Phi(y)$, so Φ (and consequently $co\, F$) has a fixed point. □

The analogue of Theorem 17 for $HYACC(X,X)$ maps is now immediate.

Theorem 18. *Let X be a compact subset of a Hausdorff topological space and $F \in HYACC(X,X)$ (so in particular there exists a map $\Phi \in AC(X,X)$ with $\Phi(x) \subseteq F(x)$ for $x \in X$). Also assume $X \in GNES(compact)$ (w.r.t. Φ). Then the Lefschetz number $\mathbf{\Lambda}(\Phi)$ is well defined and if $\mathbf{\Lambda}(\Phi) \neq 0$ then Φ (so consequently F) has a fixed point.*

A special case of Theorem 18 (with $\Phi = F$) is the following.

Theorem 19. *Let X be a compact subset of a Hausdorff topological space and $F \in AC(X,X)$. Also assume $X \in GNES(compact)$ (w.r.t. F). Then the Lefschetz number $\mathbf{\Lambda}(F)$ is well defined and if $\mathbf{\Lambda}(F) \neq 0$ then F has a fixed point.*

Remark 20. *In [18, Theorem 2.4] we note that the sAd maps there should be replaced by the AC maps (in [18, Theorem 2.4] one point was inadvertently overlooked in the proof).*

Let X be a subset of a Hausdorff topological space and $\Phi \in AC(X,X)$.

Definition 21. *We say $X \in GANES(compact)$ (w.r.t. Φ) if for any compact subset Z of a Hausdorff topological space and $A \subseteq Z$ closed in Z, any homeomorphism $g : X \to A$, and any $\alpha \in Cov_X(X)$, there exists a neighborhood U_α of A in Z and a map $\Psi_\alpha \in AC(U_\alpha, X)$ with $(\Psi_\alpha)_\star (j_{U_\alpha})_\star = \Phi_\star(g^{-1})_\star$ (here $j_{U_\alpha} : A \hookrightarrow U_\alpha$ is the natural embedding) and such that if $x \in A$ with $x \in g\,\Psi_\alpha(x)$ then Φ has an α–fixed point.*

Remark 22. *(i). In Definition 21 one can think of $GANES(compact)$ (w.r.t. Φ) as an approximative $GNES(compact)$ (w.r.t. Φ); to see this see (ii) below and also Remark 24.*

(ii). In Definition 21 note $\Psi_\alpha\, j_{U_\alpha} \in AC(A,X)$ and $\Phi\, g^{-1} \in AC(A,X)$ so from Theorem 11 (iii) we have $(\Psi_\alpha\, j_{U_\alpha})_\star = (\Psi_\alpha)_\star (j_{U_\alpha})_\star$ and $(\Phi\, g^{-1})_\star = \Phi_\star(g^{-1})_\star$. Now if $\Psi_\alpha\, j_{U_\alpha} \sim \Phi\, g^{-1}$ (in the sense of [10, Definition 32.5] (see also [9]) i.e. an acyclic homotopy) then [10, Proposition 32.6] (see also [7]) guarantees that $(\Psi_\alpha\, j_{U_\alpha})_\star = (\Phi\, g^{-1})_\star$ and so $(\Psi_\alpha)_\star (j_{U_\alpha})_\star = (\Psi_\alpha\, j_{U_\alpha})_\star = (\Phi\, g^{-1})_\star = \Phi_\star(g^{-1})_\star$. For other comments see Remark 24.

Example 23. *Let X be a subset of a Hausdorff topological space, $\Phi \in AC(X,X)$ and $X \in SANES(compact)$. Then $X \in GANES(compact)$ (w.r.t. Φ).*

To see this let Z be a compact subset of a Hausdorff topological space, $A \subseteq Z$ closed in Z, $\alpha \in Cov_X(X)$ and let $g : X \to A$ be a homeomorphism. Note $g^{-1} : A \to X$ is continuous. Now since $X \in SANES(compact)$ there exists a neighborhood U_α of A in Z and a continuous function $h_\alpha : U_\alpha \to X$ such that $h_\alpha|_A$ and g^{-1} are α–close and homotopic. Note $h_\alpha j_{U_\alpha} g : X \to X$ and $i : X \to X$ are α–close (here $j_{U_\alpha} : A \hookrightarrow U_\alpha$ is the natural embedding). To see this let $x \in X$. Now let $y = j_{U_\alpha} g(x)$ so $y \in A$. Then there exists a $V \in \alpha$ with $g^{-1}(y) \in V$ and $h_\alpha(y) \in V$ i.e. $x \in V$ and $h_\alpha j_{U_\alpha} g(x) \in V$. Also note $h_\alpha j_{U_\alpha} g : X \to X$ and $i : X \to X$ are homotopic. To see this note since $h_\alpha|_A$ and g^{-1} are homotopic then there exists a continuous map $\lambda : A \times [0,1] \to X$ with $\lambda(x,0) = h_\alpha(x)$ and $\lambda(x,1) = g^{-1}(x)$ and note if $\eta(x,t) = \lambda(g(x),t)$ then $\eta : X \times [0,1] \to X$ with $\eta(x,0) = h_\alpha j_{U_\alpha} g(x)$ and $\eta(x,1) = g^{-1} g(x) = x$.

Let $\Psi_\alpha = \Phi\, h_\alpha$ and note $\Psi_\alpha \in AC(U_\alpha, X)$. Note from Theorem 11 (iii) that $(\Psi_\alpha)_\star = \Phi_\star (h_\alpha)_\star$. Also note since $h_\alpha j_{U_\alpha} g$ and i are homotopic then from [10] we have $(h_\alpha)_\star (j_{U_\alpha})_\star g_\star = i_\star$ and so

$$(\Psi_\alpha)_\star (j_{U_\alpha})_\star g_\star = \Phi_\star (h_\alpha)_\star (j_{U_\alpha})_\star g_\star = \Phi_\star,$$

i.e. $(\Psi_\alpha)_\star (j_{U_\alpha})_\star = \Phi_\star (g^{-1})_\star$.

Now suppose $x \in A$ and $x \in g\,\Psi_\alpha(x)$. Then $x \in g\,\Phi\,h_\alpha(x)$. Let $y = h_\alpha(x)$ so $y \in h_\alpha g\,\Phi(y)$ i.e. $y = h_\alpha g(w)$ for some $w \in \Phi(y)$. Now since $h_\alpha g : X \to X$ and $i : X \to X$ are α–close then there exists a $U \in \alpha$ with $h_\alpha g(w) \in U$ and $w \in U$ i.e. $y (= h_\alpha g(w)) \in U$ and $w \in U$. Thus $y \in U$ and $\Phi(y) \cap U \neq \emptyset$ (since $w \in U$ and $w \in \Phi(y)$) i.e. Φ has an α–fixed point.

Remark 24. *In Definition 21 if we assume (a). for each $x \in A$ there exists a $U \in \alpha$ with $\Psi_\alpha(x) \subseteq U$ and $\Phi\, g^{-1}(x) \cap U \neq \emptyset$, then we immediately have (b). if $x \in A$ with $x \in g\,\Psi_\alpha(x)$ then Φ has an α–fixed point.*

To see this let $x \in A$ with $x \in g\,\Psi_\alpha(x)$. Then there exists a $y \in \Psi_\alpha(x)$ with $x = g(y)$ and note $g(y) \in A$ so $x \in A$. Now (a) implies that there exists a $U \in \alpha$ with $\Psi_\alpha(x) \subseteq U$ and $\Phi\, g^{-1}(x) \cap U \neq \emptyset$. Thus since $y \in \Psi_\alpha(x)$ we have $y \in U$ and $\Phi(y) \cap U \neq \emptyset$ (recall $g^{-1}(x) = y$), so Φ has an α–fixed point.

In fact in the proof of the above we just need a condition to guarantee "for each $x \in A$ and each $y \in X$ with $y \in \Psi_\alpha(x)$ and $x = g(y)$ there exists a $U \in \alpha$ with $y \in U$ and $\Phi\, g^{-1}(x) \cap U \neq \emptyset$".

(ii). In (i) if $\Phi = \phi$ and $\Psi_\alpha = \psi_\alpha$ are single valued maps then (a) reads "ψ_α and $\phi\, g^{-1}$ are α–close".

Theorem 25. *Let X be a compact subset of a Hausdorff topological vector space and $F \in HYAC(X,X)$ (so in particular there exists a map $\Phi \in AC(X,X)$ with $\Phi(x) \subseteq co\,(F(x))$ for $x \in X$). Also assume $X \in GANES(compact)$ (w.r.t. Φ). Then the Lefschetz number $\mathbf{\Lambda}(\Phi)$ is well defined and if $\mathbf{\Lambda}(\Phi) \neq 0$ then Φ (so consequently $co\,F$) has a fixed point.*

Proof. Let Φ be as in the statement of Theorem 25. Let $\alpha \in Cov_X(X)$ and (p,q) be a selected pair of $\Phi \in AC(X,X)$ so $\Phi_\star = q_\star p_\star^{-1}$. Note X can be embedded as a closed subset $K^\star$ of T and let $s : X \to K^\star$ be a homeomorphism. Since $X \in GANES(\text{compact})$ (w.r.t. Φ) then there exists a neighborhood U_α of $K^\star$ in T and a map $\Psi_\alpha \in AC(U_\alpha, X)$ with $(\Psi_\alpha)_\star (j_{U_\alpha})_\star = \Phi_\star (s^{-1})_\star$ (here $j_{U_\alpha} : K^\star \hookrightarrow U_\alpha$ is the natural embedding) and such that if $x \in K^\star$ with $x \in s\,\Psi_\alpha(x)$ then Φ has an α–fixed point. Let (p'_α, q'_α) be a selected pair of Ψ_α and note $(\Psi_\alpha)_\star = (q'_\alpha)_\star (p'_\alpha)_\star^{-1}$. Let $G_\alpha = j_{U_\alpha}\, s\,\Psi_\alpha$ and note $G_\alpha \in Ad(U_\alpha, U\alpha)$ is a compact map and $(p'_\alpha, j_{U_\alpha}\, s\, q'_\alpha)$ is a selected pair of G_α (as an aside note $G_\alpha \in AC(U_\alpha, U_\alpha)$ since the homeomorphic image of an acyclic set is acyclic). Now Theorem 9 guarantees that $(j_{U_\alpha}\, s\, q'_\alpha)_\star (p'_\alpha)_\star^{-1}$ is a Leray endomorphism. Also since $(\Psi_\alpha)_\star (j_{U_\alpha})_\star = \Phi_\star (s^{-1})_\star$ we have

$$(q'_\alpha)_\star (p'_\alpha)_\star^{-1} (j_{U_\alpha})_\star s_\star = (\Psi_\alpha)_\star (j_{U_\alpha})_\star s_\star = \Phi_\star (s^{-1})_\star s_\star = \Phi_\star = q_\star p_\star^{-1}$$

and

$$(j_{U_\alpha})_\star s_\star (q'_\alpha)_\star (p'_\alpha)_\star^{-1} = (j_{U_\alpha}\, s\, q'_\alpha)_\star (p'_\alpha)_\star^{-1}.$$

Now [8, pp 214 (see (1.3))] (here we have $E' = H(U_\alpha)$, $E'' = H(X)$, $u = (q'_\alpha)_\star (p'_\alpha)_\star^{-1}$, $v = (j_{U_\alpha})_\star s_\star$, $f' = (j_{U_\alpha}\, s\, q'_\alpha)_\star (p'_\alpha)_\star^{-1}$, $f'' = q_\star p_\star^{-1}$ and note

$$\begin{aligned} u\, f' &= (q'_\alpha)_\star (p'_\alpha)_\star^{-1} (j_{U_\alpha})_\star s_\star (q'_\alpha)_\star (p'_\alpha)_\star^{-1} \\ &= q_\star p_\star^{-1} (q'_\alpha)_\star (p'_\alpha)_\star^{-1} = f''\, u) \end{aligned}$$

guarantees that $q_\star p_\star^{-1}\,(= \Phi_\star)$ is a Leray endomorphism (so $\mathbf{\Lambda}\,(\Phi)$ is well defined) and $\mathbf{\Lambda}\,(q_\star p_\star^{-1}) = \mathbf{\Lambda}\,((j_{U_\alpha}\, s\, q'_\alpha)_\star (p'_\alpha)_\star^{-1})$.

Now suppose $\mathbf{\Lambda}(\Phi) \neq 0$. Let $\alpha \in Cov_X(X)$ and let (p, q) be a selected pair of Φ. Now let s, Ψ_α, p'_α, q'_α, j_{U_α} and G_α be as described above and note we have $\mathbf{\Lambda}((j_{U_\alpha}\, s\, q'_\alpha)_\star (p'_\alpha)_\star^{-1}) = \mathbf{\Lambda}(q_\star p_\star^{-1}) = \mathbf{\Lambda}(\Phi) \neq 0$. Thus $\mathbf{\Lambda}(G_\alpha) \neq \{0\}$ (as an aside note $\mathbf{\Lambda}(G_\alpha) \neq 0$). Now Theorem 9 guarantees that there exists a $x \in U_\alpha$ with $x \in G_\alpha(x) = j_{U_\alpha}\, s\, \Psi_\alpha(x)$. Note $s\Psi_\alpha(x) \subseteq K^\star$ so $x \in K^\star$ with $x \in s\, \Psi_\alpha(x)$. From the above Φ has an $\alpha-$ fixed point (for each $\alpha \in Cov_X(X)$). Now Theorem 1 and Remark 2 (note Hausdorff topological vector spaces are uniform spaces) guarantee that Φ (so consequently $co\, F$) has a fixed point. □

The analogue of Theorem 25 for $HYACC(X, X)$ maps is now immediate.

Theorem 26. *Let X be a compact subset of a Hausdorff topological vector, let X be a uniform space and let $F \in HYACC(X, X)$ (so in particular there exists a map $\Phi \in AC(X, X)$ with $\Phi(x) \subseteq F(x)$ for $x \in X$). Also assume $X \in GANES(compact)$ (w.r.t. Φ). Then the Lefschetz number $\mathbf{\Lambda}(\Phi)$ is well defined and if $\mathbf{\Lambda}(\Phi) \neq 0$ then Φ (so consequently F) has a fixed point.*

A special case of Theorem 26 (with $\Phi = F$) is the following.

Theorem 27. *Let X be a compact subset of a Hausdorff topological space, let X be a uniform space and let $F \in AC(X, X)$. Also assume $X \in GANES(compact)$ (w.r.t. F). Then the Lefschetz number $\mathbf{\Lambda}(F)$ is well defined and if $\mathbf{\Lambda}(F) \neq 0$ then F has a fixed point.*

References

[1] R.P. Agarwal, D.O'Regan, *Fixed point theory for maps with lower semicontinuous selections and equilibrium theory for abstract economies*, J. Nonlinear Convex Anal. **2**(2001), 31–46. MR1828157. Zbl 1007.47020.

[2] H. Ben-El-Mechaiekh, *The coincidence problem for compositions of set valued maps*, Bull. Austral. Math. Soc. **41**(1990), 421–434. MR1071044. Zbl 0688.54030.

[3] H. Ben-El-Mechaiekh, *Spaces and maps approximation and fixed points*, J. Comput. Appl. Math. **113**(2000), 283–308. MR1735830. Zbl 0949.54059.

[4] H. Ben-El-Mechaiekh, P. Deguire, *Approachability and fixed point for nonconvex set-valued maps*, J. Math. Anal. Appl. **170**(1992), 477–500. MR1188567. Zbl 0762.54033.

[5] X.P. Ding, W.K. Kim, K.K. Tan, *A selection theorem and its applications*, Bull. Australian Math. Soc. **46**(1992), 205–212. MR1183778. Zbl 0762.47030.

[6] R. Engelking, *General Topology*, Heldermann Verlag, Berlin, 1989. MR1039321. Zbl 0684.54001.

[7] G. Fournier, L. Gorniewicz, *The Lefschetz fixed point theorem for multi–valued maps of non metrizable spaces*, Fundamenta Math. **92**(1976), 213-222. MR0420600. Zbl 0342.55006.

[8] G. Fournier, A. Granas, *The Lefschetz fixed point theorem for some classes of non metrizable spaces*, J. Math. Pures Appl. **52**(1973), 271–283. MR0339116. Zbl 0294.54034.

[9] L. Gorniewicz , A. Granas, *Fixed point theorems for multi–valued mappings of the absolute neighborhood retracts*, J. Math. Pures et Appl. **49**(1970), 257–271. MR0285004. Zbl 0203.25203.

[10] L. Gorniewicz, *Topological fixed point theory of multivalued mappings*, Kluwer Acad. Publishers, Dordrecht, 1999. MR2238622. Zbl 0937.55001.

[11] W. He, N.C. Yannelis, *Equilibria with discontinuous preferences: new fixed point theorems*, J. Math. Anal. Appl. **450**(2017), 1421–1433. MR3639109. Zbl 1510.91007.

[12] J.L. Kelley, *General Topology*, D.Van Nostrand, New York, 1955. MR0070144. Zbl 0066.16604.

[13] M. A. Khan, R.P. McLean, M. Uyanik, *On equilibria in constrained generalized games with the weak continuous inclusion property*, J. Math. Anal. Appl. **537**(2024), Art. No. 128258, 19pp. MR4717711. Zbl 1536.91014.

[14] L.J. Lin, S. Park, Z.T. Yu, *Remarks on fixed points, maximal elements and equilibria of generalized games*, J. Math. Anal. Appl. **233**(1999), 581–596. MR1689602. Zbl 0949.91004.

[15] E. Michael, *Continuous selections I*, Ann. of Math. **63**(1956), 361–382. MR0077107. Zbl 0071.15902.

[16] D. O'Regan, *A note on maps with upper semicontinuous selections on extension type spaces*, The Journal of Analysis. To appear.

[17] D. O'Regan, *Fixed point theory for compact absorbing contractions in extension type spaces*, CUBO, A Mathematical Journal. **12**(2010), 199-215. MR2724888. Zbl 1218.54045.

[18] D.O'Regan, *Fixed point theory for extension type maps in topological spaces*, Applicable Analysis. **88**(2009), 301–308. MR2536035. Zbl 1218.54044.

[19] M.J. Powers, *Multi–valued mappings and Lefschetz fixed point theorems*, Proc. Camb. Phil. Soc. **68**(1970), 619–630. MR0264657. Zbl 0201.25001.

[20] V. Scalzo, *Existence of doubly strong equilibria in generalized games and quasi–Ky Fan minimax inequalities*, J. Math. Anal. Appl. **514**(2022), Art. No. 126258, 11pp. MR4412223. Zbl 1492.91023.

[21] X. Wu, *A new fixed point theorem and its applications*, Proc. Amer. Math. Soc. **125**(1997), 1779–1783. MR1397000. Zbl 0871.47038.

Donal O'Regan
School of Mathematical and Statistical Sciences,
University of Galway,
Galway, Ireland.
e-mail: donal.oregan@nuigalway.ie
orchid: https://orcid.org/0000-0002-4096-1469

License

Received: September 01, 2025; Accepted: December 18, 2025; Published: December 21, 2025.