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GEOMETRIC FEATURES OF NORMAL CURVES IN EUCLIDEAN SPACES

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Abstract. A normal curve in the Euclidean space $\mathbb{E}^3$ is defined as a curve whose position vector lies entirely within the normal plane, ensuring orthogonality at each point to the tangent vector. In this article, we examine normal curves in $\mathbb{E}^3$ and $\mathbb{E}^4$. For $\mathbb{E}^3$, we derive a differential equation relating curvature and torsion to characterize normal curves. We also demonstrate that for unit-speed normal curves, the distance function remains constant, and there exist specific relationships between the curvatures and different components of the curve. In $\mathbb{E}^4$, we extend these results by characterizing normal curves in terms of the curvatures k_1, k_2, and k_3, establishing conditions for constant curvatures, and analyzing their normal and binormal components.

1 Introduction

Curve theory in differential geometry is a well-studied topic in mathematics and physics, with numerous applications in computer science. This theory is rich with remarkable results and still presents many unanswered questions. In animation, curves define the paths along which objects move, and various curve models are used to interpolate these paths and ensure smooth motion. In geometry, curves represent smooth and continuous paths in space. Generally, a curve in space is defined as a continuous function that maps an interval of real numbers to points in space. It can be thought of as a path traced by a point moving through space. During this motion, several vectors namely the tangent vector, the binormal vector, and the normal vector are associated with the curve, providing insight into its geometric behavior. A detailed discussion of these vectors is provided in the preliminary section of this paper.

The relationship between the position vectors of a curve and the associated vectors leads to three distinct types of curves: rectifying curves, normal curves and

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osculating curves. A rectifying curve is characterized by position vectors that lie in the orthogonal complement of the normal vector. A normal curve is characterized by position vectors that are orthogonal to the tangent vector. An osculating curve has position vectors that lie in the orthogonal complement of the binormal vector. For more information on these curves, one can refer to [6, 11].

Chen [3] explored the conditions under which the position vector of a space curve lies within its rectifying plane, laying the groundwork for understanding these curves. Later on Chen and Dillen [4] expanded on this work by examining rectifying curves that act as centrodes and extremal curves, which provided a deeper insight into their geometric properties. Additionally, Deshmukh et al.[5] made significant contributions by examining the unique characteristics and applications of rectifying curves in $\mathbb{E}^3$. Extending these ideas to higher dimensions, Ílarslan and Nešović [7, 9] explored the behavior of rectifying and osculating curves in $\mathbb{E}^4$, providing a better understanding of how curvature and torsion can be understood in higher dimensions.

The authors of [2, 14, 13, 1, 15, 8], have studied curves by restricting their position vectors to the rectifying, osculating, and normal plane on a surface and obtained their characterization under isometry and conformal maps of smooth surfaces. Their work, along with studies by Ílarslan and Nešović, who explored rectifying curves in $\mathbb{E}^4$, provides a comprehensive background for understanding these curves in higher-dimensional spaces. Lone[10] studied the geometric properties of normal curves subjected to conformal transformations in $\mathbb{E}^3$. This paper builds on these foundational studies by focusing on normal curves in both $\mathbb{E}^3$ and $\mathbb{E}^4$, presenting new findings on their geometric properties and characteristics.

This article is organized as follows. In Section 2, we discuss the basic notation used throughout the article, along with essential definitions and concepts necessary for understanding the subsequent work. Sections 3 and 4 are dedicated to the study of normal curves in $\mathbb{E}^3$ and $\mathbb{E}^4$, respectively. Finally, the conclusion and future work of this article is presented in Section 5.

2 Preliminary

Let $\mathbb{E}^n$ be the n-dimensional Euclidean space, particularly, when $n = 3$, it represents the 3-dimensional Euclidean space, and when $n = 4$, it represents the 4-dimensional Euclidean space. If $u = (a_1, a_2, \ldots, a_n)$ and $v = (b_1, b_2, \ldots, b_n)$ are two vectors in $\mathbb{E}^n$, then the inner product of these vectors in $\mathbb{E}^n$ is given by

$$\langle u, v\rangle = a_1b_1 + a_2b_2 + \ldots + a_nb_n.$$

And a norm of any vector say, $u = (a_1, a_2, \ldots, a_n)$ in $\mathbb{E}^n$, is given as

$$||u|| = \sqrt{\langle u, u\rangle} = a_1^2 + a_2^2 + \ldots + a_n^2.$$

A smooth curve ξ parameterized by arc-length "w" in $\mathbb{E}^n$ is said to be of a unit-speed if the norm of their velocity vector is constant and is equal to 1, i.e., $||\xi'|| = 1$, where '$\prime$' denotes the derivative along the parameter of arc length.

Consider a unit-speed curve $\xi : I \to \mathbb{E}^3$ parameterized by arc-length w in $\mathbb{E}^3$. The position vectors of this curve are associated with three unit vectors, namely, the tangent vector, the normal vector, and the binormal vector. The tangent vector $T(w)$ indicates in the direction of the curve's motion. The normal vector $N(w)$ is perpendicular to the tangent vector and lies in the plane of curvature of the curve. The binormal vector $B(w)$ is orthogonal to both the tangent and the normal vectors. These three vectors constitute an orthogonal frame in $\mathbb{E}^3$ known as the Serret-Frenet frame, in which these vectors are related by the Serret-Frenet equations [12, 16], given as:

$$\begin{aligned} \frac{dT}{dw} &= kN, \\ \frac{dN}{dw} &= -kT + \tau B, \\ \frac{dB}{dw} &= -\tau N, \end{aligned} \tag{2.1}$$

where k is the curvature and τ is the torsion of the curve.

Whereas in $\mathbb{E}^4$, the position vectors of a curve $\xi : I \to \mathbb{E}^4$ parameterized by arc parameter "w" are associated with four unit vectors: the tangent vector $T(w)$, the normal vector $N(w)$, the first binormal vector $B_1(w)$, and the second binormal vector $B_2(w)$. Here, $T(w)$ indicates the direction of the curve's motion, $N(w)$ is perpendicular to $T(w)$ and lies in the plane of curvature, and $B_1(w)$ and $B_2(w)$ both are orthogonal to both $T(w)$ and $N(w)$. These four vectors form an orthonormal frame in $\mathbb{E}^4$ known as the Serret-Frenet frame. The Serret-Frenet formula in $\mathbb{E}^4$ is given as:

$$\begin{aligned} T' &= \kappa_1 N, \\ N' &= -\kappa_1 T + \kappa_2 B_1, \\ B_1' &= -\kappa_2 N + \kappa_3 B_2, \\ B_2' &= -\kappa_3 B_1, \end{aligned} \tag{2.2}$$

where k_1, k_2, and k_3 are the curvatures of the curve, respectively known as the first, second, and third curvatures of the curve.

3 Characterizations of normal curve in $\mathbb{E}^3$

A normal curve in $\mathbb{E}^3$ is characterized by position vectors that are orthogonal to the tangent vector, i.e., $\langle \xi(w), T(w)\rangle = 0$. The orthogonal complement of a tangent vector in $\mathbb{E}^3$ is given as

$$T^{\perp}(w) = \{v \in \mathbb{E}^3 | \langle v, T(w)\rangle = 0\}.$$

Thus a complement of a tangent vector in $\mathbb{E}^3$ is a plane normal to the tangent vector and spanned by normal and the binormal vector known as the normal plane.
Hence, a normal vector is also defined as a curve whose position vectors are continuously moving in the normal plane and must satisfy the equation

$$\xi(w) = \alpha(w)N(w) + \beta(w)B(w). \tag{3.1}$$

Theorem 1. *Let $\xi : I \to \mathbb{E}^3$ be a unit-speed curve. Then $\xi(w)$ is a normal curve if and only if its torsion $\tau(w)$ and curvature $k(w)$ satisfy the following differential equation:*

$$\left(\frac{1}{\tau}\frac{k'(w)}{k^2(w)}\right)' - \frac{1}{k(w)}\tau = 0.$$

Proof. Let $\xi : I \to \mathbb{E}^3$ be a normal curve. Then, by definition, it must satisfy equation (3.1).
On differentiating (3.1) with respect to "w", we obtain

$$\xi'(w) = \alpha'(w)N(w) + \alpha(w)N'(w) + \beta'(w)B(w) + \beta(w)B'(w). \tag{3.2}$$

From Serret-Frenet equations (2.1) in $\mathbb{E}^3$, we get

$$\frac{dN}{dw} = -kT + \tau B, \quad \textit{and} \quad \frac{dB}{dw} = -\tau N.$$

Therefore, equation (3.2) becomes

$$\begin{aligned} \xi'(w) &= \alpha'(w)N(w) + \alpha(w)(-kT(w) + \tau B(w)) + \beta'(w)B(w) + \beta(w)(-\tau N(w)). \\ \Rightarrow T(w) &= -\alpha(w)kT(w) + (\alpha'(w) - \beta(w)\tau)N(w) + (\alpha(w)\tau + \beta'(w))B(w). \end{aligned}$$

Equating both sides, we get the following expressions

$$-\alpha(w)k(w) = 1, \tag{3.3}$$

$$\alpha'(w) - \beta(w)\tau = 0, \tag{3.4}$$

$$\alpha(w)\tau + \beta'(w) = 0. \tag{3.5}$$

By using these three equations, we can find the value of $\alpha(w)$ and $\beta(w)$.
From (3.3) and (3.4), we get

$$\alpha(w) = -\frac{1}{k(w)}, \tag{3.6}$$

and

$$\begin{aligned} \beta(w)\tau &= \alpha'(w), \\ \Rightarrow \beta(w) &= \frac{1}{\tau}\frac{k'(w)}{k^2(w)}. \end{aligned} \tag{3.7}$$

By using the values of $\alpha(w)$ and $\beta(w)$ (from (3.6) and (3.7)) in (3.5), we get a relation between curvature and torsion of a normal curve ξ, which is

$$\left(\frac{1}{\tau}\frac{k'(w)}{k^2(w)}\right)' - \frac{1}{k(w)}\tau = 0. \tag{3.8}$$

Thus, it is clear from the above that if a unit-speed curve ξ is a normal curve, then their curvature and torsion must satisfy the relation (3.8).

Conversely, we need to show that ξ is a normal curve, assuming that the equation (3.8) must be satisfied by the curvature k and the torsion τ for an arbitrary unit-speed curve ξ in $\mathbb{E}^3$.
Consider the vector $\chi \in \mathbb{E}^3$ defined by

$$\chi(w) = \xi(w) + \frac{1}{k(w)}N(w) - \frac{k'(w)}{\tau k^2(w)}B(w). \tag{3.9}$$

Differentiating with respect to w, we get

$$\begin{aligned} \chi'(w) &= \xi'(w) + \left(\frac{1}{k(w)}\right)' N(w) + \frac{1}{k(w)}N'(w) - \left(\frac{k'(w)}{\tau k^2(w)}\right)' B(w) \\ &\quad - \frac{k'(w)}{\tau k^2(w)}B'(w), \\ &= T(w) + \left(\frac{1}{k(w)}\right)' N(w) + -T(w) + \frac{\tau}{k(w)}B(w) - \left(\frac{k'(w)}{\tau k^2(w)}\right)' B(w) \\ &\quad - \frac{k'(w)}{k^2(w)}(N(w)), \\ &= 0. \end{aligned}$$

Therefore, the vector χ taken above is constant. This implies that

$$\langle \chi(w), \chi(w) \rangle = C, \quad \text{for some constant } C \in \mathbb{R}.$$

Differentiating with respect to 'w', we get

$$\langle \chi(w), T(w) \rangle = 0.$$

From equation (3.9), we can write

$$\begin{aligned}\langle\chi(w),T(w)\rangle &= \langle\xi(w),T(w)\rangle+\frac{1}{k(w)}\langle N(w),T(w)\rangle-\frac{k'(w)}{\tau k^2(w)}\langle B(w),T(w)\rangle,\\ \Rightarrow\langle\xi(w),T(w)\rangle &= 0.\end{aligned}$$

This indicates that ξ is a normal curve. □

This theorem defines the relationship between the curvature and torsion of the normal curve. If the curvature of the normal curve is a non-zero constant, then the above differential equation implies that the torsion of the curve ξ must be zero, i.e., $\tau = 0$. This implies that, in this case, a normal curve forms a circle within the normal plane. Therefore, we can deduce the following corollary from Theorem 1.

Corollary 2. *A normal curve ξ with a non-zero constant curvature in $\mathbb{E}^3$ is congruent to a circle in the normal plane.*

Corollary 3. *A helix in $\mathbb{E}^3$ can never be a normal curve.*

Proof. Let $\xi : I \to \mathbb{E}^3$ represents a helix in $\mathbb{E}^3$. We claim that it can never be a normal curve. If possible, let us suppose that the helix ξ is a normal curve in $\mathbb{E}^3$, then it must satisfy equation (3.8). Since ξ represents a helix in $\mathbb{E}^3$, and according to the property of a helix, it must have constant curvature $k \neq 0$ and constant torsion $\tau \neq 0$, which easily implies a contradiction when we substitute these non-zero constant values of k and τ in (3.8). Hence a helix ξ in $\mathbb{E}^3$ can never be a normal curve. □

Theorem 4. *Let $\xi : I \to \mathbb{E}^3$ be a unit speed normal curve in $\mathbb{E}^3$ with non-zero curvature k and torsion τ. Then the following conditions must hold:*

(i) The distance function $\rho = ||\xi||$ of the curve ξ must be constant.

(ii) The normal component of the curve ξ is directly proportional to its radius of curvature, and the binormal component of the curve ξ is non-constant and is given as:

$$\langle\xi(w),B(w)\rangle=\frac{k'(w)}{\tau k^2(w)},$$

where $k'(w)$ represents the derivative of the curvature $k(w)$.

Conversely, if any of the above conditions hold, then a unit-speed curve ξ becomes a normal curve.

Proof. Let ξ be a normal curve in $\mathbb{E}^3$ having non-zero curvature k and torsion τ. Let us proceed to prove these two conditions one by one.

(i) In condition (i), we need to show that the distance function of the curve ξ given as $\rho = ||\xi||$ is constant.
Since, ξ is a normal curve in $\mathbb{E}^3$, it must satisfy equations (3.3), (3.4) and (3.5). By multiplying equation (3.4) with $\beta'(w)$ and equation (3.5) with $\alpha'(w)$ and, then subtracting their resultant, we obtain

$$\begin{aligned} \alpha'(w)\beta'(w) - \beta'(w)\beta(w)\tau - \alpha(w)\alpha'(w)\tau + \alpha'(w)\beta'(w) &= 0, \\ \Rightarrow (\beta'(w)\beta(w) + \alpha(w)\alpha'(w))\tau &= 0, \\ \Rightarrow \beta'(w)\beta(w) + \alpha(w)\alpha'(w) &= 0, \\ \Rightarrow \beta^2(w) + \alpha^2(w) &= a^2, \quad (3.10) \end{aligned}$$

for some constant $a \in \mathbb{R}^* = \mathbb{R}\setminus\{0\}$.
As we know that, $\rho(w) = ||\xi(w)|| = \sqrt{\langle \xi(w), \xi(w) \rangle}$.
This implies,

$$\rho^2(w) = \langle \xi(w), \xi(w) \rangle = \alpha^2(w) + \beta^2(w) = a^2,$$

which yields the distance function $\rho(w) = ||\xi(w)||$ is constant.

Conversely, assume that the distance function of a unit-speed parameterized curve ξ is constant, i.e., $\rho = c$, for some constant $c \in \mathbb{R}^*$. This implies,

$$\langle \xi(w), \xi(w) \rangle = c^2.$$

Differentiating with respect to 'w', we get

$$\langle \xi(w), T(w) \rangle = 0.$$

Thus, a unit-speed parameterized curve ξ is a normal curve.

(ii) To prove (ii), we need to show that the normal component of a normal curve is proportional to its radius of curvature, and the binormal component of a normal curve is non-constant and depends upon the curvature and torsion of the curve.
From equation (3.1), we get

$$\langle \xi(w), N(w) \rangle = \alpha(w), \quad (3.11)$$
$$\text{and} \quad \langle \xi(w), B(w) \rangle = \beta(w). \quad (3.12)$$

Substituting the value of $\alpha(w)$ and $\beta(w)$, respectively, from equations (3.6) and (3.7), to equations (3.11) and (3.12), we get the normal and binormal components of the curve ξ, which are respectively given as

$$\langle \xi(w), N(w) \rangle = -\frac{1}{k(w)}, \tag{3.13}$$

and

$$\langle \xi(w), B(w) \rangle = \frac{k'(w)}{\tau k^2(w)}. \tag{3.14}$$

Thus, the normal component of a normal curve is directly proportional to the radius of curvature, and the binormal component of a normal curve is non-constant and depends upon the curvature and torsion of the curve ξ.

Conversely, assume that the normal and binormal components of a unit-speed parameterized curve ξ satisfy equations (3.13) and (3.14).
Differentiating (3.13) with respect to parameter w, we have

$$\begin{aligned}
\langle \xi'(w), N(w) \rangle + \langle \xi(w), N'(w) \rangle &= \frac{1}{k^2(w)} k'(w), \\
\Rightarrow -k(w)\langle \xi(w), T(w) \rangle + \tau \langle \xi(w), B(w) \rangle &= \frac{1}{k^2(w)} k'(w), \\
\Rightarrow -k(w)\langle \xi(w), T(w) \rangle &= \frac{1}{k^2(w)} k'(w) - \tau \frac{k'(w)}{\tau k^2(w)}, \\
\Rightarrow \langle \xi(w), T(w) \rangle &= 0.
\end{aligned}$$

Thus a unit-speed parameterized curve ξ is a normal curve.

□

4 Characterizations of normal curve in $\mathbb{E}^4$

A normal curve in $\mathbb{E}^4$ is a space curve whose position vector is entirely contained within the region orthogonal to its tangent vector field T. In $\mathbb{E}^4$, the orthogonal complement $T^\perp$ is expressed as follows

$$T^\perp = \{W \in \mathbb{E}^4 \mid \langle W, T \rangle = 0\},$$

which is a 3-dimensional subspace of $\mathbb{E}^4$, normal to the tangent vector and spanned by the normal vector, the first binormal vector, and the second binormal vector.
Therefore, a normal curve in $\mathbb{E}^4$ is characterized by its position vectors lying within a 3-dimensional subspace $T^\perp$. This implies that the position vector can be expressed as a linear combination of the normal vector, the first binormal vector, and the second binormal vector, i.e.,

$$\xi(w) = \alpha(w)N(w) + \beta(w)B_1(w) + \gamma(w)B_2(w), \tag{4.1}$$

where $\alpha(w)$, $\beta(w)$, and $\gamma(w)$ are differentiable functions of the arc-length parameter w.

Theorem 5. *A unit-speed curve $\xi(w)$ in $\mathbb{E}^4$ with non-zero curvatures $k_1(w)$, $k_2(w)$ and $k_3(w)$ is a normal curve if and only if it satisfies the following condition:*

$$\left(\frac{k_1'(w)}{k_2(w)k_1^2(w)}\right)k_3(w)+ \left(\frac{k_1(w)k_2(w)k_1''(w)-k_1^2(w)k_2^3(w)-k_1(w)k_1'(w)k_2'(w)-2k_2(w)k_1'^2(w)}{k_1^3(w)k_2^2(w)k_3(w)}\right)'=0. \tag{4.2}$$

Proof. Let us consider a unit-speed parameterized normal curve $\xi : I \to \mathbb{E}^4$ with non-zero curvatures $k_1(w)$, $k_2(w)$ and $k_3(w)$. Then it satisfy equation (4.1).

Differentiating equation (4.1) in terms of w, and applying the Serret-Frenet equations (2.2) for $\mathbb{E}^4$, we acquire the following:

$$\begin{aligned}
\xi'(w) &= \alpha'(w)N(w)+\alpha(w)N'(w)+\beta'(w)B_1(w)+\beta(w)B_1'(w)+\gamma'(w)B_2(w)\\
&\quad +\gamma(w)B_2'(w),\\
\Rightarrow T(w) &= \alpha'(w)N(w)+\alpha(w)(-k_1(w)T(w)+k_2(w)B_1(w))+\beta'(w)B_1(w)+\beta(w)\\
&\quad (-k_2(w)N(w)+k_3(w)B_2(w))+\gamma'(w)B_2(w)+\gamma(w)(-k_3(w)B_1(w)),\\
\Rightarrow T(w) &= -\alpha(w)k_1(w)T(w)+(\alpha'(w)-\beta(w)k_2(w))N(w)+(\alpha(w)k_2(w)+\beta'(w)\\
&\quad -\gamma(w)k_3(w))B_1(w)+(\beta(w)k_3(w)+\gamma'(w))B_2(w),
\end{aligned}$$

which yields the following expressions:

$$-\alpha(w)k_1(w) = 1, \tag{4.3}$$
$$\alpha(w)k_2(w)+\beta'(w)-\gamma(w)k_3(w) = 0, \tag{4.4}$$
$$\alpha'(w)-\beta(w)k_2(w) = 0, \tag{4.5}$$
$$\beta(w)k_3(w)+\gamma'(w) = 0. \tag{4.6}$$

By using these above equations, we can calculate the value of differentiable functions $\alpha(w)$, $\beta(w)$ and $\gamma(w)$ used in (4.1).
From (4.3), it follows that

$$\alpha(w) = -\frac{1}{k_1(w)}, \tag{4.7}$$

and equation (4.5) entails

$$\begin{aligned}
\beta(w) &= \frac{1}{k_2(w)}\alpha'(w),\\
\Rightarrow \beta(w) &= \frac{k_1'(w)}{k_2(w)\,k_1^2(w)}.
\end{aligned} \tag{4.8}$$

Using (4.7) and (4.8) in (4.4), we obtain

$$\gamma(w) = \frac{k_1(w)k_2(w)k_1''(w) - k_1^2(w)k_2^3(w) - k_1(w)k_1'(w)k_2'(w)}{k_1^3(w)k_2^2(w)k_3(w)} - \frac{2k_2(w)k_1'^2(w)}{k_1^3(w)k_2^2(w)k_3(w)}. \tag{4.9}$$

By using these derived values of the differentiable functions $\alpha(w)$, $\beta(w)$ and $\gamma(w)$ from (4.7),
(4.8) and (4.9) in (4.6), we get a relation between curvatures $k_1(w)$, $k_2(w)$ and $k_3(w)$, which is given as

$$\left(\frac{k_1'(w)}{k_2(w)k_1^2(w)}\right)k_3(w) + \left(\frac{k_1(w)k_2(w)k_1''(w) - k_1^2(w)k_2^3(w)}{k_1^3(w)k_2^2(w)k_3(w)} - \frac{k_1(w)k_1'(w)k_2'(w) + 2k_2(w)k_1'^2(w)}{k_1^3(w)k_2^2(w)k_3(w)}\right)' = 0. \tag{4.10}$$

Thus, it is clear from the above that if a unit-speed curve ξ is a normal curve, then their curvatures satisfy the relation (4.10).

Conversely, we need to show that $\xi(w)$ is a normal curve, assuming that the equation (4.10) must be satisfied by the curvatures $k_1(w)$, $k_2(w)$ and $k_3(w)$ for an arbitrary unit-speed curve $\xi(w)$ in $\mathbb{E}^4$.
Consider the vector $\chi(w) \in \mathbb{E}^3$ defined by

$$\begin{aligned} \chi(w) &= \xi(w) - \frac{k_1(w)k_2(w)k_1''(w) - k_1^2(w)k_2^3(w) - k_1(w)k_1'(w)k_2'(w) - 2k_2(w)k_1'^2(w)}{k_1^3(w)k_2^2(w)k_3(w)}B_2(w) \\ &\quad + \frac{1}{k_1(w)}N(w) - \frac{k_1'(w)}{k_2(w)\ k_1^2(w)}B_1(w). \end{aligned} \tag{4.11}$$

Differentiating (4.11) in terms of 'w', and considering the equations (2.2) and (4.10), we easily get $\chi'(w) = 0$. Thus, $\chi(w)$ is constant, which implies

$$\langle \chi(w), \chi(w)\rangle = C, \quad \text{for some constant } C \in \mathbb{R}.$$

Differentiating with respect to 'w', we get

$$\langle \chi(w), T(w)\rangle = 0.$$

From equation (4.11), we can write

$$\begin{aligned} \langle \chi(w), T(w)\rangle &= \langle \xi(w), T(w)\rangle + \frac{1}{k_1(w)}\langle N(w), T(w)\rangle - \frac{k_1'(w)}{k_2(w)\ k_1^2(w)}\langle B_1(w), T(w)\rangle \\ &\quad - \frac{k_1(w)k_2(w)k_1''(w) - k_1^2(w)k_2^3(w) - k_1(w)k_1'(w)k_2'(w) - 2k_2(w)k_1'^2(w)}{k_1^3(w)k_2^2(w)k_3(w)} \\ &\quad \langle B_2(w), T(w)\rangle, \\ \Rightarrow \langle \xi(w), T(w)\rangle &= 0. \end{aligned}$$

Thus $\xi(w)$ is a normal curve. □

This theorem provides a necessary and sufficient condition for a unit-speed curve in $\mathbb{E}^4$ to be congruent to a normal curve by obtaining a relation among curvatures $k_1(w)$, $k_2(w)$ and $k_3(w)$. To further illustrate the significance of this relation, we analyze the cases where one or more curvatures are held constant.
The following theorem addresses these cases and demonstrates how the constancy of certain curvatures influences the others, thus providing a clearer picture of the structure of normal curves under these special conditions.

Theorem 6. *Let $\xi(w)$ represents a normal curve in $\mathbb{E}^4$ with non-zero curvatures $k_1(w)$, $k_2(w)$ and $k_3(w)$. Then the following statements hold:*

(i) If $k_1(w)$ and $k_2(w)$ are non-zero constant curvatures, then $k_3(w)$ is also constant and is given by $k_3(w) = c\frac{k_2}{k_1}$, $c \in \mathbb{R}^*$.

(ii) If $k_1(w)$ and $k_3(w)$ are non-zero constant curvatures, then $k_2(w)$ is also constant and is given by $k_2(w) = ck_1k_3$, $c \in \mathbb{R}^*$.

(iii) If $k_2(w)$ and $k_3(w)$ are non-zero constant curvatures, then $k_1(w)$ is non constant and is given by

$$k_1(w) = \frac{k_3^2 + k_2^2}{A(k_3^2 + k_2^2)e^{w\sqrt{k_3^2+k_2^2}} + B(k_3^2 + k_2^2)e^{-w\sqrt{k_3^2+k_2^2}} - C},$$

where A, B, and C are arbitrary constants.

Proof. Let $\xi(w)$ be a normal curve in $\mathbb{E}^4$ with non-zero curvatures $k_1(w)$, $k_2(w)$ and $k_3(w)$.

(i) Suppose $k_1(w)$ and $k_2(w)$ are non-zero constant curvatures, say $k_1(w) = k_1$ and $k_2(w) = k_2$. Then, from (4.10), we obtain

$$\begin{aligned} \left(\frac{k_1^2 k_2^3}{k_1^3 k_2^2 k_3(w)}\right)' &= 0, \\ \Rightarrow \frac{k_2}{k_1 k_3(w)} &= c, \\ \Rightarrow k_2 &= ck_1k_3(w), \\ \Rightarrow k_3(w) &= c\frac{k_2}{k_1}. \end{aligned}$$

Hence (i) is true.

(ii) Suppose $k_1(w) = k_1$ and $k_3(w) = k_3$ are non-zero constant curvatures. Then (4.10) yields

$$\begin{aligned}\left(\frac{k_1^2 k_2^3(w)}{k_1^3 k_2^2(w) k_3}\right)' &= 0,\\ \Rightarrow \frac{k_2(w)}{k_1 k_3} &= c,\\ \Rightarrow k_2(w) &= c k_1 k_3.\end{aligned}$$

Hence (ii) is true.

(iii) Suppose $k_2(w)$ and $k_3(w)$ are non-zero constant curvatures k_2 and k_3, respectively. Then (4.10) entails

$$\begin{aligned}\left(\frac{k_1'(w)}{k_2 k_1^2(w)}\right) k_3 + \left(\frac{k_1(w) k_2 k_1''(w) - k_1^2(w) k_2^3 - 2k_2 k_1'^2(w)}{k_1^3(w) k_2^2 k_3}\right)' &= 0,\\ \Rightarrow -\frac{k_3}{k_2}\left(\frac{1}{k_1(w)}\right)' + \left(\frac{k_1(w) k_1''(w) - 2k_1'^2(w)}{k_1^3(w) k_2 k_3} - \frac{k_2}{k_1(w) k_3}\right)' &= 0,\\ \Rightarrow -\frac{k_3}{k_2}\left(\frac{1}{k_1(w)}\right)' + \frac{1}{k_2 k_3}\left(\frac{k_1'(w)}{k_1^2(w)}\right)'' - \frac{k_2}{k_3}\left(\frac{1}{k_1(w)}\right)' &= 0,\\ \Rightarrow \frac{1}{k_2 k_3}\left(\frac{k_1'(w)}{k_1^2(w)}\right)'' - \left(\frac{k_3^2 + k_2^2}{k_2 k_3}\right)\left(\frac{1}{k_1(w)}\right)' &= 0,\\ \Rightarrow \left(\frac{k_1'(w)}{k_1^2(w)}\right)'' - (k_3^2 + k_2^2)\left(\frac{1}{k_1(w)}\right)' &= 0,\\ \Rightarrow \left(\frac{1}{k_1(w)}\right)''' - (k_3^2 + k_2^2)\left(\frac{1}{k_1(w)}\right)' &= 0.\end{aligned}$$

Integrating both sides, we obtain

$$\left(\frac{1}{k_1(w)}\right)'' - (k_3^2 + k_2^2)\left(\frac{1}{k_1(w)}\right) = c,$$

which is a second-order non-homogeneous differential equation whose solution is given by

$$k_1(w) = \frac{k_3^2 + k_2^2}{A(k_3^2 + k_2^2) e^{w\sqrt{k_3^2 + k_2^2}} + B(k_3^2 + k_2^2) e^{-w\sqrt{k_3^2 + k_2^2}} - C},$$

which holds (iii).

□

Theorem 7. *Let $\xi : I \to \mathbb{E}^4$ be a normal curve in $\mathbb{E}^4$ with non-zero curvatures $k_1(w)$, $k_2(w)$ and $k_3(w)$. Then the following conditions must be satisfied:*

(i) *The distance function* $\rho(w) = ||\xi(w)||$ *of a normal curve* ξ *with constant curvature* $k_1(w)$ *is constant and satisfies*

$$\rho^2(w) = \frac{1}{2k_1^2(w)} + c, \; c \in \mathbb{R}.$$

(ii) *The normal component of a normal curve* $\xi(w)$ *is directly proportional to its radius of curvature.*

(iii) *The first binormal component and the second binormal component of a normal curve* ξ *are non-constant and are respectively given as:*

$$\begin{aligned} \langle \xi(w), B_1(w) \rangle &= -\frac{k_1'(w)}{k_2(w)k_1^2(w)}, \\ \textit{and} \quad \langle \xi(w), B_2(w) \rangle &= \frac{k_1(w)k_2(w)k_1''(w) - k_1^2(w)k_2^3(w) - k_1(w)k_1'(w)k_2'(w)}{k_1^3(w)k_2^2(w)k_3(w)} \\ &\quad - \frac{2k_2(w)k_1'^2(w)}{k_1^3(w)k_2^2(w)k_3(w)}. \end{aligned}$$

Conversely, if a unit-speed arc-length parameterized curve $\xi(w)$ *in* $\mathbb{E}^4$ *with non-zero curvatures* $k_1(w)$, $k_2(w)$ *and* $k_3(w)$ *satisfies any one of the above conditions (i)-(iii), then it is a normal curve.*

Proof. Let $\xi(w)$ represents a normal curve in $\mathbb{E}^4$ parameterized by arc-length w. Let $k_1(w)$, $k_2(w)$ and $k_3(w)$ be the non-zero curvatures of $\xi(w)$. We will prove the required conditions one by one:

(i) Since $\xi(w)$ is a normal curve in $\mathbb{E}^4$, it must be of the form (4.1) where the differentiable functions $\alpha(w)$, $\beta(w)$ and $\gamma(w)$ satisfy equations (4.3), (4.4), (4.5) and (4.6).

Multiplying equation (4.4) with $\beta(w)$ and equation (4.6) with $\gamma(w)$ and then adding the resultant, we obtain

$$\begin{aligned} \alpha(w)\beta(w)k_2(w) + \beta(w)\beta'(w) + \gamma(w)\gamma'(w) &= 0, \\ \Rightarrow \beta(w)\beta'(w) + \gamma(w)\gamma'(w) &= -\alpha(w)\beta(w)k_2(w), \\ \Rightarrow \beta(w)\beta'(w) + \gamma(w)\gamma'(w) &= \frac{1}{k_1(w)} \frac{k_1'(w)}{k_2(w)k_1^2(w)} k_2(w), \\ \Rightarrow \beta(w)\beta'(w) + \gamma(w)\gamma'(w) &= \frac{k_1'(w)}{k_1^3(w)}, \end{aligned}$$

which implies by integration

$$\begin{aligned} \beta^2(w) + \gamma^2(w) &= \int k_1^{-3}(w)k_1'(w)dw, \\ \Rightarrow \beta^2(w) + \gamma^2(w) &= \frac{-1}{2k_1^2(w)} + c. \end{aligned} \tag{4.12}$$

The distance function $\rho(w) = ||\xi(w)|| = \sqrt{\langle\xi(w), \xi(w)\rangle}$ gives

$$\rho^2(w) = \langle\xi(w), \xi(w)\rangle.$$

By virtue of (4.1), we have

$$\langle\xi(w), \xi(w)\rangle = \alpha^2(w) + \beta^2(w) + \gamma^2(w). \tag{4.13}$$

In view of (4.12) and (4.7), (4.13) entails

$$\begin{aligned}\langle\xi(w), \xi(w)\rangle &= \frac{1}{k_1^2(w)} - \frac{1}{2k_1^2(w)} + c,\\ \Rightarrow \langle\xi(w), \xi(w)\rangle &= \frac{1}{2k_1^2(w)} + c,\\ \Rightarrow \rho^2(w) &= \frac{1}{2k_1^2(w)} + c.\end{aligned}$$

Conversely, assume that the distance function of a unit-speed parameterized curve $\xi(w)$ with non-zero constant curvature satisfies

$$\begin{aligned}\rho^2(w) &= \frac{1}{2k_1^2(w)} + c,\\ \Rightarrow \langle\xi(w), \xi(w)\rangle &= \frac{1}{2k_1^2(w)} + c.\end{aligned}$$

Differentiating with respect to 'w', we get

$$2\langle\xi(w), \xi'(w)\rangle = \frac{-2}{2k_1^3(w)}k_1'(w).$$

Since $k_1(w)$ is constant, their derivative $k_1'(w) = 0$ and hence

$$\langle\xi(w), T(w)\rangle = 0,$$

consequently $\xi(w)$ is a normal curve.

(ii) To prove (ii), we need to show the normal component of $\xi(w)$ is proportional to its radius of curvature.
Since $\xi(w)$ is a normal curve, it must satisfy equation (4.1) and hence

$$\langle\xi(w), N(w)\rangle = \alpha(w). \tag{4.14}$$

Substituting the value of $\alpha(w)$ (from equation (4.7)) in equation (4.14), we get a normal component of the curve $\xi(w)$ as

$$\langle\xi(w), N(w)\rangle = -\frac{1}{k_1(w)}.$$

This proves that the normal component of $\xi(w)$ is directly proportional to its radius of curvature.
Conversely, assume that the normal component of a unit-speed parameterized normal curve $\xi(w)$ is

$$\langle \xi(w), N(w) \rangle = -\frac{1}{k_1(w)}.$$

Differentiating both sides in terms of arc parameter w, we get

$$\begin{aligned}
\langle \xi'(w), N(w) \rangle + \langle \xi(w), N'(w) \rangle &= \frac{1}{k_1^2(w)} k_1'(w), \\
\Rightarrow \langle T(w), N(w) \rangle + \langle \xi(w), -k_1(w)T(w) + k_2(w)B_1(w) \rangle &= \frac{1}{k_1^2(w)} k_1'(w), \\
\Rightarrow -k_1(w)\langle \xi(w), T(w) \rangle + k_2(w)\beta(w) &= \frac{1}{k_1^2(w)} k_1'(w), \\
\Rightarrow -k_1(w)\langle \xi(w), T(w) \rangle &= \frac{1}{k_1^2(w)} k_1'(w) \\
&\quad - \frac{k_2(w)k_1'(w)}{k_2(w)k_1^2(w)}, \\
\Rightarrow \langle \xi(w), T(w) \rangle &= 0.
\end{aligned}$$

Thus ξ is a normal curve.

(iii) Let ξ be a normal curve. Then equation (4.1) yields

$$\langle \xi(w), B_1(w) \rangle = \beta(w), \tag{4.15}$$

and hence

$$\langle \xi(w), B_2(w) \rangle = \gamma(w). \tag{4.16}$$

Using the derived values of $\beta(w)$ and $\gamma(w)$, respectively, (from (4.8) and (4.9)) in (4.15) and (4.16), we obtain the first and the second binormal component of a normal curve ξ as

$$\langle \xi(w), B_1(w) \rangle = -\frac{k_1'(w)}{k_2(w)k_1^2(w)}, \tag{4.17}$$

$$\begin{aligned}
\langle \xi(w), B_2(w) \rangle &= \frac{k_1(w)k_2(w)k_1''(w) - k_1^2(w)k_2^3(w)}{k_1^3(w)k_2^2(w)k_3(w)} \\
&\quad - \frac{k_1(w)k_1'(w)k_2'(w) + 2k_2(w)k_1'^2(w)}{k_1^3(w)k_2^2(w)k_3(w)}.
\end{aligned} \tag{4.18}$$

Conversely, suppose that for a unit-speed parameterized curve $\xi(w)$, the equations (4.17) and (4.18) are satisfied. Then

$$\langle \xi(w), B_1(w)\rangle = -\frac{k_1'(w)}{k_2(w)k_1^2(w)}.$$

Differentiating both sides with respect to w, we get

$$\begin{aligned}
\langle \xi'(w), B_1(w)\rangle + \langle \xi(w), B_1'(w)\rangle &= \left(\frac{k_1'(w)}{k_2(w)k_1^2(w)}\right)', \\
\Rightarrow \langle \xi(w), B_1'(w)\rangle &= \left(\frac{k_1'(w)}{k_2(w)k_1^2(w)}\right)', \\
\Rightarrow -k_2(w)\langle \xi(w), N(w)\rangle + k_3(w)\langle \xi(w), B_2(w)\rangle &= \left(\frac{k_1'(w)}{k_2(w)k_1^2(w)}\right)' \quad (4.19)
\end{aligned}$$

In view of (4.18) and (4.19), we get

$$\begin{aligned}
\Rightarrow -k_2(w)\langle \xi(w), N(w)\rangle &= \frac{(k_2(w)k_1^2(w))k_1''(w) - k_1'(w)k_2'(w)k_1'(w)}{(k_2(w)k_1^2(w))^2} \\
&\quad -\frac{2k_1(w)k_2(w)k_1'^2(w)}{(k_2(w)k_1^2(w))^2} \\
&\quad -\frac{k_1(w)k_2(w)k_1''(w) - k_1^2(w)k_2^3(w)}{k_1^3(w)k_2^2(w)k_3(w)} \\
&\quad -\frac{k_1(w)k_1'(w)k_2'(w) + 2k_2(w)k_1'^2(w)}{k_1^3(w)k_2^2(w)k_3(w)}, \\
\Rightarrow \langle \xi(w), N(w)\rangle &= -\frac{1}{k_1(w)}.
\end{aligned}$$

Differentiating both sides in terms of “w” and using (2.2), we obtain

$$\begin{aligned}
-k_1(w)\langle \xi(w), T(w)\rangle + k_2(w)\langle \xi(w), B_1(w)\rangle &= \frac{k_1'(w)}{k_1^2(w)}, \\
\Rightarrow -k_1(w)\langle \xi(w), T(w)\rangle &= \frac{k_1'(w)}{k_1^2(w)} - \frac{k_2(w)k_1'(w)}{k_1^2(w)k_2(w)}, \\
\Rightarrow \langle \xi(w), T(w)\rangle &= 0.
\end{aligned}$$

Thus, ξ is a normal curve.

□

Conclusion 8. *This study has provided a comprehensive analysis of normal curves in both $\mathbb{E}^3$ and $\mathbb{E}^4$. We have obtained differential equations that govern the behavior* *of normal curves and derived significant properties and relationships between curvature, torsion and the components of these curves, and also results including the characterization of normal curves with constant curvature and the impossibility of helices being normal curves in $\mathbb{E}^3$ as well as the intricate relationships among curvatures in $\mathbb{E}^4$, which offer valuable insights into the geometric structure of normal curves. These findings not only advance the theoretical understanding of normal curves but also pave the way for further exploration in higher-dimensional differential geometry.*

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