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ON THE USE OF THE SCHWARZIAN DERIVATIVE IN REAL ONE-DIMENSIONAL DYNAMICS

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Abstract. In the study of properties within one-dimensional dynamics, the negative Schwarzian derivative condition has been shown to be very useful. However, this condition may seem somewhat arbitrary, as it is not inherently a dynamical condition, except for the fact that it is preserved under iteration. In this brief work, we show that the negative Schwarzian derivative condition is not arbitrary in any sense but is instead strictly related to the fulfillment of the Minimum Principle for the derivative of the map and its iterates, which plays a key role in the proof of Singer's Theorem.

1 Introduction

The *Schwarzian derivative* appears in a wide range of mathematical topics, often in areas that seem unrelated at first glance [3]. It was first formulated in 1869 by Hermann A. Schwarz in his work on conformal mappings. David Singer was the first to apply it to one-dimensional dynamics in 1978, using it to study C^3 maps from the unit interval to itself. An initial approach was made by D.J. Allwright in [1] who studied bifurcations of C^3 maps satisfying a certain property, denoted by P, which resembles the Schwarzian derivative. In later progress, it was found that maps with negative Schwarzian derivative also possess local properties that are useful for establishing certain distortion bounds, particularly when focusing on cross-ratios [2].

At first glance, the negative Schwarzian derivative condition may appear somewhat arbitrary because it does not seem to be a dynamical condition. In this work, we show that this condition is not arbitrary in any sense, but rather strictly related to a sufficient condition that guarantees the fulfillment of the Minimum Principle for the derivative of the map and its iterates, which is the key point in the proof of Singer's Theorem. To the best of our knowledge, this is a simple and illustrative explanation of the use of the Schwarzian derivative in the context of one-dimensional differential dynamics.

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2 The Schwarzian derivative

A widely held view among mathematicians who have worked with the Schwarzian derivative is the sense of mystery surrounding its origin and the remarkable way in which it facilitates the solution of various problems in one-dimensional dynamics, driven solely by the requirement of preserving the negative condition under iterations of the map. However, its precise connection with the dynamical properties of the map remains unclear. Let us briefly recall it. Consider an interval $I = (a, b) \subseteq \mathbb{R}$ and a C^3 map $f : I \to I$. If $f'(x) \neq 0$, the *Schwarzian derivative* of f at x is defined as

$$Sf(x) := \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2.$$

We say that f satisfies the *negative Schwarzian derivative* condition on I if $Sf(x) < 0$ for all $x \in I$.

One of the main reasons the Schwarzian derivative is of interest in one-dimensional dynamics, as first observed by D. Singer, is its remarkable composition law, which follows directly from the chain rule

$$S(h \circ g)(x) = Sh(g(x)) \cdot (g'(x))^2 + Sg(x). \tag{2.1}$$

Consequently, if a map satisfies the negative Schwarzian derivative condition, then all its iterates do as well. As we shall see, this property makes it a valuable tool in one-dimensional dynamics.

3 The Minimum Principle

In this section, we introduce *The Minimum Principle*, a remarkable property of a map that carries significant dynamical implications, which will be explored in the next section.

Definition 1 (The Minimum Principle in an interval)**.** *A map g defined in an interval $J = [a, b]$ satisfies* the Minimum Principle on J *if for all $x \in (a, b)$*

$$|g(x)| > \min\{|g(a)|, |g(b)|\}.$$

Definition 2 (The Minimum Principle)**.** *A map g defined on an interval I satisfies* The Minimum Principle *if it satisfies the Minimum Principle in all intervals $J \subset I$ where the map g does not vanish.*

By definition, a map g satisfies the Minimum Principle if and only if every local maximum is positive and every local minimum is negative.

In particular, for a differentiable map f defined on an interval I, its derivative f' satisfies the Minimum Principle if for any non-vanishing critical point $x \in I$ of f', the quotient $\dfrac{f'''(x)}{f'(x)}$ is negative.

4 Singer's Theorem

An important result in one-dimensional differential dynamics was established by D. Singer in [4]. This result shows that the negative Schwarzian derivative condition restricts both the type and number of periodic orbits that a map can possess.

Before stating the theorem, we recall some necessary definitions. We say that p is a *periodic point* for a map f if, for some integer n, $f^n(p) = p$. Denote $O(p) = \{f^n(p);\, n \in \mathbb{Z}\}$ the *orbit* of p under f. The $\omega-$*limit set* is the set of accumulation points of the sequence of forward iterates of a point in this orbit. The *basin* of a periodic point p is the set of points whose $\omega-$limit set contains p. A periodic point p is *attracting* if its basin contains an interval that contains p. The *immediate basin* of a periodic point p is the union of the connected components of its basin which contain a point from $O(p)$. Finally, we say that c is a critical point of f if $f'(c) = 0$. A critical point is called *non-degenerate* if $f''(c) \neq 0$.

Theorem 3 (Singer's Theorem [4])**.** *If $f : I \to I$ is a C^3 map with negative Schwarzian derivative, then the immediate basin of any attracting periodic point contains either a critical point of f or a boundary point of I; each neutral periodic point is attracting; and there exists no interval of periodic points.*

The key point in the proof of Singer's Theorem follows from the following proposition [4].

Proposition 4. *If $Sf(x) < 0$ for all $x \in I$, then the function f' cannot have either a positive local minimum value or a negative local maximum value.*

In particular, the negative Schwarzian derivative condition for f, combined with the composition law in (2.1), implies that for all positive integers n, the derivative $(f^n)'$ satisfies the Minimum Principle.

5 The Schwarzian derivative and The Minimum Principle

Our goal is to find a condition on f such that the derivative $(f^n)'$ satisfies the Minimum Principle for all positive integers n. Unfortunately, deriving such a condition does not seem to be a straightforward task. Instead, we propose an alternative approach: finding a condition on f such that, for a given positive integer n and a non-vanishing critical point $x \in I$ of $(f^{n+1})'$, the quotient

$$\frac{(f^{n+1})'''(x)}{(f^{n+1})'(x)}$$

be negative. Indeed, as noted at the end of Section 3 this will guarantee the fulfillment of the Minimum Principle for $(f^{n+1})'$.

To achieve this, let f be a differentiable map defined on an open interval I, and let $x \in I$ be a non-vanishing critical point of $(f^{n+1})'$; that is, a point $x \in I$ such that $(f^{n+1})'(x) \neq 0$ and $(f^{n+1})''(x) = 0$.

By the chain rule, we have the following

$$(f^{n+1})'(x) = (f^n)'(f(x)) \cdot f'(x); \tag{5.1}$$

$$(f^{n+1})''(x) = (f^n)''(f(x)) \cdot (f'(x))^2 + (f^n)'(f(x)) \cdot f''(x); \tag{5.2}$$

and

$$(f^{n+1})'''(x) = (f^n)'''(f(x)) \cdot (f'(x))^3 + 3\,(f^n)''(f(x)) \cdot f'(x) \cdot f''(x) + (f^n)'(f(x)) \cdot f'''(x). \tag{5.3}$$

Thus, from (5.1) and (5.3), we obtain

$$\frac{(f^{n+1})'''(x)}{(f^{n+1})'(x)} = \frac{(f^n)'''(f(x))}{(f^n)'(f(x))} \cdot (f'(x))^2 + 3\,\frac{(f^n)''(f(x)) \cdot f''(x)}{(f^n)'(f(x))} + \frac{f'''(x)}{f'(x)}. \tag{5.4}$$

Since we have assumed that $(f^{n+1})''(x) = 0$, it follows from (5.2) that

$$(f^n)''(f(x)) \cdot (f'(x))^2 = -(f^n)'(f(x)) \cdot f''(x). \tag{5.5}$$

First, multiply (5.5) by $f''(x)$ and rearrange terms, we obtain

$$\frac{(f^n)''(f(x)) \cdot f''(x)}{(f^n)'(f(x))} = -\left(\frac{f''(x)}{f'(x)}\right)^2.$$

Substituting this into the second term on the right-hand side of (5.4) gives

$$\frac{(f^{n+1})'''(x)}{(f^{n+1})'(x)} = \frac{(f^n)'''(f(x))}{(f^n)'(f(x))} \cdot (f'(x))^2 - 3\left(\frac{f''(x)}{f'(x)}\right)^2 + \frac{f'''(x)}{f'(x)}. \tag{5.6}$$

Second, again from (5.5), dividing by $((f^n)'(f(x)))^2$, we obtain

$$\frac{(f^n)''(f(x))}{((f^n)'(f(x)))^2} \cdot (f'(x))^2 = -\frac{f''(x)}{(f^n)'(f(x))}.$$

Substituting this into the second term on the right-hand side of (5.4) gives

$$\frac{(f^{n+1})'''(x)}{(f^{n+1})'(x)} = \frac{(f^n)'''(f(x))}{(f^n)'(f(x))} \cdot (f'(x))^2 - 3\left(\frac{(f^n)''(f(x))}{(f^n)'(f(x))}\right)^2 \cdot (f'(x))^2 + \frac{f'''(x)}{f'(x)}. \tag{5.7}$$

Thus, by adding (5.6) and (5.7), dividing by 2 and rearranging terms, we obtain

$$\frac{(f^{n+1})'''(x)}{(f^{n+1})'(x)} = \left(\frac{(f^n)'''(f(x))}{(f^n)'(f(x))} - \frac{3}{2}\left(\frac{(f^n)''(f(x))}{(f^n)'(f(x))}\right)^2\right) \cdot (f'(x))^2 + \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2. \quad (5.8)$$

Hence, assuming that $(f^{n+1})''(x) = 0$ and that both expressions

$$\frac{(f^n)'''(f(x))}{(f^n)'(f(x))} - \frac{3}{2}\left(\frac{(f^n)''(f(x))}{(f^n)'(f(x))}\right)^2 \quad (5.9)$$

and

$$\frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2 \quad (5.10)$$

are negative, it follows from (5.8) that the quotient

$$\frac{(f^{n+1})'''(x)}{(f^{n+1})'(x)}$$

is negative, as desired.

Thus, for a given positive integer n, the Minimum Principle holds for $(f^{n+1})'$ provided that, for any non-vanishing critical point $x \in I$ of $(f^{n+1})'$, the expressions in (5.9) and (5.10) are negative.

Note that the expressions in (5.9) and (5.10) correspond to $S(f^n)(f(x))$ and $Sf(x)$, respectively. Therefore, for any non-vanishing critical point $x \in I$ of $(f^{n+1})'$, (5.8) can be rewritten as

$$\frac{(f^{n+1})'''(x)}{(f^{n+1})'(x)} = S(f^n)(f(x)) \cdot (f'(x))^2 + Sf(x), \quad (5.11)$$

which resembles the composition law given in (2.1), with $h = f^n$ and $g = f$. In fact, by the definition of the Schwarzian derivative for f^{n+1} and (5.11), it follows directly that, for any non-vanishing critical point $x \in I$ of $(f^{n+1})'$,

$$S(f^{n+1})(x) = S(f^n)(f(x)) \cdot (f'(x))^2 + Sf(x). \quad (5.12)$$

On the other hand, a straightforward computation shows that (5.12) holds for all $x \in I$, leading to a remarkable consequence: the negative Schwarzian derivative condition for f is preserved by iterates. Therefore, the Minimum Principle for the iterates of f is guaranteed only by the requirement that the expression in (5.10) is negative for all $x \in I$, which precisely corresponds to the negative Schwarzian derivative condition for f.

6 Conclusion

The negative Schwarzian derivative condition naturally emerges when looking for a condition on f that guarantees that $(f^n)'$ satisfies the Minimum Principle for all positive integers n. This principle has significant dynamical implications, particularly with respect to the type and number of positively oriented fixed points of f, as stated in Singer's Theorem. To the best of our knowledge, in the literature, there is no satisfactory and simple explanation about the use of the Schwarzian derivative in real one-dimensional dynamics. Therefore, we believe that this brief note is valuable in order to shed light on this point.

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