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$(C,2)(E,q)$ PRODUCT MEANS AND FOURIER-STIELTJES SERIES

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Abstract. In this paper, as a sequel to [3], we generalize Fejer's theorem on the determination of jumps of functions of bounded variation to Fourier-Stieltjes series in terms of $(C,2)(E,q)$ product means.

1 Introduction

Let f be a real-valued function on the closed and bounded interval $[a,b]$ and let $P=\{x_0,x_1,...,x_k\}$ be a partition of $[a,b]$. Then the variation of f with respect to P is

$$V(f;P)=\sum_{i=1}^{k}|f(x_i)-f(x_{i-1})|$$

and the total variation of f on $[a,b]$ is

$$TV(f)=supV(f;P)$$

for all partition P of $[a,b]$. A real-valued function f on the closed and bounded interval $[a,b]$ is said to be a function of bounded variation if $TV(f)$ is finite. From now on let f be a function of bounded variation on $[0,2\pi]$. It is well known that such an f may have only discontinuities of the first kind, i.e., the left-hand limit $f(x^-)$ and the right-hand limit $f(x^+)$ exist. Throughout this paper, a function f of bounded variation is normalized by the condition

$$f(x)=\frac{f(x^+)+f(x^-)}{2}.$$

The Fourier-Stieltjes coefficients of f (equivalently, the Fourier coefficients of df) are defined by

$$\hat{df}(k)=\frac{1}{2\pi}\int_0^{2\pi}e^{-ikx}df(x),$$

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where $k \in Z$ and the integral is Riemann-Stieltjes integral. We write

$$df(x) \sim \sum_{k\in Z} \hat{df}(k)e^{ikx} \tag{1.1}$$

and call this series the Fourier-Stieltjes series of f (equivalently, the Fourier series of df). The nth symmetric partial sum of series in (1.1) is defined as

$$s_n(df,x) = \sum_{|k|\le n} \hat{df}(k)e^{ikx}.$$

The following result is attributed to Fejer [1] (see the details in [5]): If f is a periodic function of bounded variation on $[0, 2\pi]$, then for every $0 < x < 2\pi$, we have

$$\lim_{n\to\infty} \frac{1}{n} s_n(df,x) = \frac{1}{\pi}(f(x^+) - f(x^-)),$$

while for $x = 0$ or $x = 2\pi$, we have

$$\lim_{n\to\infty} \frac{1}{n} s_n(df,x) = \frac{1}{\pi}(f(0^+) - f(2\pi^-) + c(f)),$$

where $c(f) = 2\pi\hat{df}(0) = f(2\pi) - f(0)$.
Given a sequence $\{s_n\}$, we put

$$s_n^o = s_n,$$

$$s_n^1 = s_o^o + s_1^o + ... + s_n^o,$$

$$s_n^2 = s_o^1 + s_1^1 + ... + s_n^1.$$

Similarly set

$$A_n^o = 1,$$

$$A_n^1 = A_o^o + A_1^o + ... + A_n^o,$$

$$A_n^2 = A_o^1 + A_1^1 + ... + A_n^1.$$

A sequence $\{s_n\}$ or a series $u_o + u_1 + u_2 + ...$ with partial sums s_n is $(C, 2)$ summable to the sum s if

$$t_n^{(C,2)} := \frac{s_n^2}{A_n^2} \to s$$

as $n \to \infty$. More precisely [6],

$$t_n^{(C,2)} = \frac{2}{(n+1)(n+2)} \sum_{k=0}^{n} (n-k+1)s_k \to s$$

as $n \to \infty$. It is well known that $(C,1)$ summability of a sequence involves $(C,2)$ summability to the same limit [6]. On the other hand, a sequence $\{s_n\}$ or a series $u_o + u_1 + u_2 + ...$ with partial sums s_n is (E,q) summable to the sum s [2] if

$$t_n^{(E,q)} = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} s_k \to s$$

as $n \to \infty$
The method was first applied by Euler for $q = 1$ to sum slowly convergent or divergent series and the technique was later extended to arbitrary values of q by Knopp [4]. It is well known that this method is regular for $q \geq 0$ [4]. The $(C,2)(E,q)$ product means is obtained by superimposing $(C,2)$ means on (E,q) means and is denoted by $t_n^{(C,2)(E,q)}$. More precisely,

$$t_n^{(C,2)(E,q)} = \frac{2}{(n+1)(n+2)} \sum_{k=0}^{n} (n-k+1) t_k^{(E,q)}$$

$$= \frac{2}{(n+1)(n+2)} \sum_{k=0}^{n} \{ \frac{(n-k+1)}{(1+q)^k} \sum_{i=0}^{k} \binom{k}{i} q^{k-i} s_i \}. \tag{1.2}$$

If $t_n^{(C,2)(E,q)} \to s$ as $n \to \infty$, then the infinite series $\sum_{k=0}^{\infty} u_k$ is said to be $(C,2)(E,q)$ summable to the sum s. The purpose of the present paper, as a sequel to [3], is to extend the Fejer theorem for Fourier-Stieltjes series to $(C,2)(E,q)$ product means of Fourier-Stieltjes series.

2 Main results

We recall the representation [6]

$$s_n(df, x) = \frac{1}{\pi} \int_0^{2\pi} D_n(x-t) df(t), \tag{2.1}$$

where

$$D_n(u) = \frac{1}{2} + \sum_{k=1}^{n} cosku = \frac{sin(n+\frac{1}{2})u}{2sin\frac{u}{2}}. \tag{2.2}$$

It follows from (1.2) and (2.3) that

$$t_n^{(C,2)(E,q)}(df, x) = \frac{1}{\pi} \int_0^{2\pi} L_n(x-t) df(t), \tag{2.3}$$

where

$$L_n(u) = \frac{2}{(n+1)(n+2)} \sum_{k=0}^{n} \{ \frac{(n-k+1)}{(1+q)^k} \sum_{i=0}^{k} \binom{k}{i} q^{k-i} D_i(u) \}. \tag{2.4}$$

The following lemmas will be used in the proof of our theorem. For the sake of completeness, we give the proof of [3, Lemma 2.1] here.

Lemma 1. *For any $n \in N$,*

$$\sum_{k=0}^{n} \binom{n}{k} q^{n-k} k = n(1+q)^{n-1}$$

Proof. Let $P(n)$ be the statement

$$\sum_{k=0}^{n} \binom{n}{k} q^{n-k} k = n(1+q)^{n-1}.$$

We give a proof by induction on n
Base case: For $n = 1$,

$$\binom{1}{0} q^{1-0} 0 + \binom{1}{1} q^{1-1} 1 = 1(1+q)^{1-1}.$$

Hence the statement $P(1)$ holds true.
Induction step: Assume that for $n = m$, the statement $P(m)$ holds true:

$$\sum_{k=0}^{m} \binom{m}{k} q^{m-k} k = m(1+q)^{m-1}.$$

It follows that

$$(m+1)(1+q)^m = m(1+q)^{m-1}(1+q) + (1+q)^m$$

$$= (\sum_{k=0}^{m} \binom{m}{k} q^{m-k} k)(1+q) + (1+q)^m$$

$$= (\sum_{k=0}^{m} \binom{m}{k} q^{m-k} k)(1+q) + \sum_{k=0}^{m} \binom{m}{k} q^{m-k}$$

$$= \sum_{k=0}^{m} \binom{m}{k} q^{m-k}(k+1) + \sum_{k=0}^{m} \binom{m}{k} q^{m+1-k} k$$

$$= \sum_{k=0}^{m+1} \binom{m+1}{k} q^{m+1-k} k$$

because of $\binom{m}{k} + \binom{m}{k-1} = \binom{m+1}{k}$.
Hence the statement $P(m+1)$ holds true, establishing the induction step. Therefore the statement $P(n)$ holds true for every natural number n. □

On the other hand, the following inequalities for the kernel L_n are obtained.

Lemma 2. *(i) For all n and x,*

$$|L_n(x)| \leq n + \frac{1}{2}. \tag{2.5}$$

(ii) For all n and $0 < x < 2\pi$,

$$|L_n(x)| \leq \frac{\pi}{2min\{x, 2\pi - x\}}. \tag{2.6}$$

Proof. From (2.4) it follows that for all n and x,

$$|D_n(x)| \leq n + \frac{1}{2}$$

and for all n and $0 < x < 2\pi$,

$$|D_n(x)| \leq \frac{\pi}{2min\{x, 2\pi - x\}}.$$

Since all numbers $\binom{n}{k}q^{n-k}$ are nonnegative, inequalities (2.7) and (2.8) follow immediately from (2.6) and Lemma 1. This completes the proof. □

Now we generalize a theorem of Fejer by establishing the following theorem:

Theorem 3. *Let f be a periodic function of bounded variation on $[0, 2\pi]$. Then for $0 < x < 2\pi$, we have*

$$\lim_{n\to\infty} \frac{3(1+q)}{n} t_n^{(C,2)(E,q)}(df, x) = \frac{1}{\pi}(f(x^+) - f(x^-)), \tag{2.7}$$

while for $x = 0$ or $x = 2\pi$, we have

$$\lim_{n\to\infty} \frac{3(1+q)}{n} t_n^{(C,2)(E,q)}(df, x) = \frac{1}{\pi}(f(0^+) - f(2\pi^-) + c(f)), \tag{2.8}$$

where $c(f) = 2\pi\hat{df}(0) = f(2\pi) - f(0)$.

Proof. We shall carry out the proof in four steps.
(i) We consider the particular case when f is continuous at an inner point x (i.e., $0 < x < 2\pi$). As it is well known, then the total variation of f is also continuous at x [4]. Therefore, given any $\varepsilon > 0$, we can choose $\delta = \frac{TV(f)+1}{\sqrt{n}}$ for sufficiently large n so that $0 < x - \delta < x + \delta < 2\pi$ and the total variation of f over the interval $[x - \delta, x + \delta]$ does not exceed ε. Then we decompose the integral in (2.5) as follows:

$$t_n^{(C,2)(E,q)}(df, x) = \frac{1}{\pi}(\int_0^{x-\delta} + \int_{x-\delta}^{x+\delta} + \int_{x+\delta}^{2\pi})L_n(x-t)df(t) = A + B + C.$$

Taking (2.7) and (2.8) into account, we get

$$|B| \le \frac{1}{\pi}(n + \frac{1}{2}) \int_{x-\delta}^{x+\delta} |df(t)| \le \frac{1}{\pi}(n+1)\varepsilon < \varepsilon n,$$

and

$$|A| + |C| \le \frac{1}{2\delta}(\int_0^{x-\delta} + \int_{x+\delta}^{2\pi})|df(t)| \le \frac{\sqrt{n}}{2TV(f)+2} 2TV(f) \le \sqrt{n},$$

which implies $|A| + |C| \le O(\sqrt{n})$. Hence $A + B + C = o(n)$ and this proves (2.9) with $f(x^+) - f(x^-) = 0$.
(ii) From (2.5) it follows that $t_n^{(C,2)(E,q)}(df, 0) = t_n^{(C,2)(E,q)}(df, 2\pi)$. Hence it is enough to prove (2.10) for $x = 0$. In this step, we consider the special case when

$$f(0^+) - f(2\pi^-) + c(f) = 0, \tag{2.9}$$

which means that the function $f(t) - f(2\pi - t)$ is continuous at $t = 0$ from the right. Therefore, given any $\varepsilon > 0$, we can choose $\delta = \frac{TV(f)+1}{2\sqrt{n}}$ for sufficiently large n so that the total variation of $f(t) - f(2\pi - t)$ over the interval $[0, \delta]$ does not exceed ε. Now we decompose the integral in (2.5) as follows:

$$t_n^{(C,2)(E,q)}(df, 0) = \frac{1}{\pi}(\int_0^{\delta} + \int_{\delta}^{2\pi-\delta} + \int_{2\pi-\delta}^{2\pi})L_n(t)df(t)$$

$$= \frac{1}{\pi}\int_0^{\delta} L_n(t)d(f(t) - f(2\pi - t)) + \frac{1}{\pi}\int_{\delta}^{2\pi-\delta} L_n(t)df(t) = A + B,$$

where we made use of the evenness of the kernel $L_n(t)$. By Lemma 2, we have

$$|A| \le \frac{1}{\pi}(n + \frac{1}{2}) \int_0^{\delta} |d(f(t) - f(2\pi - t))| \le \frac{1}{\pi}(n + \frac{1}{2})\varepsilon < \varepsilon n,$$

and

$$|B| \le \frac{1}{2\delta}\int_{\delta}^{2\pi-\delta} |df(t)| \le \frac{\sqrt{n}}{TV(f)+1} TV(f) \le \sqrt{n},$$

which implies $|B| \leq O(\sqrt{n})$. Hence $A+B=o(n)$ and this proves (2.10) at $x=0$ in the special case (2.11).

(iii) We shall prove (2.9) at an inner point x in the general case when f is discontinuous. Now we introduce a new function g as follows:

$$g(t)=f(t)-\frac{1}{\pi}(f(x^{+})-f(x^{-}))\phi(t-x), \tag{2.10}$$

where ϕ is defined by

$$\phi(t)=\frac{1}{2}(\pi-t) \tag{2.11}$$

for $0<t<2\pi$, $\phi(0)=\phi(2\pi)=0$, and continued periodically.

Observe that g is of bounded variation on $[0,2\pi]$ and g is continuous at $t=x$. Hence the argument in step (i) applies to g in place of f and yields

$$\lim_{n\to\infty}\frac{3(1+q)}{n}t_n^{(C,2)(E,q)}(dg,x)=0. \tag{2.12}$$

On the other hand, from (1.2) and (2.12) it follows that

$$t_n^{(C,2)(E,q)}(dg,x)=t_n^{(C,2)(E,q)}(df,x)-\frac{1}{\pi}(f(x^{+})-f(x^{-}))t_n^{(C,2)(E,q)}(d\phi,0). \tag{2.13}$$

We recall that for $0<x<2\pi$, the Fourier-Stieltjes series of ϕ is given by

$$d\phi(x)\sim\frac{1}{2}\sum_{k\in Z-\{0\}}e^{ikx}=\sum_{k=1}^{\infty}coskx. \tag{2.14}$$

From (1.2), (2.16) and Lemma 1 it follows that

$$\begin{aligned}
t_n^{(C,2)(E,q)}(d\phi,0)&=\frac{2}{(n+1)(n+2)}\sum_{k=0}^{n}\{\frac{(n-k+1)}{(1+q)^k}\sum_{i=0}^{k}\binom{k}{i}q^{k-i}s_i(d\phi,0)\}\\
&=\frac{2}{(n+1)(n+2)}\sum_{k=0}^{n}\frac{(n-k+1)}{(1+q)^k}\sum_{i=0}^{k}\binom{k}{i}q^{k-i}i\\
&=\frac{2}{(n+1)(n+2)}\sum_{k=0}^{n}\frac{(n-k+1)}{(1+q)^k}k(1+q)^{k-1}\\
&=\frac{2}{(n+1)(n+2)}\sum_{k=0}^{n}\frac{k(n-k+1)}{(1+q)}=\frac{n}{3(1+q)}.
\end{aligned}$$

Now by virtue of (2.14), (**??**)e2.15) and the last equality, we obtain (2.9).

(iv) We shall prove (2.10) at the endpoint $x=0$ (eqivalently, at $x=2\pi$) in the general case when condition (2.11) is not satisfied. We define

$$g(t)=f(t)-\frac{1}{\pi}(f(0^{+})-f(2\pi^{-})+c(f))\phi(t),$$

where $\phi(t) = \frac{1}{2}(\pi - t)$.
We see that g is of bounded variation on $[0, 2\pi]$ and condition (2.11) is satisfied with g in place of f. The rest of the proof are the same as in step (iii) above. Hence Theorem 3 is completely proved. □

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