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EXTENDED GENERALIZATION OF THE BERNOULLI EQUATION

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Abstract. This short note introduce an analytic solution for a generalized Bernoulli non-linear first-order ordinary differential equation (ODE), extending the framework proposed by Azevedo and Valentino [Int J Math Educ Sci, 2016, 47(8), 1271–1276]. The family of analytic solutions are obtained by employing a non-linear substitution that transform the extended Bernoulli ODE into a classical Bernoulli equation. We derive a family of solutions and provide illustrative examples, demonstrating the versatility of our approach for arbitrary values of the Bernoulli non-linear term power α. This work connects to applications in population growth models, such as the θ-logistic and Richards models.

1 Introduction

The classical Bernoulli differential equation, introduced by Jacob Bernoulli in 1695 [2], is a first-order non-linear ordinary differential equation (ODE) of the form:

$$\frac{dy}{dx} + P(x)y = Q(x)y^{\alpha}, \tag{1.1}$$

where $\alpha \in \mathbb{R}$ is the Bernoulli non-linear term power and $P, Q : I \to \mathbb{R}$ are continuous functions. It represents an important topic in undergraduate mathematics due to its historical significance, mathematical elegance, and wide applicability in fields such as ecological systems [7, 8], biological wave dynamics [3], diffusion models for population growth [5] and memristors dynamics [4].

Azevedo and Valentino [1] presented a generalization of the classical Bernoulli ODE, given by:

$$\frac{dy}{dx} + P(x)h(y) = Q(x)g(y), \tag{1.2}$$

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where $g(y), h(y)$ are differentiable with $g(y) \neq 0$, providing a family of solutions for a subclass of these ODEs when

$$h(y) = g(y) \int_{\mathcal{P}} g(y)^{-1}\, dy, \tag{1.3}$$

where $\int_{\mathcal{P}} f(s)\, ds = F(s)$ is the principal indefinite integral if $F'(s) = f(s)$ for $s \in I$, and showing that the equation can be transformed into a linear ODE via the substitution $u = \frac{h(y)}{g(y)}$. An alternative solution approach to the generalized Bernoulli ODE was presented by Tisdell [9], using exact differential equation methods, and transforming the generalized Bernoulli equation into an exact form using an integrating factor, offering a pedagogically alternative method without requiring linearization.

An earlier generalization for the Bernoulli ODE was also presented by Kolk and Lerman [6] in their book chapter on analytic solutions to non-linear differential equations. They considered the form:

$$\frac{dy}{dx} + a(x)G(y) + b(x)H(y) = 0, \tag{1.4}$$

which aligns with the notation used later in Azevedo and Valentino [1] by setting $P(x) = a(x)$, $Q(x) = -b(x)$, $h(y) = G(y)$, and $g(y) = H(y)$.

In this short note we further extend the generalization proposed in Azevedo and Valentino [1] and Kolk and Lerman [6], by including an explicit non-linear term power of α:

$$\frac{dy}{dx} + P(x)h(y) = Q(x)g(y)^{\alpha}, \quad \alpha \in \mathbb{R}, \tag{1.5}$$

that using the following substitution:

$$u = \left(\int_{\mathcal{P}} g(y)^{-\alpha}\, dy \right)^{\alpha}, \tag{1.6}$$

transforms the extended Bernoulli ODE, Equation (1.5), into a classical Bernoulli equation when the following connective condition is verified,

$$h(y) = g(y)^{\alpha} \int_{\mathcal{P}} g(y)^{-\alpha}\, dy, \tag{1.7}$$

offering flexibility for arbitrary α, accommodating the non-linear power term α structure, closer to the original Bernoulli ODE. The classical Bernoulli ODE, eq. (1.1), can be recovered when $h(y) = y$ and $g(y) = y$.

2 Main Theorem

Theorem 1. *Suppose $g(y)$ is differentiable, $g(y) \neq 0$, and the connectivity condition:*

$$h(y) = g(y)^{\alpha} \int_{\mathcal{P}} g(y)^{-\alpha}\, dy.$$

Then, the extended Bernoulli ODE, Equation:

$$\frac{dy}{dx} + P(x)h(y) = Q(x)g(y)^{\alpha},$$

has family of solutions $y(x) : I \to \mathbb{R}$ *satisfying:*

$$u = \left(\int_{\mathcal{P}} g(y)^{-\alpha}\, dy\right)^{\alpha} = \left(e^{-\int_{\mathcal{P}} P(x)\, dx}\left(\int_{\mathcal{P}} Q(x)e^{\int_{\mathcal{P}} P(x)\, dx}\, dx + c\right)\right)^{\alpha}, \tag{2.1}$$

for $c, \alpha \in \mathbb{R}$.

Proof. Starting from the substitution variable $u^{\frac{1}{\alpha}} = \frac{h(y)}{g(y)^{\alpha}}$ we can obtain the principal indefinite integral term as:

$$u^{\frac{1}{\alpha}} = \int_{\mathcal{P}} g(y)^{-\alpha}\, dy$$

that can be differentiate to obtain:

$$u' = \alpha u^{\frac{\alpha-1}{\alpha}} g(y)^{-\alpha} \frac{dy}{dx}, \quad \frac{dy}{dx} = \frac{u' g(y)^{\alpha}}{\alpha u^{\frac{\alpha-1}{\alpha}}}.$$

where $u' = \frac{du}{dx}$. The connectivity condition can be simplified to:

$$h(y) = g(y)^{\alpha} \int_{\mathcal{P}} g(y)^{-\alpha}\, dy = g(y)^{\alpha} u^{\frac{1}{\alpha}}.$$

Substituting the previous terms into the extended Bernoulli ODE, Equation (1.5), we obtain:

$$\frac{u' g(y)^{\alpha}}{\alpha u^{\frac{\alpha-1}{\alpha}}} + P(x)g(y)^{\alpha} u^{\frac{1}{\alpha}} = Q(x)g(y)^{\alpha},$$

that multiplied by $\frac{\alpha u^{\frac{\alpha-1}{\alpha}}}{g(y)^{\alpha}}$ (for $g(y) \neq 0$) simplifies to

$$u' + \alpha P(x)u = \alpha Q(x) u^{\frac{\alpha-1}{\alpha}},$$

which is a classical Bernoulli equation in the form of Eq. (1.1). This non-linear ODE can be linearized by using the substitution $u = v^{\alpha}, \quad u' = \alpha v^{\alpha-1} v'$,

$$v' + P(x)v = Q(x).$$

This is a first order linear classical ODE, and using an integrating factor defined as $\nu = e^{\int_{\mathcal{P}} P(x)\,dx}$, presents the following general solution:

$$v = e^{-\int_{\mathcal{P}} P(x)\,dx}\left(\int_{\mathcal{P}} Q(x)e^{\int_{\mathcal{P}} P(x)\,dx}\,dx + c\right),$$

for $c \in \mathbb{R}$. Since $u = v^{\alpha}$,

$$u = \left(e^{-\int_{\mathcal{P}} P(x)\,dx}\left(\int_{\mathcal{P}} Q(x)e^{\int_{\mathcal{P}} P(x)\,dx}\,dx + c\right)\right)^{\alpha}.$$

Thus, Equation (2.1) holds. This concludes the proof. □

3 Examples

Let us explore some examples to show the usefulness of the proposed theorem.

Example 1. *Let us present a family of solutions for*

$$\frac{dy}{dx} + \frac{1}{x}h(y) = e^{x}y^{\alpha}, \tag{3.1}$$

for $x > 0$, $y > 0$*. In this example,* $P(x) = \frac{1}{x}$*,* $Q(x) = e^{x}$*,* $g(y) = y$*, and from the connectivity relation, we can obtain* $h(y)$ *as:*

$$h(y) = y^{\alpha}\left(\int_{\mathcal{P}} y^{-\alpha}\,dy\right) = y^{\alpha}\frac{y^{1-\alpha}}{(1-\alpha)} = \frac{y}{(1-\alpha)},$$

with $u = \left(\int_{\mathcal{P}} y^{-\alpha}\,dy\right)^{\alpha} = \left(\frac{y^{1-\alpha}}{1-\alpha}\right)^{\alpha}$*. Then, the initial ODE can be written as:*

$$\frac{dy}{dx} + \frac{1}{x}\frac{y}{(1-\alpha)} = e^{x}y^{\alpha},$$

and from Theorem 1 we can obtain the general solution:

$$\left(\frac{y^{1-\alpha}}{1-\alpha}\right)^{\alpha} = \left(e^{x}\left(1-\frac{1}{x}\right)+\frac{c}{x}\right)^{\alpha}, c \in \mathbb{R}.$$

That is,

$$y = \left[(1-\alpha)\left(e^{x}\left(1-\frac{1}{x}\right)+\frac{c}{x}\right)\right]^{\frac{1}{1-\alpha}},$$

for $c \in \mathbb{R}$ *and* $\alpha \neq 1$*.*

Example 2. *The standard logistic function, $s(x)$, is the (unique) solution for the classical Bernoulli ODE of the form:*

$$\frac{ds}{dx} = s(1-s)$$

with the solution given by $s(x) = \frac{1}{1+e^{-x}} = \frac{e^x}{1+e^x}$.

In this example, we use $g(y)$ as a logistic generating type of function $g(y) = \sqrt{1+\cosh(y)}$, *and use* $P(x) = Q(x) = -1$,

$$\frac{dy}{dx} - h(y) = -(1+\cosh(y))^{\frac{\alpha}{2}}$$

and from the connectivity relation, we can obtain $h(y)$ as:

$$h(y) = (1+\cosh(y))^{\frac{\alpha}{2}} \int_{\mathcal{P}} \frac{1}{(1+\cosh(y))^{\frac{\alpha}{2}}}\, dy.$$

Hence, using $\alpha = 2$, the initial ODE can be written as:

$$\frac{dy}{dx} = \frac{e^y}{s(y)}\left(1 - \frac{1}{2s(y)}\right),$$

with $\int_{\mathcal{P}}(1+\cosh(y))^{-1}\, dy = 2s(y)$, $(1+\cosh(y)) = \frac{e^y}{2s^2(y)}$, $g(y) = \frac{e^{\frac{y}{2}}}{\sqrt{2}s(y)}$ *and* $h(y) = \frac{e^y}{s(y)}$.

From Theorem 1 we can obtain the general solution:

$$2s(y) = (1+ce^x)\,, c \in \mathbb{R},$$

that is,

$$y = s^{-1}\left(\frac{1}{2}(1+ce^x)\right) = 2\,\mathrm{arctanh}(ce^x) = \ln\left(\frac{1+ce^x}{1-ce^x}\right),$$

where $c \in \mathbb{R}$, and $s^{-1}()$ is the logit *function, defined as the inverse function of the standard logistic function.*

Example 3. *Consider the non-linear first-order ODE defined by a type of θ-logistic model used in population dynamics [7] given by the Richards ODE model [8]:*

$$\frac{dy}{dt} = \gamma_1(t)y - \gamma_2(t)y^{\theta+1} = \gamma_1(t)y\left(1 - \frac{\gamma_2(t)}{\gamma_1(t)}y^\theta\right), \tag{3.2}$$

where $\gamma_1(t) \neq 0$ and $\gamma_2(t)$: $\mathbb{R} \to \mathbb{R}$ are continuous function of the reciprocal of the characteristic times, representing the intrinsic population growth intensity.

Here we introduce a generalization of the Richards ODE model, defined as:

$$\frac{dy}{dt} = \gamma_1(t)g(y)\left(H(\theta, g(y)) - \frac{\gamma_2(t)}{\gamma_1(t)}g(y)^\theta\right), \tag{3.3}$$

with $H(\theta, g(y)) = g(y)^{\theta} \int_{\mathcal{P}} g(y)^{-(\theta+1)}\, dy$.

Assuming $\gamma_1(t) = \gamma_2(t) = \gamma(t)$, *and for a family of linear functions* $g(y) = k - y$, $k \in \mathbb{R}$, *we obtain the following Richards type of ODE:*

$$\frac{dy}{dt} = \gamma(t)(k-y)\left(\frac{1}{\theta} - (k-y)^{\theta}\right),$$

since for linear $g(y)$ *the term* $H(\theta, g(y)) = -\frac{1}{\theta}\frac{dg(y)}{dy}$.

Finally, we can reduce the modified Richards ODE into the extended Bernoulli ODE formulation,

$$\frac{dy}{dt} - \frac{\gamma(t)}{\theta}(k-y) = -\gamma(t)(k-y)^{\theta+1},$$

with $P(t) = -\frac{\gamma(t)}{\theta}$, $Q(t) = -\gamma(t)$, $g(y) = k - y$, *and* $h(y) = k - y$. *From Theorem 1 we can obtain the general solution:*

$$\frac{(k-y)^{-\theta}}{\theta} = e^{\int_{\mathcal{P}} \frac{\gamma(t)}{\theta}\, dt}\left(c - \int_{\mathcal{P}} \gamma(t) e^{\int_{\mathcal{P}} -\frac{\gamma(t)}{\theta}\, dt}\, dt\right), c \in \mathbb{R}.$$

That is,

$$y = k - \left[\theta\left(e^{\int_{\mathcal{P}} \frac{\gamma(t)}{\theta}\, dt}\left(c - \int_{\mathcal{P}} \gamma(t) e^{\int_{\mathcal{P}} -\frac{\gamma(t)}{\theta}\, dt}\, dt\right)\right)\right]^{-\frac{1}{\theta}},$$

for $c, k \in \mathbb{R}$.

The extended generalization of the Bernoulli ODE can encourage students and scientists to explore beyond the classical Bernoulli form, promoting critical thinking about substitution techniques. Example 1 is an introductory example to illustrate how modern research extends classical methods, connecting historical and contemporary mathematics, while Examples 2 and 3 can be used as connection to applications in population dynamics, such as the logistic and θ-logistic model in population dynamics [7] and Richards model [8].

4 Concluding Remarks

The presented extension enhances Azevedo and Valentino [1] and Kolk and Lerman [6] generalized Bernoulli ODEs by accommodating the non-linear power term α structure, closer to the original Bernoulli ODE. Future work could apply this to extend diffusion models for population growth [5] and memristors dynamics [4] or explore other types of connectivity conditions relating $h(y)$ and $g(y)$ family functions.

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