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SUGGESTIONS TO STUDY AFFINE AND GIT QUOTIENTS OF THE EXTENDED GROTHENDIECK–SPRINGER RESOLUTION

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Abstract. We define filtered ADHM data and connect a notion of filtered quiver representations to Grothendieck–Springer resolutions. We also provide current developments and give a list of research problems to further study filtered ADHM equation.

1 Introduction

Springer resolution and the Grothendieck–Springer resolution are fundamental and important in geometric representation theory. In type A, the Springer correspondence gives a bijection between irreducible representations of the symmetric group and unipotent conjugacy classes of the general linear group. That is, given a unipotent conjugacy class and a fixed element u in the conjugacy class, the corresponding irreducible representation of S_n is the cohomology group $H^{2\dim\mathcal{B}_u}(\mathcal{B}_u,\mathbb{Q})$, where $\mathcal{B}_u$ is the set of Borel subgroups of $\mathsf{GL}_n(\mathbb{C})$ in the Springer resolution of the unipotent group containing u (cf. [1, 2, 11]).

The Springer resolution could be thought of as embedded in the Grothendieck–Springer resolution (cf. [2, 18]), and the enhanced Grothendieck–Springer resolution could be viewed as the Hamiltonian reduction of a certain parabolic moment map (cf. [36, 18]), where the latter is easier to visualize, manipulate and study, using a notion called filtered quiver representations (cf. §2.1.2 and §2.4.1; also see [36, Prop. 3.2], [18, Prop. 1.1]).

Also in representation theory, representations of quivers and quiver varieties arise in many context and have deep connections to mathematical physics, string theory, cluster algebras, Kac–Moody algebras, to name a few. Lusztig's [29, 30, 26]

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and Nakajima's [33, 34] are some of many foundational grounds to study quiver representations.

The construction of Nakajima quiver varieties involves an action by reductive groups, but in our generalized setting, we shift our focus to parabolic group actions. Studying parabolic group actions on vector spaces with a fixed filtration and connecting their affine and GIT constructions to well-known varieties, revealing different ways to view the same geometric object and often providing a deeper insight into commonly known varieties, are important in mathematics. First, consider a classical example: let $G = \mathsf{GL}_n(\mathbb{C})$ and $\mathfrak{g} = \mathfrak{gl}_n = \mathsf{Lie}(G)$. In the case of the G-action on its Lie algebra $\mathfrak{g}$ by conjugation, the Jordan quiver in §2.1.1 is the natural quiver associated to this G-equivariant geometry. The orbit space $\mathfrak{g}/G$ is not a variety, but it is an orbifold. The affine quotient $\mathfrak{g}/\!\!/G$ consists of equivalence classes of semisimple, i.e., diagonalizable, matrices, so it is isomorphic to $\mathbb{C}^n$. Algebrically, $\mathfrak{g}/\!\!/G := \mathsf{Spec}(\mathbb{C}[\mathfrak{g}]^G)$, with the generators of the G-invariant ring being coefficients of the characteristic polynomial of $n \times n$ matrices. Now, let $B \leq G$ be the standard Borel subgroup, i.e., the set of invertible upper triangular matrices, and let $\mathfrak{b} = \mathsf{Lie}(B)$. The B-action on $\mathfrak{b}$ has a natural quiver representation $\mathsf{F}^\bullet \mathsf{Rep}(Q, n)$ but with a notion of filtration modifying the representation space of the Jordan quiver, which one may think of $\mathsf{F}^\bullet \mathsf{Rep}(Q, n)$ to be the set of all those linear maps $W_r \in \mathsf{End}(\mathbb{C}^n)$ fixing the complete standard flag $\mathsf{F}^\bullet$ in $\mathbb{C}^n$. The subspace $\mathsf{F}^\bullet \mathsf{Rep}(Q, n) \subseteq \mathsf{Rep}(Q, n)$ is acted upon by those matrices in G which preserve this flag of vector spaces, which is precisely B (cf. §2.1.2).

In this paper, we generalize the classical construction of Nakajima quiver varieties (cf. §2.1.2), provide current development (as a generalization of classical results), and give a list of open problems (cf. §6). Current development includes using a new technique through filtered ADHM data, where we explicitly give in §3 (semi-)invariant polynomials for certain finite acyclic and cyclic quiver representations with a fixed filtration of vector spaces (see [17, 16] for the strategies). We then focus on the Grothendieck–Springer resolution setting and describe the B (and P-)orbits on the Hamiltonian reduction of the cotangent bundle of extended Lie algebra $\mathfrak{b} = \mathsf{Lie}(B)$ (and $\mathfrak{p} = \mathsf{Lie}(P)$) in §4 and §5, respectively. In §6, we construct affine and GIT quotients using filtered ADHM equations (cf. §6.2), discovering connections to modified rational Cherednik algebras (cf. §6.9) and corresponding $\mathcal{D}$-modules, and investigating variations of Hilbert schemes, which may be related to the isospectral Hilbert scheme (cf. §6.7,[14]). Some motivations to generalize quiver varieties in this way come from the study of quiver Hecke algebras, also known as KLR-algebras (cf. [21, 22, 39]), universal quiver Grassmannians and universal quiver flag varieties (cf. [3]), Lusztig's triangular decomposition of the upper half of the universal enveloping algebra of a Kac–Moody algebra (cf. [24, 25, 31]), and the Grothendieck–Springer resolution (cf. [2, 36]).

Throughout this paper, we will work over $\mathbb{C}$, and we will assume that the set $\mathbb{N}$ of natural numbers includes 0.

2 Representation theory

2.1 Parabolic equivariant geometry

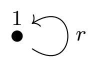
Studying $\mathsf{GL}_n(\mathbb{C})$-orbits on the set $\mathfrak{gl}_n$ of $n \times n$ complex matrices by conjugation amounts to putting the matrices into Jordan canonical form, up to a permutation of its elementary blocks. In such area of mathematics, one hopes to construct interesting moduli spaces associated to a pair $(\mathfrak{X}, \mathfrak{G})$, where $\mathfrak{X}$ is a space and $\mathfrak{G}$ is a group.

Let B be invertible upper triangular matrices in $G := \mathsf{GL}_n(\mathbb{C})$ and let $\mathfrak{b}$ be the set of upper triangular matrices in $\mathfrak{g} = \mathfrak{gl}_n$. Letting B act on $\mathfrak{b}$ by adjoint action, we can ask what are the B-orbits on $\mathfrak{b}$? More generally, how does one manage a general parabolic group action on a variety that has a certain notion of filtrations associated to it?

Returning to the example of the B-action on $\mathfrak{b}$, such pair $(\mathfrak{b}, B)$ can be thought of as a variety with a natural notion of filtration. The set $\mathbb{C}^n \xrightarrow{a} \mathbb{C}^n$ of all linear homomorphisms can be identified with $\mathfrak{g}$, and changing domain and codomain basis vectors amounts to a certain action by $\mathsf{GL}_n(\mathbb{C})$. Now suppose we impose a complete filtration on both the domain and the codomain with respect to the standard basis vectors. That is, letting $\mathsf{F}^\bullet : 0 \subseteq \mathbb{C}^1 \subseteq \mathbb{C}^2 \subseteq \ldots \subseteq \mathbb{C}^n$ be the complete standard filtration of vectors, where $\mathbb{C}^k$ is the subspace of $\mathbb{C}^n$ spanned by the first k standard basis vectors, consider the set $\mathsf{F}^\bullet \xrightarrow{r} \mathsf{F}^\bullet$ of all linear maps preserving the filtration; such maps consist of r such that $r|_{\mathbb{C}^k} : \mathbb{C}^k \to \mathbb{C}^k$ is linear for each k. Since $\mathsf{F}^\bullet$ is a filtration with respect to standard basis vectors, r has the form of an upper triangular matrix, and changing domain and codomain basis vectors (which preserves the filtration) amounts to an action by B.

2.1.1 Quiver representations

We refer to [5, 12, 16] for an extensive background on quivers and their representations. Quivers Q are a directed graph with finite number of vertices and arrows. We will denote the set of vertices as Q_0 and the set of arrows as Q_1. An example is the Jordan quiver:



with one vertex 1 and one arrow r; in this example, the head hr of r and the tail tr of r end and begin at the same vertex. We call the Jordan quiver cyclic since it has an oriented cycle, i.e., r is called a cycle since the tail of r equals the head of r. Another example is the A_2-quiver:

$$\overset{1}{\bullet} \xrightarrow{\quad a \quad} \overset{2}{\bullet}$$

with two vertices 1 and 2 and one arrow a whose tail of a equals 1 and the head of a equals 2. We call this quiver the A_2-quiver since the underlying graph has the structure of an A_2-Dynkin diagram. Furthermore, we call the A_2-quiver acyclic since it does not have any oriented cycles.

Let $\mathbf{v} = (v_1, \ldots, v_{Q_0}) \in \mathbb{N}^{Q_0}$, a dimension vector. A representation V of a quiver with dimension vector $\mathbf{v}$ assigns a finite dimensional vector space V_i of dimension v_i to each vertex and a linear map V_a to each arrow $a \in Q_1$. The representation space $\mathsf{Rep}(Q, \mathbf{v})$ consists of all representations of the given quiver Q with dimension vector $\mathbf{v}$. Consider the Jordan quiver, and let $\mathbf{v} = n$. Then $\mathsf{Rep}(Q, n) = \mathsf{Spec}(\mathbb{C}[V_r]) \cong \mathbb{C}^{n^2}$. If $\mathbf{v} = (n, m)$ for the A_2-quiver, then $\mathsf{Rep}(Q, (n, m)) = \mathsf{Spec}(\mathbb{C}[V_a]) \cong \mathbb{C}^{mn}$.

Studying isomorphism classes of representations of a quiver with a prescribed dimension vector amounts to a change-of-basis of the vector space at each vertex, i.e., it amounts to a certain natural quotient of the representation space by a group, and the study of the orbit structure is equivalent to equivariant geometry in geometric invariant theory (GIT; see §2.4).

2.1.2 Filtered ADHM data

In this section, we generalize the constructions in [35], giving a refined analogue of quiver representations called filtered quiver representations. A **filtered quiver representation** is a finite quiver with a fixed filtration of vector spaces at each vertex whose linear map associated to each arrow (of the quiver) preserves the filtration. Furthermore, there is a natural group action by the parabolic group $\mathfrak{P}_{\mathbf{v}}$ of $\mathfrak{G}_{\mathbf{v}} = \prod_\iota \mathsf{GL}_{v_\iota}$ acting on and preserving filtered vector spaces.

Let $Q = (Q_0, Q_1)$, where cycles are allowed, and let Q_1^{op} be the same set of arrows as Q_1 but in opposite orientation. Let $\overline{Q} = (Q_0, Q_1 \sqcup Q_1^{\mathsf{op}})$, a **double quiver**. Write $\overline{Q}_1 := Q_1 \sqcup Q_1^{\mathsf{op}}$. Let $\mathbf{v} = (v_\iota) \in \mathbb{N}^{Q_0}$ and let $V = (V_\iota)_{\iota \in Q_0}$ be a collection of vector spaces such that $\dim V_\iota = v_\iota$ for each $\iota \in Q_0$. For $V^1 = (V_\iota^1)_{\iota \in Q_0}$ and $V^2 = (V_\iota^2)_{\iota \in Q_0}$, define

$$L(V^1, V^2) := \bigoplus_{\iota \in Q_0} \mathsf{Hom}(V_\iota^1, V_\iota^2) \quad \text{and} \quad E(V^1, V^2) := \bigoplus_{a \in \overline{Q}_1} \mathsf{Hom}(V_{ta}^1, V_{ha}^2).$$

Given $B = (B_a)_{a \in \overline{Q}_1} \in E(V^1, V^2)$ and $C = (C_a)_{a \in \overline{Q}_1} \in E(V^2, V^3)$, we also define

$$CB := \left(\sum_{ta = \iota} C_a B_{a^{\mathsf{op}}} \right)_{\iota \in Q_0} \in L(V^1, V^3),$$

where a^{op} has the same endpoints as a but is in reverse orientation.

Now, choose a sequence $\gamma^1, \gamma^2, \ldots, \gamma^N = \mathbf{v} \in \mathbb{N}^{Q_0}$ of dimension vectors such that for all k and $\iota \in Q_0$, $\gamma_\iota^k \leq \gamma_\iota^{k+1}$. For each $\iota \in Q_0$, we get a filtration of $\mathbb{C}^{v_\iota}$:

$$\{0\} \subseteq \mathbb{C}^{\gamma_\iota^1} \subseteq \mathbb{C}^{\gamma_\iota^2} \subseteq \ldots \subseteq \mathbb{C}^{v_\iota},$$

where each $\mathbb{C}^l$ is spanned by l basis elements of $\mathbb{C}^{v_\iota}$. Define

$$\mathsf{F}^\bullet \, \mathsf{Rep}(Q, \mathbf{v}, \mathbf{w}) := \mathsf{F}^\bullet \, E(V, V) \oplus L(W, V) \oplus L(V, W),$$

where

$$\mathsf{F}^\bullet E(V,V) := \bigoplus_{a\in\overline{Q}_1} \mathsf{F}^\bullet \mathsf{Hom}(V_{ta}, V_{ha})$$

such that if $B_a \in \mathsf{F}^\bullet \mathsf{Hom}(V_{ta}, V_{ha})$ where $a \in Q_1$, then B_a preserves the fixed sequence of vector spaces at every level, i.e., $B_a(\mathbb{C}^{\gamma^k_{ta}}) \subseteq \mathbb{C}^{\gamma^k_{ha}}$ for every k, and if $B_c \in \mathsf{F}^\bullet \mathsf{Hom}(V_{tc}, V_{hc})$, where $c = a^{\mathsf{op}} \in Q_1^{\mathsf{op}}$, then $\mathsf{F}^\bullet \mathsf{Hom}(V_{tc}, V_{hc})$ is the dual of $\mathsf{F}^\bullet \mathsf{Hom}(V_{ta}, V_{ha})$ such that the trace map

$$\mathsf{F}^\bullet \mathsf{Hom}(V_{ta}, V_{ha}) \times \mathsf{F}^\bullet \mathsf{Hom}(V_{tc}, V_{hc}) \xrightarrow{\mathsf{tr}} \mathbb{C}, \quad (B_a, B_c) \mapsto \mathsf{tr}(B_a B_c)$$

is a nondegenerate pairing. We denote

$$C = (A, B) \in \mathsf{F}^\bullet E(V,V), \quad \text{where}$$
$$A \in \bigoplus_{a\in Q_1} \mathsf{F}^\bullet \mathsf{Hom}(V_{ta}, V_{ha}) \quad \text{and} \quad B \in \bigoplus_{c\in Q_1^{\mathsf{op}}} \mathsf{F}^\bullet \mathsf{Hom}(V_{tc}, V_{hc}),$$

$i \in L(W, V)$, and $j \in L(V, W)$. We call an element of $\mathsf{F}^\bullet \mathsf{Rep}(Q, \mathbf{v}, \mathbf{w})$ a **filtered ADHM datum**, while the filtered representation space $\mathsf{F}^\bullet \mathsf{Rep}(Q, \mathbf{v}, \mathbf{w})$ is called **filtered ADHM data**.

2.2 Invariant and semi-invariant polynomials

Invariant and semi-invariant polynomials play a fundamental role in classical and geometric invariant theory (§2.4). In fact, studying orbit spaces precisely amounts to describing invariant and semi-invariant polynomials.

2.3 Moment maps and complete intersection

Moment maps arise in symplectic geometry as a tool to construct conserved quantities. The action of a Lie group G on a vector space X is induced to the cotangent bundle T^*X of X. Taking the derivative of the group action induces an infinitesimal action $\mathfrak{g} \xrightarrow{a} TX$ on X, given by tangent vectors. By dualizing this action, one obtains the moment map $T^*X \xrightarrow{\mu} \mathfrak{g}^*$, where $\mu = a^*$. If the zero fiber of μ is a complete intersection, then this means $\mu^{-1}(0)$ has the expected number of irreducible components, with $\mu^{-1}(0)$ having dimension $2\dim X - \dim \mathfrak{g}$.

2.3.1 Moment maps for filtered ADHM data

Recall filtered ADHM data from §2.1.2.

Let $\varepsilon : \overline{Q}_1 \to \{\pm 1\}$, where

$$\varepsilon(a) = \begin{cases} 1 & \text{if } a \in Q_1, \\ -1 & \text{if } a \in Q_1^{\mathsf{op}}. \end{cases}$$

Then $\varepsilon C \in \mathsf{F}^\bullet E(V^1, V^2)$ is defined as $(\varepsilon C)_a = \varepsilon(a) C_a$ for $a \in \overline{Q}_1$. We define a symplectic form ω on $\mathsf{F}^\bullet \mathsf{Rep}(Q, \mathbf{v}, \mathbf{w})$ by

$$\omega((C,i,j),(C',i',j')) = \mathsf{tr}(\varepsilon C C') + \mathsf{tr}(ij' - i'j), \quad \text{where} \quad \mathsf{tr}(A) = \sum_k \mathsf{tr}(A_k).$$

The product $\mathfrak{P}_{\mathbf{v}} := \prod_{\iota \in Q_0} P_{v_\iota}$ of parabolic groups acts on $\mathsf{F}^\bullet \mathsf{Rep}(Q, \mathbf{v}, \mathbf{w})$ via

$$p \circ (C, i, j) = (pCp^{-1}, pi, jp^{-1}),$$

which preserves the symplectic form ω on the filtered ADHM data. The $\mathfrak{P}_{\mathbf{v}}$-action induces the moment map

$$\mu_{\mathfrak{P}_{\mathbf{v}}} : T^* \mathsf{F}^\bullet \mathsf{Rep}(Q, \mathbf{v}, \mathbf{w}) \to \mathsf{Lie}(\mathfrak{P}_{\mathbf{v}})^* = \bigoplus_{\iota \in Q_0} \mathfrak{p}^*_{v_\iota}, \quad \text{where} \quad (C, i, j) \mapsto \varepsilon CC + ij \tag{2.1}$$

and $\mathfrak{p}_{v_\iota} = \mathsf{Lie}(P_{v_\iota})$.

2.4 Geometric invariant theory

The orbit space for a G-action on a variety X may not exist since X/G may not necessarily be an algebraic variety. In order to remedy this, invariant polynomials are used to construct new and interesting varieties called affine quotients; one may think of the affine quotient as being generated by orbit closures (or alternatively, closed orbits). Procesi in [38] proved that invariant polynomials of quiver varieties arise as traces of oriented paths in characteristic zero, and Donkin in [8] and [9] later proved an analogous result in characteristic p.

To produce other interesting (and sometimes projective) varieties, one uses GIT techniques (cf. [32, 37]), where one aims to produce all semi-invariant polynomials for various character of the group. Derksen–Weyman in [6], Schofield–van den Bergh in [40], and Domokos–Zubkov in [7] used long exact sequences of the Ringel resolution, representation-theoretic techniques, and combinatorial techniques, respectively, to give a strategy to produce semi-invariant polynomials for quiver representations.

Recall that given a $\mathfrak{G}$-action on $\mathfrak{X}$, affine and GIT quotients are

$$\mathfrak{X}/\!\!/\mathfrak{G} := \mathsf{Spec}(\mathbb{C}[\mathfrak{X}]^{\mathfrak{G}}) \quad \text{and} \quad \mathfrak{X}/\!\!/_{\chi}\mathfrak{G} := \mathsf{Proj}\left(\bigoplus_{i=0}^{\infty} \mathbb{C}[\mathfrak{X}]^{\mathfrak{G}, \chi^i}\right),$$

respectively, where $\chi : \mathfrak{G} \to \mathbb{C}^*$ is a character of $\mathfrak{G}$. For more detail, see [32, 37].

2.4.1 Hamiltonian reduction of filtered ADHM data

Recall $\mu_{\mathfrak{P}_{\mathbf{v}}}$ defined in (2.1). The locus $\mu_{\mathfrak{P}_{\mathbf{v}}} = 0$ is called **filtered ADHM equation**. We say

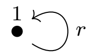
$$\mathfrak{M}_0^{\mathsf{F}^\bullet} = \mathfrak{M}_0^{\mathsf{F}^\bullet}(\mathbf{v}, \mathbf{w}) := \mu_{\mathfrak{P}_{\mathbf{v}}}^{-1}(0)/\!\!/\mathfrak{P}_{\mathbf{v}} = \mathsf{Spec}\left(\mathbb{C}[\mu_{\mathfrak{P}_{\mathbf{v}}}^{-1}(0)]^{\mathfrak{P}_{\mathbf{v}}}\right)$$

is the **filtered affine quotient**, and

$$\mathfrak{M}_\chi^{\mathsf{F}^\bullet} = \mathfrak{M}_\chi^{\mathsf{F}^\bullet}(\mathbf{v},\mathbf{w}) := \mu_{\mathfrak{P}_\mathbf{v}}^{-1}(0)^s/\mathfrak{P}_\mathbf{v} = \mathsf{Proj}\left(\bigoplus_{i\geq 0}\mathbb{C}[\mu_{\mathfrak{P}_\mathbf{v}}^{-1}(0)]^{\mathfrak{P}_\mathbf{v},\chi^i}\right)$$

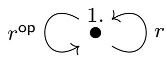
is called a **filtered quiver variety**.

In the case when Q is the Jordan quiver



(2.2)

then the double quiver $\overline{Q}$ is



(2.3)

Let $\mathbf{v} = n$ and $\mathbf{w} = 1$. We impose the complete standard filtration of vector spaces on the representation $V_1 = \mathbb{C}^n$ at vertex 1 to obtain that $\mu_B^{-1}(0)/B \cong T^*(\widetilde{\mathfrak{g}}\times\mathbb{C}^n/\,\mathsf{GL}_n(\mathbb{C}))$ (cf. [36, Prop. 3.2], [18, Prop. 1.1]).

3 (Semi)-invariants of filtered quiver representations

We obtain the following in [16, Thm. 5.1.2]:

Theorem 1 (Im)**.** *Consider a quiver $Q = (Q_0, Q_1)$ of finite Dynkin type with dimension vector $\mathbf{v} = (n,\ldots,n)$. Impose the complete standard filtration of vector spaces on the representation at each vertex of the quiver. Let $\mathfrak{U}$ be the product of maximal unipotent subgroups of the product $\mathfrak{B} = B^{Q_0}$ of standard Borels. Then $\mathbb{C}[\mathfrak{b}^{Q_1}]^{\mathfrak{U}} \cong \mathbb{C}[\mathfrak{t}^{Q_1}]$.*

Theorem 1 tells us that invariant polynomials cannot arise from off-diagonal coordinates of $\mathfrak{b}$. Instead, only the eigenvalues of $\mathfrak{b}^{Q_1}$ remain invariant under the $\mathfrak{U}$-action.

Next, we state [16, Thm. 5.2.12].

Theorem 2 (Im)**.** *Consider an affine quiver $\widetilde{A}_r$ with a framing, and let $\mathbf{v} = (n,\ldots,n,m)$, the dimension vector. Assume the complete standard filtration of vector spaces on the representation at each vertex of $\widetilde{A}_r$ (except at the framed vertex). Let $\mathfrak{U}$ be the product of maximal unipotent subgroups of the product $\mathfrak{B} = B^r$ of standard Borels. Then the invariant subring $\mathbb{C}[\mathfrak{b}^{\oplus r+1}\oplus M_{n\times m}]^{\mathfrak{U}}$ has finitely-many generators.*

We give an explicit description of the subalgebra in Theorem 2, thus generalizing [13, Thm. 13.3], which states that for the maximal unipotent group U of the standard Borel $B \subseteq \mathsf{GL}_n(\mathbb{C})$, the algebra $\mathbb{C}[M_{n\times m}]^U$ is generated by bideterminants of standard Young bitableaux $(D|E)$, where each row of D has the form $p, p+1, \ldots, n$ for a suitable p, $1 \leq p \leq n$. Note that one may associate a bideterminant to a Young bitableau in a natural way, i.e., it is a product of minors, each of which is determined by i-th row of D and i-th row of E.

Also see [17, Thm. 1.1, Thm. 1.3] as further extensions of Theorem 1 and Theorem 2.

4 Hamiltonian reduction of the Borel moment map

Throughout this section, we will restrict to the Jordan quiver. Let $\mathbf{v} = n$ and $\mathbf{w} = 1$. Let $\mathsf{F}^\bullet$ be the complete standard filtration of vector spaces. Then $\mathsf{F}^\bullet \operatorname{\mathsf{Rep}}(Q, n, 1) = T^*(\mathfrak{b} \times \mathbb{C}^n)$, which is associated to the moment map

$$\mu_B : T^*(\mathfrak{b} \times \mathbb{C}^n) \to \mathfrak{b}^*, \quad \text{where} \quad (r, s, i, j) \mapsto [r, s] + ij.$$

Let $\mu_B^{-1}(0)^{\mathsf{rss}}$ be the locus of points whose eigenvalues of r are pairwise distinct. Let $\Delta_n = \{(r_1, \ldots, r_n, 0, \ldots, 0) : r_\iota = r_\gamma \text{ for some } \iota \neq \gamma\}$. Then [18, Thm. 1.5] says that:

Theorem 3 (Im)**.** *The map*

$$P : \mu_B^{-1}(0)^{\mathsf{rss}} \twoheadrightarrow \mathbb{C}^{2n} \setminus \Delta_n \quad \text{given by} \quad (r, s, i, j) \mapsto (r_{11}, \ldots, r_{nn}, s'_{11}, \ldots, s'_{nn})$$

is a regular, well-defined surjection separating B-orbit closures.

Furthermore, we have [18, Thm. 1.6]:

Theorem 4 (Im)**.** *The surjection P in Theorem 3 descends to an isomorphism*

$$\mu_B^{-1}(0)^{\mathsf{rss}} /\!\!/ B \cong \mathbb{C}^{2n} \setminus \Delta_n$$

of algebraic varieties.

We prove that the components of μ_B form a regular sequence using a certain monomial ordering and the resulting initial ideal.

In the process of proving Theorem 3 and Theorem 4, rational B-invariant polynomials appearing as traces of products of matrices were discovered (cf. [16, §6.2]): for $1 \leq \iota \leq n$ and $1 \leq \gamma < \nu \leq n$,

$$\begin{aligned} F_\iota(r, s, i, j) &= \operatorname{\mathsf{tr}}(jL^\iota i), \\ G_\iota(r, s, i, j) &= \operatorname{\mathsf{tr}}(L^\iota s), \\ H_\iota(r, s, i, j) &= \operatorname{\mathsf{tr}}(L^\iota r), \\ K_{\gamma\nu}(r, s, i, j) &= [\operatorname{\mathsf{tr}}((L^\nu - L^\gamma) r)]^{-1}. \end{aligned} \tag{4.1}$$

These rational functions should play an important role in the construction of the affine quotient $\mu_B^{-1}(0) /\!\!/ B$ (see §6.2 for more detail).

5 Hamiltonian reduction of the parabolic moment map

In this section, we continue to work with the Jordan quiver. Let $\mathbf{v} = n$ and $\mathbf{w} = 1$. Here, rather than working with a complete flag as in §4, we work with a partial standard flag

$$\mathsf{F}^\bullet : \mathbb{C}^{\alpha_1} \subseteq \mathbb{C}^{\alpha_1+\alpha_2} \subseteq \ldots \subseteq \mathbb{C}^{\alpha_1+\ldots+\alpha_\ell}, \quad \text{where} \quad \ell \leq 5. \tag{5.1}$$

Then we have [19, Thm. 1.1]:

Theorem 5 (Im–Scrimshaw)**.** *Let $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ such that $\ell \leq 5$. Let P be the parabolic subgroup of $\mathsf{GL}_n(\mathbb{C})$ with block size vector α. The components of μ_P form a complete intersection.*

We also explicitly describe the irreducible components in [19, Thm. 1.2].

6 Open problems

We will now list some research problems related to filtered ADHM data.

6.1 (Semi-)invariant polynomials for a general quiver

As a result of the B-invariant functions in (4.1), let us now focus on the n rational functions L^ι enumerated by $\iota = 1, \ldots, n$. For $l_k(r) = r - r_{kk}\,\mathrm{I}_n$,

$$M^\iota := \left(\mathsf{tr}\left(\prod_{1\leq k\leq n, k\neq\iota} l_k(r)\right)\right) L^\iota = \prod_{1\leq k\leq n, k\neq\iota} l_k(r),$$

which is an operator whose matrix entries are zero except in coordinates (μ, ν) for $\mu \leq \iota$ and $\nu \geq \iota$. One should think of these operators acting on elements in $\mathfrak{b}$ as killing off columns of a matrix, or as creating new coordinate entries from matrices in $\mathfrak{b}^* = \mathfrak{gl}_n/\mathfrak{u}^+$ such that powers of the trace of the product of these matrices give the desired invariant polynomials for the filtered affine quotient setting.

Problem 6. Describe $\mathbb{C}[\mathfrak{b}^{\oplus k}]^{U_n\times U_n}$ for $k \geq 3$, and $\mathbb{C}[\mathfrak{b}^{\oplus \ell}]^{U_n}$ for $\ell \geq 2$.

By making progress on Problem 6, together with Theorem 2, one can then describe the unipotent invariant subring for a filtered ADHM data for *any* quiver Q.

6.2 Affine and GIT quotients

Let Q be the Jordan quiver. By clearing the denominators of the rational functions in (4.1), they are *some* of the generators of $\mathbb{C}[\mu_B^{-1}(0)/\!\!/B]$. More generally, we have:

Problem 7. Find generators and relations for $\mu_{\mathfrak{P}_{\mathbf{v}}}^{-1}(0)/\!\!/\mathfrak{P}_{\mathbf{v}}$ and $\mu_{\mathfrak{P}_{\mathbf{v}}}^{-1}(0)/\!\!/_{\chi}\mathfrak{P}_{\mathbf{v}}$ to describe interesting quotients.

Problem 7 is of interest for constructing new moduli spaces, and it remains an open problem to study the algebro-geometric structure of these quotients for various quivers, filtrations, and stability conditions, i.e., for different characters.

6.3 Birational morphisms

Problem 8. Construct birational morphisms between $\mu_{\mathfrak{P}_{\mathbf{v}}}^{-1}(0)/\!\!/_{\chi}\mathfrak{P}_{\mathbf{v}}$ and $\mu_{\mathfrak{P}_{\mathbf{v}}}^{-1}(0)/\!\!/_{\chi'}\mathfrak{P}_{\mathbf{v}}$ for two characters χ and χ' of $\mathfrak{P}_{\mathbf{v}}$.

Problem 8 is known as variational GIT. Fixing a character of the group and constructing GIT quotients is known as a polarization on the variety, and changing one polarization to another is known as wall-crossing. Since a parabolic group has many more characters than $\mathsf{GL}_n(\mathbb{C})$, this problem now becomes even more interesting than a reductive group setting. Thus what happens when we fix a character of the parabolic group and the phenomenon that arises when we cross a wall? Are the resulting GIT quotients birational or isomorphic?

Even when we restrict the quiver to the Jordan quiver, studying variational GIT and constructing Fourier automorphism, thus giving an isomorphism between certain open loci of GIT quotients, are doable yet a difficult task (one needs to choose various 1-parameter subgroups and show that certain points are unstable with respect to a fixed character).

6.4 Complete intersection for $\mu_P^{-1}(0)$

It remains to prove that when $\ell > 5$ for ℓ in (5.1), the associated parabolic moment map is flat, i.e., $\mu_P^{-1}(0)$ is a complete intersection or equivalently, the components of μ_P form a regular sequence. Furthermore, one should use the well-known fact that μ_P is flat if and only if $\dim \mu_P^{-1}(0)/\!\!/P = 2(\dim(\mathfrak{p} \times \mathbb{C}^n) - \dim \mathfrak{p})$ if and only if

$$\operatorname{codim}\{y \in \mathfrak{p} \times \mathbb{C}^n : \dim G_y = k\} \geq k \quad \text{for all } k \geq 1.$$

Results in [18] coincide with the classical notion that the trace of an oriented cycle of a quiver, as well as the trace of a path that begin and end at a framed vertex, is an invariant function (cf. [4, 23, 26]). These strategies are appliable to Nakajima's affine and quiver varieties. This leads us to believe, together with [10, Proof of Prop. 8.2.1], that $\mathbb{C}[\mu_P^{-1}(0)]^P$ is generated by traces of products of matrices.

Problem 9. The filtered ADHM equation μ_P is a complete intersection for $\ell > 5$.

6.5 Complete intersection for the general quiver

Regular sequences for the filtered ADHM equation play an important role in affine and GIT quotients. A set of polynomials $f_1, \ldots, f_N$ is $\mathbb{C}[x_1, \ldots, x_m]$-**regular** if the scheme defined by the vanishing locus of $f_1, \ldots, f_N$ form a complete intersection. That is, the variety has the expected dimension $m - N$, and in such a case, we say that the components form a **regular sequence**. The affine quotient of a variety is generated by invariant polynomials, and GIT quotients are generated by semi-invariant polynomials. In the case that the components of a moment map form a regular sequence, the affine and the GIT quotients are finitely generated; this means we only need to produce finitely-many invariant and semi-invariant polynomials since the quotients will have the expected dimension.

Problem 10. Describe when components of $\mu_{\mathfrak{P}_\mathsf{v}}$ form a regular sequence.

As mentioned in §2.1.2, connecting the notion of the moment map for filtered quiver representations and Gröbner basis is plausible since there appears to be a direct link between the matrix generators $[r, s] + ij = 0$ and Gröbner basis elements in such a way that Gröbner basis techniques elegantly show us that the variety is a complete intersection for small $n \leq 5$. But it is also very likely that Crawley-Boevey's [4] may provide easier and more direct techniques since $T^*(\mathfrak{b} \times \mathbb{C}^n) \xrightarrow{\mu_B} \mathfrak{b}^*$ can be thought of as a map between filtered quiver representations.

The following results are in [16].

Proposition 11. *Let Q be any finite acyclic quiver and let β be a dimension vector. Then invariant polynomials $I(\mathsf{F}^\bullet \operatorname{Rep}(Q, \beta))$ in the filtered quiver representation space are the constants.*

Proposition 12. *Consider any finite cyclic quiver and fix a dimension vector. Then invariant polynomials in the filtered quiver representation space arise as determinants of oriented paths.*

We will now discuss the semi-invariant setting. At each vertex i, impose the filtration

$$\mathsf{F}_i^\bullet : 0 \subseteq \mathbb{C}^{\gamma_1} \subseteq \mathbb{C}^{\gamma_2} \subseteq \ldots \subseteq \mathbb{C}^{\gamma_k} \subseteq \mathbb{C}^{\beta_i}, \tag{6.1}$$

where the filtration (except the top dimensional vector space) is the same at each vertex. Furthermore, we will restrict to those filtrations and dimension vectors such that if $\mathsf{F}_i^\bullet$ at vertex i (with the top dimensional vector space omitted) is not the same as the filtration $\mathsf{F}_j^\bullet$ at vertex j (with the top dimensional vector space omitted) where $i \neq j$, then $\beta_i = 0$ or $\beta_j = 0$.

For the sake of simplicity, let P be a parabolic group of $\mathsf{GL}_n(\mathbb{C})$ with $\chi(p) = \prod_{k=1}^{l} \det(p_k)^{i_k}$. When the exponent vector of χ lies in the lattice $\{(j, j, \ldots, j) : j \in \mathbb{Z}\}$, we will say that the character χ is a **full power of the determinant**. We will assume (6.1) for Theorems 13 and 14.

Theorem 13 (Im)**.** *The determinants of the diagonal blocks of the representations in acyclic filtered quiver representations generate the semi-invariant coordinate ring.*

Since the dimension vector β_i at vertex i need not be the same as the dimension vector β_j at vertex j, we define the notion of the diagonal blocks by embedding a general representation of $\mathsf{F}^\bullet \mathsf{Rep}(Q, \beta)$ into $\mathfrak{gl}_N$ for some large $N \gg 0$ and reduce our parabolic setting to the reductive group setting by showing an isomorphism of graded rings $\bigoplus_\chi \mathbb{C}[\mathfrak{p}^k]^{P \times P, \chi} \cong \bigoplus_\chi \mathbb{C}[\mathfrak{l}^k]^{L \times L, \chi}$, where L is the Levi subgroup of P and $\mathfrak{l} = \mathsf{Lie}(L)$.

Note that when χ is a full zero power of det, we are basically looking for invariant polynomials since we are taking the character χ to be the trivial character 1, which is essentially the reason to assume in Theorem 13 that the exponent vector of χ varies over all tuples. Standard short exact sequences for quivers are called Ringel resolutions. In the reductive group setting, one applies the contravariant functor $\mathsf{Hom}(-, X)$ to the short exact sequence and then obtains a long exact sequence so that under certain conditions on the weight and the dimension vector, the determinant associated to this long exact sequence gives semi-invariant polynomials.

We now come to one of the most interesting aspects of filtered quiver representations.

Theorem 14 (Im)**.** *Semi-invariants for filtered quiver representations arise as determinants of certain square matrices but with extra paths (including a notion of multiple trivial paths at each vertex) added to the classical case.*

See [16, §2.1.2] or [17] for more detail.

6.6 Comparing filtered ADHM data and quivers with relations

There is a close relation to quivers with relations but examples show that they are not equivalent. It would be interesting to check under what conditions are irreducible components of quivers with relations normal. In the case that components of quivers with relations are normal, the function theory for quivers with relations can be extended to all of the component, and, thus, is comparable to filtered ADHM data.

6.7 Hilbert schemes

Writing $T^*(\mathfrak{b} \times \mathbb{C}^n)$ as $\mathfrak{b} \times \mathfrak{b}^* \times \mathbb{C}^n \times (\mathbb{C}^n)^*$, it contains the set of all quadruples of the form (r, s, i, j), where s takes the form of lower triangular matrices (technically, $\mathfrak{b}^* = \mathfrak{gl}_n/\mathfrak{u}$ where $\mathfrak{u}$ is the set of nilpotent matrices in $\mathfrak{b}$), i is a vector, and j is a covector. Framing means that there is no group action on the vector space at that vertex. Restricting to those points satisfying the relation $[r, s] + ij = 0$ should remind the experts of the Hilbert scheme $(\mathbb{C}^2)^{[n]}$ of n points on the complex plane.

Problem 15. Construct $\mu_P^{-1}(0)/\!\!/_\chi P \twoheadrightarrow \mu_P^{-1}(0)/\!\!/P$ and relate it to the Hilbert–Chow morphism $(\mathbb{C}^2)^{[n]} \twoheadrightarrow \mathbb{C}^{2n}/S_n$, where S_n is a permutation group of n letters.

6.8 Idempotents in filtered quiver representations and quivers with relations

The L^{ι}'s in [18] (also see [16]) are n rational idempotents. They are pairwise orthonormal and their sum equals the identity matrix. These idempotents are all upper triangular matrices whose coordinate are rational functions. In §6.6, we compare filtered ADHM data with quivers with relations. In view of the Borel subalgebra corresponding to the Jordan quiver embedded in a quiver with relations, whose quiver has n vertices and each vertex has a trivial path (also known as idempotents in its path algebra), it is highly likely that these idempotent matrices are directly related to the n idempotents appearing in quivers with relations.

6.9 Modified rational Cherednik algebras

In [10], the authors consider the spherical subalgebra of $\mathfrak{gl}_n$-type and realize it as a quantum Hamiltonian reduction of the algebra $\mathcal{D}(\mathfrak{g})$ of polynomial differential operators on $\mathfrak{g}$. Since studying the Hamiltonian reduction of $\mathcal{D}(\mathfrak{g})$ with respect to P is the same as performing the Hamiltonian reduction of $\mathcal{D}(\mathfrak{g} \times \mathbb{P}^n)$ with respect to G (acting diagonally on $\mathfrak{g} \times \mathbb{P}^n$), this leads us to an analogous investigation for parabolic groups:

Problem 16. Construct a parabolic analog of the Cherednik algebra and realize it as a quantization of $\mathcal{D}(\mathfrak{p})$.

6.10 Parabolic $\mathcal{D}$-modules

In a similar spirit as in [20], we believe that the quantization of the Hilbert scheme associated to $\mu_P^{-1}(0)/\!\!/_\chi P$ may be realized as a microlocalization of a modified rational Cherednik algebra. Furthermore, Gan–Ginzburg construct a Lagrangian subvariety $\mathfrak{M}_{\mathsf{nil}}$ in $\mathfrak{M}$ in the classical case of $\mathsf{GL}_n(\mathbb{C})$, and then studies a category of holonomic $\mathcal{D}$-modules whose characteristic variety is contained in $\mathfrak{M}_{\mathsf{nil}}$.

Problem 17. Construct a category of $\mathcal{D}$-modules whose characteristic variety is contained in the parabolic analogue $\mathfrak{M}^P_{\mathsf{nil}}$ and compare its simple objects to Lusztig's (parabolic) character sheaves (cf. [27, 28]).

6.11 Derived category of coherent sheaves on $\mu_P^{-1}(0)/\!\!/_\chi P$

There is an equivalence between derived categories of coherent sheaves on $(\mathbb{C}^2)^{[n]}$ and finitely-generated modules over a noncommutative crepant resolution of $\mathbb{C}^{2n}/S_n$ (cf. [15]).

Problem 18. Construct an equivalence between derived categories of coherent sheaves on $\mu_P^{-1}(0)/\!\!/_\chi P$ and the category of finitely-generated modules over a modified rational Cherednik algebra.

6.12 Generalized Grothendieck–Springer resolution

It is a classical result in representation theory that the Grothendieck–Springer resolution $\widetilde{\mathfrak{g}} \twoheadrightarrow \mathfrak{g}$, where $(x, \mathfrak{b}) \mapsto x$, is generically finite and dominant of degree $|W|$, where W is the Weyl group.

Problem 19. Extend $\mu_{\mathfrak{P}_{\mathrm{v}}}$ from the Jordan quiver to a general quiver as it is interesting to study the resolution from a filtered quiver variety point of view.

Directly studying the parabolic orbits on the Grothendieck–Springer resolution is very difficult, and surprisingly, no one seems to have studied such geometric objects from the quiver variety point of view.

Filtered ADHM data represent a new technique for studying objects in geometric representation theory and algebraic geometry. They are nice because of their concrete nature and because of many applications in mathematics. Using filtered quiver variety techniques, we hope that new and different insights to important varieties will be identified and discovered.

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References

[1] W. Borho, R. MacPherson, *Représentations des groupes de Weyl et homologie d'intersection pour les variétés nilpotentes*, C. R. Acad. Sci. Paris Sér. I Math. **292** (1981), no. 15, 707–710. MR618892. Zbl 0467.20036. DOI: https://cir.nii.ac.jp/crid/1573387449508494592.

[2] N. Chriss, V. Ginzburg, *Representation theory and complex geometry*, Birkhäuser, Boston, 2010. x+495 pp. MR2838836. Zbl 1185.22001. DOI: https://doi.org/10.1007/978-0-8176-4938-8.

[3] A. Craw, *Quiver flag varieties and multigraded linear series*, Duke Math. J. **156** (2011), no. 3, 469–500. MR2772068. Zbl 1213.14026. DOI: https://doi.org/10.1215/00127094-2010-217.

[4] W. Crawley-Boevey, *Geometry of the moment map for representations of quivers*, Compos. Math. **126** (2001), no. 3, 257–293. MR1834739. Zbl 1037.16007. DOI: https://doi.org/10.1023/A:1017558904030.

[5] W. Crawley-Boevey, *Lectures on Representations of Quivers*, 1992, 1–38. https://www.math.uni-bielefeld.de/~wcrawley/quivlecs.pdf.

[6] H. Derksen, J. Weyman, *Semi-invariants of quivers and saturation for Littlewood–Richardson coefficients*, J. Amer. Math. Soc. **13** (2000), no. 3, 467–479. MR1758750. Zbl 0993.16011. DOI: https://doi.org/10.1090/S0894-0347-00-00331-3.

[7] M. Domokos, A. N. Zubkov, *Semi-invariants of quivers as determinants*, Transform. Groups **6** (2001), no. 1, 9–24. MR1825166. Zbl 0984.16023. DOI: https://doi.org/10.1007/BF01236060.

[8] S. Donkin, *Invariants of several matrices*, Invent. Math. **110** (1992), no. 2, 389–401. MR1185589. Zbl 0826.20036. DOI: https://doi.org/10.1007/BF01231338.

[9] S. Donkin, *Polynomial invariants of representations of quivers*, Comment. Math. Helv. **69** (1994), no. 1, 137–141. MR1259609. Zbl 0816.16015. DOI: https://doi.org/10.1007/BF02564477.

[10] W. L. Gan, V. Ginzburg, *Almost-commuting variety, $\mathscr{D}$-modules, and Cherednik algebras*, Int. Math. Res. Pap. **2006** (2006), Article ID 26439, 1–54. MR2210660. Zbl 1158.14006. DOI: https://doi.org/10.1155/imrp/2006/26439.

[11] V. Ginzburg, *Geometric methods in the representation theory of Hecke algebras and quantum groups*, in Representation Theories and Algebraic Geometry, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. **514** (1998), 127–183. MR1649626. Zbl 1009.17011. DOI: https://doi.org/10.1007/978-94-015-9131-7_4.

[12] V. Ginzburg, *Lectures on Nakajima's quiver varieties*, 2009, 1–40. http://arxiv.org/pdf/0905.0686v2.

[13] F. D. Grosshans, *Algebraic homogeneous spaces and invariant theory*, Lecture Notes in Math., **1673**, Springer-Verlag, Berlin, 1997. vi+148 pp. MR1489234. Zbl 0886.14020. DOI: https://doi.org/10.1007/BFb0093525.

[14] M. Haiman, *Hilbert schemes, polygraphs and the Macdonald positivity conjecture*, J. Amer. Math. Soc. **14** (2001), no. 4, 941–1006. MR1839919. Zbl 1009.14001. DOI: https://doi.org/10.1090/S0894-0347-01-00373-3.

[15] M. Haiman, *Vanishing theorems and character formulas for the Hilbert scheme of points in the plane*, Invent. Math. **149** (2002), no. 2, 371–407. MR1918676. Zbl 1053.14005. DOI: https://doi.org/10.1007/s002220200219.

[16] M.S. Im, *On semi-invariants of filtered representations of quivers and the cotangent bundle of the enhanced Grothendieck–Springer resolution*, Ph.D. thesis, University of Illinois at Urbana–Champaign, ProQuest

LLC, Ann Arbor, MI, 2014. vi+127. MR3312842. Zbl 1607.02209. DOI: https://hdl.handle.net/2142/49392.

[17] M.S. Im, *Semi-invariants of filtered quiver representations with at most two pathways*, Contemp. Math. **835** (2026), 215–229. MR5023831. Zbl 1409.0702. DOI: https://doi.org/10.1090/conm/835.

[18] M.S. Im, *The regular semisimple locus of the affine quotient of the cotangent bundle of the Grothendieck–Springer resolution*, J. Geom. Phys. **132** (2018), 84–98. MR3836769. Zbl 1403.14081. DOI: https://doi.org/10.1016/j.geomphys.2018.05.022.

[19] M.S. Im, T. Scrimshaw, *The regularity of almost-commuting partial Grothendieck–Springer resolutions and parabolic analogs of Calogero–Moser varieties*, J. Lie Theory **31** (2021), no. 1, 127–148. MR4161536. Zbl 1469.14100. DOI: https://www.heldermann.de/JLT/JLT31/JLT311/jlt31007.htm.

[20] M. Kashiwara, R. Rouquier, *Microlocalization of rational Cherednik algebras*, Duke Math. J. **144** (2008), no. 3, 525–573. MR2444305. Zbl 1147.14002. DOI: https://doi.org/10.1215/00127094-2008-043.

[21] M. Khovanov, A. D. Lauda, *A diagrammatic approach to categorification of quantum groups. I*, Represent. Theory **13** (2009), 309–347. MR2525917. Zbl 1188.81117. DOI: https://doi.org/10.1090/S1088-4165-09-00346-X.

[22] M. Khovanov, A. D. Lauda, *A diagrammatic approach to categorification of quantum groups. II*, Trans. Amer. Math. Soc. **363** (2011), no. 5, 2685–2700. MR2763732. Zbl 1214.81113. DOI: https://doi.org/10.1090/S0002-9947-2010-05210-9.

[23] L. Le Bruyn, C. Procesi, *Semisimple representations of quivers*, Trans. Amer. Math. Soc. **317** (1990), no. 2, 585–598. MR958897. Zbl 0693.16018. DOI: https://doi.org/10.1090/S0002-9947-1990-0958897-0.

[24] G. Lusztig, *Canonical bases arising from quantized enveloping algebras*, J. Amer. Math. Soc. **3** (1990), no. 2, 447–498. MR1035415. Zbl 0703.17008. DOI: https://doi.org/10.1090/S0894-0347-1990-1035415-6.

[25] G. Lusztig, *Canonical bases arising from quantized enveloping algebras. II*, Progr. Theoret. Phys. Suppl. **102** (1990), 175–201. MR1182165. Zbl 0776.17012. DOI: https://doi.org/10.1143/PTP.102.175.

[26] G. Lusztig, *On quiver varieties*, Adv. Math. **136** (1998), no. 1, 141–182. MR1623674. Zbl 0915.17008. DOI: https://doi.org/10.1006/aima.1998.1729.

[27] G. Lusztig, *Parabolic character sheaves. I*, Mosc. Math. J. **4** (2004), no. 1, 153–179, 311. MR2074987. Zbl 1102.20030. DOI: https://doi.org/10.17323/1609-4514-2004-4-1-153-179. DOI: https://doi.org/10.17323/1609-4514-2004-4-1-153-179.

[28] G. Lusztig, *Parabolic character sheaves. II*, Mosc. Math. J. **4** (2004), no. 4, 869–896, 981. MR2124170. Zbl 1103.20041. DOI: https://doi.org/10.17323/1609-4514-2004-4-4-869-896.

[29] G. Lusztig, *Quiver varieties and Weyl group actions*, Ann. Inst. Fourier (Grenoble) **50** (2000), no. 2, 461–489. MR1775358. Zbl 0958.20036. DOI: https://doi.org/10.5802/aif.1762.

[30] G. Lusztig, *Quivers, perverse sheaves, and quantized enveloping algebras*, J. Amer. Math. Soc. **4** (1991), no. 2, 365–421. MR1088333. Zbl 0738.17011. DOI: https://doi.org/10.1090/S0894-0347-1991-1088333-2.

[31] G. Lusztig, *Semicanonical bases arising from enveloping algebras*, Adv. Math. **151** (2000), no. 2, 129–139. MR1758244. Zbl 0983.17009. DOI: https://doi.org/10.1006/aima.1999.1873.

[32] D. Mumford, J. Fogarty, F. Kirwan, *Geometric invariant theory*, 3rd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete, **34**, Springer-Verlag, Berlin, 1994. xiv+292 pp. MR1304906. Zbl 0797.14004. DOI: https://link.springer.com/book/9783540569633.

[33] H. Nakajima, *Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras*, Duke Math. J. **76** (1994), no. 2, 365–416. MR1302318. Zbl 0826.17026. DOI: https://doi.org/10.1215/S0012-7094-94-07613-8.

[34] H. Nakajima, *Lectures on Hilbert schemes of points on surfaces*, University Lecture Series, **18**, Amer. Math. Soc., Providence, RI, 1999. xii+132 pp. MR1711344. Zbl 0949.14001. DOI: https://doi.org/10.1090/ulect/018.

[35] H. Nakajima, *Quiver varieties and Kac-Moody algebras*, Duke Math. J. **91** (1998), no. 3, 515–560. MR1604167. Zbl 0970.17017. DOI: https://doi.org/10.1215/S0012-7094-98-09120-7.

[36] T. A. Nevins, *Stability and Hamiltonian reduction for Grothendieck–Springer resolutions*, University of Illinois at Urbana-Champaign, 2011, 1–25. `http://www.math.uiuc.edu/~nevins/papers/b-hamiltonian-reduction-2011-0316.pdf`.

[37] P. E. Newstead, *Geometric invariant theory*, in Moduli spaces and vector bundles, London Math. Soc. Lecture Note Ser. **359** (2009), 99–127. MR2537067. Zbl 1187.14054. DOI: https://doi.org/10.1017/CBO9781139107037.005.

****

[38] C. Procesi, *The invariant theory of* $n \times n$ *matrices*, Adv. Math. **19** (1976), no. 3, 306–381. MR0419491. Zbl 0331.15021. DOI: https://doi.org/10.1016/0001-8708(76)90027-X.

[39] R. Rouquier, *2-Kac-Moody algebras*, 2008, 1–66. http://arxiv.org/pdf/0812.5023v1.

[40] A. Schofield, M. van den Bergh, *Semi-invariants of quivers for arbitrary dimension vectors*, Indag. Math. (N.S.) **12** (2001), no. 1, 125–138. MR1908144. Zbl 1004.16012. DOI: https://doi.org/10.1016/S0019-3577(01)80010-0.

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