Surveys in Mathematics and its Applications

ISSN 1842-6298 (electronic), 1843-7265 (print)
Volume 21 (2026), 219 – 232

KKM MAPS AND LERAY–SCHAUDER TYPE FIXED POINT THEORY

Donal O'Regan

Abstract. In this paper we establish new fixed point theory and Leray-Schauder type alternatives for KKM type maps. The arguments are based on new composition and product properties for KKM type maps and an Urysohn type lemma in completely regular spaces.

1 Introduction

In this paper we begin by establishing new composition rules for KKM maps. We will then use these results in Section 2 to obtain new Leray-Schauder alternatives for KKM type maps in a variety of setting, the most general being in Hausdorff topological vector spaces (see Theorem 15 and Theorem 16). These results are crucial from an application point of view since a Leray–Schauder alternative presents a fixed point result for a map, F say, by considering the natural homotopy (i.e. $t\,F$, $0 \leq t \leq 1$) between F and the zero map. In our new results the map F is a KKM map and the setting considered is very general. In addition from this Leray–Schauder setting we present a new fixed point result for KKM self maps (see Theorem 26). This fixed point result is an extension and a generalization of the Schauder–Tychonoff fixed point theorem for KKM maps in a general setting. The results in this paper are new and extend previous results in the literature (see [1, 2] and the references therein) and in particular we note that the KKM class includes for example the Kakutani maps, the acyclic maps, the maps admissible with respect to Gorniewicz, the approachable maps and the Park maps (see [2, 3, 6]).

Now we describe the KKM maps. Let X be a convex subset of a Hausdorff topological vector space and Y a Hausdorff topological space. If $S, T : X \to 2^Y$ (nonempty subsets of Y) are two set valued maps such that $T(co\,(A)) \subseteq S(A)$ for each finite subset A of X then we call S a generalized KKM mapping w.r.t. T. Now the set valued map $T : X \to 2^Y$ is said to have the KKM property if for any generalized KKM map $S : X \to 2^Y$ w.r.t. T the family $\{\,\overline{S(x)} : \; x \in X\}$

2020 Mathematics Subject Classification: 47H10; 54H25.
Keywords: Fixed points; KKM maps; Leray–Schauder alternatives

has the finite intersection property (i.e. the intersection of each finite subfamily is nonempty). We let

$$KKM(X,Y) = \{T : X \to 2^Y \,|\, T \text{ has the } KKM \text{ property}\}.$$

First we recall the following result [2].

Theorem 1. *Let X be a convex subset of a Hausdorff topological vector space and Y, Z be Hausdorff topological spaces.*
(i). $T \in KKM(X,Y)$ iff $T|_{\triangle} \in KKM(\triangle, Y)$ for each polytope $\triangle$ in X;
(ii). if $T \in KKM(X,Y)$ and $f \in C(Y,Z)$ then $f\,T \in KKM(X,Z)$;
(iii). if Y is a normal space, $\triangle$ a polytope of X and if $T : \triangle \to 2^Y$ is a set valued map such that for each $f \in C(Y,\triangle)$ we have that $f\,T$ has a fixed point in $\triangle$, then $T \in KKM(\triangle, Y)$.

Next we note a fixed point result for KKM maps given in [1]. Recall a nonempty subset W of a Hausdorff topological vector space E is said to be admissible if for any nonempty compact subset K of W and every neighborhood V of 0 in E there exists a continuous map $h : K \to W$ with $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace of E (for example every nonempty convex subset of a locally convex space is admissible).

Theorem 2. *Let X be an admissible convex set in a Hausdorff topological vector space E and $T \in KKM(X,X)$ be a closed compact map. Then T has a fixed point in X.*

From Theorem 1 and Theorem 2 one has the following result.

Theorem 3. *Let X be an admissible convex set in a Hausdorff topological vector space and Z a subset of a Hausdorff topological space. If $T \in KKM(X,Z)$ is a upper semicontinuous compact map with compact values and $f \in C(Z,X)$ then $T\,f$ has a fixed point in Z.*

Proof. Now $T \in KKM(X,Z)$, $f \in C(Z,X)$ and Theorem 1 (ii) implies $f\,T \in KKM(X,X)$. Also $f\,T$ is a compact upper semicontinuous map with compact values (so $f\,F$ is a closed map). Now Theorem 2 guarantees that $f\,T$ has a fixed point in X and consequently $T\,f$ has a fixed point in Z. □

Next we present an analogue of Theorem 1 (ii) for $T\,f$. This composition result, whose proof follows immediately from Theorem 1 and Theorem 3, will be needed later for our general composition rule.

Theorem 4. *(i). Let X be an admissible convex set in a Hausdorff topological vector space, Y a convex set in a Hausdorff topological vector space and Z a normal topological space. If $T \in KKM(X,Z)$ is an upper semicontinuous map with compact values and $f \in C(Y,X)$ then $T\,f \in KKM(Y,Z)$.*

(ii). Let X be an admissible convex set in a Hausdorff topological vector space, Y a convex set in a Hausdorff topological vector space and Z a Hausdorff topological space. Suppose $T \in KKM(X,Z)$ is an upper semicontinuous map with compact values, $f \in C(Y,X)$ and there exists a compact subset K of Z with $T(X) \subseteq K$. Then $T\,f \in KKM(Y,K)$.

Proof. (i). Note $T\,f : Y \to 2^Z$. To establish our result note from Theorem 1 (i), (iii) we just need to show for each polytope $\triangle$ in Y that $g(T\,f)$ has a fixed point in $\triangle$ for any $g \in C(Z,\triangle)$. Note from Theorem 1 (ii) since $T \in KKM(X,Z)$ and $g \in C(Z,\triangle)$ then $g\,T \in KKM(X,\triangle)$. Now from Theorem 3 (note $Z = \triangle$ is compact and $g\,T : X \to 2^{\triangle}$ is an upper semicontinuous compact map with compact values) we have that $(g\,T)\,f$ has a fixed point in $\triangle$.

(ii). Note $T\,f : Y \to 2^K$ and from Theorem 1 (i), (iii) (note K is normal) we just need to show for each polytope $\triangle$ in Y that $g(T\,f)$ has a fixed point in $\triangle$ for any $g \in C(K,\triangle)$. The proof is exactly the same as in (i) with Z replaced by K. $\square$

We next note the following two properties which will be applied throughout this paper. Let C and X be convex subsets of a Hausdorff topological vector space E with $C \subseteq X$ and Y a Hausdorff topological space.

(i). If $T \in KKM(X,Y)$ then $G \equiv T|_C \in KKM(C,Y)$.

This can be seen from Theorem 1 (i). Note $T \in KKM(X,Y)$ so $T|_{\triangle} \in KKM(\triangle,Y)$ for each polytope $\triangle$ in X from Theorem 1 (i). Thus in particular for any polytope $\triangle$ in C we have $T|_{\triangle} \in KKM(\triangle,Y)$ so from Theorem 1 (i) we have $T|_C \in KKM(C,Y)$.

Alternatively we can prove it directly as follows. Let $S : C \to 2^Y$ be a generalized KKM map w.r.t. G i.e. $G(co\,(A)) \subseteq S(A)$ for each finite subset A of C. We must show $\{\,\overline{S(x)} :\ x \in C\}$ has the finite intersection property. To see this let $S^\star : X \to 2^Y$ be given by

$$S^\star(x) = \begin{cases} S(x), & x \in C \\ Y, & x \in X\setminus C. \end{cases}$$

We claim $T(co\,(D)) \subseteq S^\star(D)$ for each finite subset D of X. Now either (a). $x \in C$ for all $x \in D$ or (b). there exists a $y \in D$ with $y \notin C$. Suppose first case (b) occurs. Then since $S^\star(y) = Y$ we have

$$T(co\,(D)) \subseteq Y = S^\star(y) = S^\star(D).$$

It remains to consider case (a). Then since C is convex we have $co\,(D) \subseteq C$ and since $S^\star(z) = S(z)$ for $z \in C$ we have

$$T(co\,(D)) = G(co\,(D)) \subseteq S(D) = S^\star(D).$$

Thus $S^\star : X \to 2^Y$ is a generalized KKM map w.r.t. T. Since $T \in KKM(X,Y)$ then $\{\,\overline{S^\star(x)} :\ x \in X\}$ has the finite intersection property. Now for any finite subset

Ω of C (note $S^\star(z) = S(z)$ for $z \in C$) we have

$$\cap_{x\in\Omega} \overline{S(x)} = \cap_{x\in\Omega} \overline{S^\star(x)} \neq \emptyset,$$

so $G = T|_C \in KKM(C,Y)$.

(ii). If $T \in KKM(X,Y)$, $T(X) \subseteq Z \subseteq Y$ and Z is closed in Y then $T \in KKM(X,Z)$.

Let $S : X \to 2^Z$ be a generalized KKM map w.r.t. T i.e. $T(co\,(A)) \subseteq S(A)$ for each finite subset A of X. We must show $\{\, \overline{S(x)}^Z : x \in X\}$ has the finite intersection property. Note since $S : X \to 2^Y$ is a generalized KKM map w.r.t. T then since $T \in KKM(X,Y)$ we have that $\{\, \overline{S(x)}\, (= \overline{S(x)}^Y) : x \in X\}$ has the finite intersection property. However note for $x \in X$ that

$$\overline{S(x)}^Z = \overline{S(x)}^Y \cap Z = \overline{S(x)}^Y\, (= \overline{S(x)})$$

since Z is closed in Y (note $S(X) \subseteq Z$ so $\overline{S(x)}^Y \subseteq Z$). Thus $\{\, \overline{S(x)}^Z : x \in X\} = \{\, \overline{S(x)}\, (= \overline{S(x)}^Y) : x \in X\}$ has the finite intersection property.

A product rule for the KKM class is unknown. However a product rule is known for most subclasses of KKM maps. As a result in this paper we consider subclasses of KKM maps (for simplicity in notation we call these KKM also) where the following Property 5 holds. We also assume throughout Section 2 (except in Remark 25) that the analogue of Theorem 1, Theorem 4, properties (i) and (ii) above hold (in the relevant setting) for this subclass of KKM maps (see Remark 6).

Property 5. *Let I be a finite set, X be a convex set in a Hausdorff topological vector space and $\{Y_i\}_{i\in I}$ be a family of Hausdorff topological spaces. For each $i \in I$ suppose $T_i \in KKM(X,Y_i)$ is an upper semicontinuous map with compact values and let $T : X \to 2^Y$ (here $Y = \prod_{i\in I} Y_i$) be defined by $T(x) = \prod_{i\in I} T_i(x)$ for $x \in X$. Also assume for each $i \in I$ that there exists a compact set $K_i \subseteq Y_i$ with $T_i(X) \subseteq K_i$. Then $T \in KKM(X,K)$ where $K = \prod_{i\in I} K_i$.*

Remark 6. *Note we also have $T \in KKM(X,Y)$; consider $i \circ T$ where $i : K \to Y$ is the inclusion map. Note Property 5 holds (in the relevant setting) for the usual classes of maps in the literature, for example the maps admissible with respect to Gorniewicz, selectionable maps, approximable maps etc (there are also analogues of Theorem 1, Theorem 4, properties (i) and (ii) above for these maps). Note Theorem 1, Theorem 4, and property (i) are a lot less restrictive for Gorniewicz maps, selectionable maps and approximable maps (i.e., the analogue of Theorem 4 is a lot more general for these maps). Property (ii) above is immediate for the Gorniewicz maps but the set Z is more special for the selectionable and approximable maps. As a result some of the conditions on the spaces can be removed in Section 2 for these subclasses. This will be discussed in a future paper.*

2 Composition and existence results

We consider the subclass of KKM maps (again for simplicity we call these KKM) for the remainder of Section 2 (except Remark 25) and we assume Theorem 1, Theorem 4, properties (i) and (ii) in Section 1 and Property 5 hold for the subclass. We begin by presenting a result for the composition of KKM maps which will be needed later. To prove this composition result we will need the following fixed point result whose proof relies on composition (Theorem 4) and product (Property 5).

Theorem 7. *Let X and Y be subsets in Hausdorff topological vector spaces, $F : X \to 2^Y$ and $G : Y \to 2^X$. Suppose there exists a convex admissible compact subset Ω of X with $G(Y) \subseteq \Omega$ and there exists a convex admissible compact subset K of Y with $F(X) \subseteq K$. Also assume $F \in KKM(\Omega, K)$ and $G \in KKM(K, \Omega)$ are upper semicontinuous maps with compact values. Then FG has a fixed point in Y.*

Proof. Let $P_1 : \Omega\times K \to \Omega$ be the projection of $\Omega\times K$ onto Ω and $P_2 : \Omega\times K \to K$ be the projecton of $\Omega\times K$ onto K. Now Theorem 4 guarantees that $F\,P_1 \in KKM(\Omega\times K, K)$ and $G\,P_2 \in KKM(\Omega\times K, \Omega)$ (recall compact Hausdorff spaces are normal) are upper semicontinuous compact maps with compact values. Let $H : \Omega\times K \to \Omega\times K$ be given by $H(x, y) = G\,P_2(x, y)\times F\,P_1(x, y) = G(y)\times F(x)$ for $(x, y) \in \Omega\times K$. Now Property 5 guarantees that $H \in KKM(\Omega \times K, \Omega \times K)$ is an upper semicontinuous compact map with compact values. Then Theorem 2 guarantees that there exists a $(x, y) \in X \times Y$ with $(x, y) \in G(y) \times F(x)$. □

Remark 8. *If X in Theorem 7 is a convex set and $F \in KKM(X, K)$ (or $F \in KKM(X, Y)$) then (see property (i) and (ii) in Section 1) we have $F \in KKM(\Omega, K)$.*

Now we present our result for the composition of KKM maps whose proof follows immediately from Theorem 7.

Theorem 9. *Let X_0 be a convex subset in a Hausdorff topological vector space, Y_0 be a subset in a Hausdorff topological vector space, Z_0 be a normal topological space, $T : X_0 \to 2^{Y_0}$ and $H : Y_0 \to 2^{Z_0}$. Suppose there exists a convex admissible compact subset K of Y with $T(X_0) \subseteq K$. Also assume $T \in KKM(X_0, K)$ is an upper semicontinuous map with compact values and $H \in KKM(K, Z_0)$ is an upper semicontinuous map with compact values. Then $H\,T \in KKM(X_0, Z_0)$.*

Proof. Note $H\,T : X_0 \to 2^{Z_0}$. From Theorem 1 (i), (iii) (note Z_0 is normal) we just need to show for each polytope $\triangle$ in X that $f(H\,T)$ has a fixed point in $\triangle$ for any $f \in C(Z_0, \triangle)$. Note from Theorem 1 (ii) since $H \in KKM(K, Z_0)$ and $f \in C(Z_0, \triangle)$ that $f\,H \in KKM(K, \triangle)$ is an upper semicontinuous compact map with compact values. Also note $T \in KKM(\triangle, K)$ is an upper semicontinuous compact map with compact values. Then Theorem 7 (with $F = T$, $G = f\,H$, $\Omega = \triangle$, $K = K$ and note $\triangle$ is admissible) guarantees that $(f\,H)\,T$ has a fixed point in $\triangle$. □

Remark 10. *(i). Note if we replace "Z_0 is a normal topological space" with "Z_0 is a Hausdorff topological space" then we can deduce $HT \in KKM(X_0, \Omega)$ (where $\Omega = H(K)$) in Theorem 9. This is immediate since $HT : X_0 \to 2^{\Omega}$ and from Theorem 1 (i), (iii) (note Ω is compact) we just need to show for each polytope $\triangle$ in X that $f(HT)$ has a fixed point in $\triangle$ for any $f \in C(\Omega, \triangle)$. The proof is then exactly the same as the above with Z_0 replaced by Ω.*

(ii). Note in Theorem 9 if $T \in KKM(X_0, Y_0)$ then we have $T \in KKM(X_0, K)$ and if Y_0 is a convex set and $H \in KKM(Y_0, Z_0)$ then $H \in KKM(K, Z_0)$ (see property (i) and (ii) in Section 1).

Next we present a collectively coincidence result whose proof follows immediately from Theorem 2. In this result we need Property 5 to hold when I is an index set (instead of I a finite set).

Theorem 11. *Let I and J be index sets, $\{X_i\}_{i\in I}$ be a family of sets each in a Hausdorff topological vector space E_i and $\{Y_i\}_{i\in J}$ be a family of sets each in a Hausdorff topological vector space Z_i. For each $i \in J$ suppose $F_i : X \equiv \prod_{j\in I} X_j \to 2^{Y_i}$ and there exists a convex campact subset Ω_i of Y_i with $F_i(X) \subseteq \Omega_i$. For each $j \in I$ assume $G_j : Y \equiv \prod_{i\in J} Y_i \to 2^{X_j}$ and there exists a convex campact subset K_j of X_j with $G_j(Y) \subseteq K_j$. Let $K = \prod_{i\in I} K_i$, $\Omega = \prod_{j\in J} \Omega_j$ and for each $i \in J$ suppose $F_i \in KKM(K, \Omega_i)$ is an upper semicontinuous map with compact values and for each $j \in J$ suppose $G_j \in KKM(\Omega, K_j)$ is an upper semicontinuous map with compact values. Also suppose Ω is an admissible subset of $\prod_{j\in J} Z_j$ and K is an admissible subset of $\prod_{i\in I} E_i$. Then there exists a $x \in X$, a $y \in Y$ with $y_j \in F_j(x)$ for all $j \in J$ and $x_i \in G_i(y)$ for all $i \in I$ (here x_i (respectively, y_j) is the projection of x (respectively, y) on X_i (respectively, Y_j)).*

Proof. Let $F(x) = \prod_{j\in J} F_j(x)$ for $x \in K$ and $G(y) = \prod_{i\in I} G_i(y)$ for $y \in \Omega$. Note Property 5 guarantees that $F \in KKM(K, \Omega)$ and $G \in KKM(\Omega, K)$. Now Theorem 9 (with $T = G$, $H = F$, $X_0 = \Omega$, $K = K$ and $Z_0 = \Omega$) guarantees that $FG \in KKM(\Omega, \Omega)$ is an upper semicontinuous map with compact values. Finally Theorem 2 implies that there exists a $y \in \Omega$ with $y \in FG(y)$. Let $x \in G(y)$ with $y \in F(x)$ and the result is now immediate. □

Remark 12. *(i). Note in Theorem 11 if X_i is convex (for each $i \in I$) and for each $j \in J$ we have $F_j \in KKM(X, \Omega_j)$ (or $F_j \in KKM(X, Y_j)$), then $F_j \in KKM(K, \Omega_j)$ (see property (i) and (ii) in Section 1).*

(ii). One could also consider GF instead of FG in the proof of Theorem 11.

We next present some Leray–Schauder alternatives [3] for KKM maps which will then enable us to improve Theorem 2 in certain circumstances. Our proof relies on our composition (Theorem 4) result, Property 5 (I a finite set), an Urysohn type lemma in completely regular spaces and a fixed point result in the literature (Theorem 2).

Theorem 13. *Let E be a normal topological vector space, U an open convex subset of E, $0 \in U$, $\overline{U}$ an admissible subset of E, $\overline{U}$ a retract of E (i.e. there exists a retraction $r : E \to \overline{U}$), $F : \overline{U} \to 2^E$ and there exists a convex compact admissible subset K of E with $0 \in K$ and $F(\overline{U}) \subseteq K$; here $\overline{U}$ denotes the closure of U in E. Also assume $F \in KKM(\overline{U}, E)$ is an upper semicontinuous map with compact values and*

$$x \notin t\,F(x) \quad \text{for} \quad x \in \partial U \quad \text{and} \quad t \in (0,1]; \tag{2.1}$$

here ∂U denotes the boundary of U in E. Then there exists a $x \in U$ with $x \in F(x)$.

Proof. Let

$$\Omega = \left\{x \in \overline{U} : \; x \in \lambda\, F(x) \;\text{ for some }\; \lambda \in [0,1]\right\}.$$

Now $\Omega \neq \emptyset$ (take $\lambda = 0$ and note $0 \in U$) and Ω is closed and compact (note F is an upper semicontinuous compact map with compact values). Also from (2.1) note $\Omega \cap \partial U = \emptyset$. Now since Hausdorff topological vector spaces are completely regular then there exists a continuous map (a Urysohn type map) $\mu : E \to [0,1]$ with $\mu(E\backslash U) = 0$ and $\mu(\Omega) = 1$. Let $N : E \to 2^E$ be given by $N(x) = \mu(x)\, F(r(x))$; in fact note $N : E \to 2^K$ since $F\,r(E) \subseteq F(\overline{U}) \subseteq K$, K is convex and $0 \in K$. We will now show $N \in KKM(E, K)$. Note $F \in KKM(\overline{U}, E)$, $r \in C(E, \overline{U})$ so Theorem 4 (i) (note $\overline{U}$ is admissible and E is a normal space) guarantees that $F\,r \in KKM(E, E)$ is an upper semicontinuous compact map (note $F\,r(E) \subseteq F(\overline{U}) \subseteq K$) with compact values. Let $\Psi : E \to 2^{[0,1]\times E}$ be given by $\Psi(x) = J(x) \times F\,r(x)$, $x \in E$ where $J(x) = \mu(x)$. From Property 5 (with $T_1 = J$, $T_2 = F\,r$, $X = E$, $K_1 = [0,1]$ and $K_2 = K$) we have that $\Psi \in KKM(E, [0,1] \times K)$ is an upper semicontinuous compact (note $\Psi(E) \subseteq [0,1] \times K$) with compact values. Let $p : [0,1] \times K \to K$ be given by $p(t,x) = t\,x$ for $(t,x) \in [0,1] \times K$ (note K is convex and $0 \in K$) and note $p \in C([0,1] \times K, K)$. From Theorem 1 (ii) note $N = \mu\,F\,r = p\,\Psi \in KKM(E, K)$ is an upper semicontinuous compact map (note $[0,1]\,\overline{F(\overline{U})}$ is compact) with compact values. Thus (see property (i) in Section 1) $N \in KKM(K, K)$ is an upper semicontinuous compact map with compact values and Theorem 2 guarantees that there exists a $x \in K$ with $x \in N(x)$. Now there are two cases to consider, $x \notin U$ or $x \in U$. If $x \notin U$ then $\mu(x) = 0$, so $x = 0$ (note $x \in N(x) = \mu(x)\, F(r(x)) = \{0\}$), a contradiction since $0 \in U$. Thus $x \in U$ so $r(x) = x$ and $x \in N(x) = \mu(x)\, F(r(x)) = \mu(x)\, F(x)$. As a result $x \in \Omega$ so $\mu(x) = 1$ and thus $x \in F(x)$. □

In applications E is usually admissible so the above result is more general than what is needed from an application viewpoint. Note if we examine the proof in Theorem 13 we see that we could remove "K is convex, admissible and $0 \in K$" if we assume "E is admissible". To see this let Ω, μ and N be as in the proof of Theorem 13 and note here that $N : E \to 2^E$ (note $N(x) = \mu(x)\, F(r(x))$). We claim $N \in KKM(E, E)$. To see this let Ψ be as in the proof of Theorem 13 and note $\Psi \in KKM(E, [0,1] \times K)$ is an upper semicontinuous compact with compact values.

Let $p : [0,1] \times K \to E$ be given by $p(t,x) = t\,x$ for $(t,x) \in [0,1] \times K$ and note $p \in C([0,1] \times K, E)$. From Theorem 1 (ii) we see that $N = p\,\Psi \in KKM(E,E)$ is an upper semicontinuous compact with compact values. Now apply Theorem 2 and we can deduce our result as in Theorem 13. Thus we have the following.

Theorem 14. *Let E be an admissible normal topological vector space, U an open convex subset of E, $0 \in U$, $\overline{U}$ an admissible subset of E, $\overline{U}$ a retract of E, $F : \overline{U} \to 2^E$ and there exists a compact subset K of E with $F(\overline{U}) \subseteq K$. Also assume $F \in KKM(\overline{U}, E)$ is an upper semicontinuous map with compact values and suppose (2.1) holds. Then F has a fixed point in U.*

It is also of interest to note that we can remove E normal in Theorem 13 and Theorem 14. The idea here is to use Theorem 4 (ii) instead of Theorem 4 (i). To see this let Ω, μ and N be as in the proof of Theorem 13 and note from Theorem 4 (ii) that $F\,r \in KKM(\overline{U}, K)$. Now proceed exactly the same as in the above. Thus we have the following.

Theorem 15. *Let E be a Hausdorff topological vector space, U an open convex subset of E, $0 \in U$, $\overline{U}$ an admissible subset of E, $\overline{U}$ a retract of E, $F : \overline{U} \to 2^E$ and there exists a convex compact admissible subset K of E with $0 \in K$ and $F(\overline{U}) \subseteq K$. Also assume $F \in KKM(\overline{U}, E)$ is an upper semicontinuous map with compact values and suppose (2.1) holds. Then F has a fixed point in U.*

Theorem 16. *Let E be an admissble Hausdorff topological vector space, U an open convex subset of E, $0 \in U$, $\overline{U}$ an admissible subset of E, $\overline{U}$ a retract of E, $F : \overline{U} \to 2^E$ and there exists a compact subset K of E with $F(\overline{U}) \subseteq K$. Also assume $F \in KKM(\overline{U}, E)$ is an upper semicontinuous map with compact values and suppose (2.1) holds. Then F has a fixed point in U.*

Theorem 13 immediately yields the following two results (these are results for maps which have KKM selectors).

Theorem 17. *Let E be a normal topological vector space, U an open convex subset of E, $0 \in U$, $\overline{U}$ an admissible subset of E, $\overline{U}$ a retract of E, $\Phi : \overline{U} \to 2^E$ and there exists a map $F : \overline{U} \to 2^E$ with $F(x) \subseteq \Phi(x)$ for $x \in \overline{U}$, and there exists a convex compact admissible subset K of E with $0 \in K$ and $F(\overline{U}) \subseteq K$ (or alternatively, $\Phi(\overline{U}) \subseteq K$). Also assume $F \in KKM(\overline{U}, E)$ is an upper semicontinuous map with compact values and suppose (2.1) holds (or alternatively, $x \notin t\,\Phi(x)$ for $x \in \partial U$ and $t \in (0,1]$). Then F (so consequently Φ) has a fixed point in U.*

Theorem 18. *Let E be a normal topological vector space, U an open convex subset of E, $0 \in U$, $\overline{U}$ an admissible subset of E, $\overline{U}$ a retract of E, $\Phi : \overline{U} \to 2^E$ and there exists a map $F : \overline{U} \to 2^E$ with $F(x) \subseteq co\,\Phi(x)$ for $x \in \overline{U}$, and there exists a convex compact admissible subset K of E with $0 \in K$ and $F(\overline{U}) \subseteq K$ (or alternatively, $co\,\Phi(\overline{U}) \subseteq K$). Also assume $F \in KKM(\overline{U}, E)$ is an upper semicontinuous map with* *compact values and suppose (2.1) holds (or alternatively, $x \notin t\,co\,\Phi(x)$ for $x \in \partial U$ and $t \in (0,1]$). Then F (so consequently $co\,\Phi$) has a fixed point in U.*

Remark 19. *There are obvious analogues of Theorem 17 and Theorem 18 if we use either Theorem 14, Theorem 15 or Theorem 16.*

Remark 20. *One of the conditions in all the Leray–Schauder alternatives given above is that "$\overline{U}$ is a retract of E". For example if E is a locally convex topological vector space and U is an open convex subset of E then $\overline{U}$ is a retract of E. We either quote Dugundji's extension theorem [3] or alternatively let $r : E \to \overline{U}$ be given by*

$$r(x) = \frac{x}{\max\{1, \eta(x)\}} \quad \textit{for} \quad x \in E$$

where η is the Minkowski functional on $\overline{U}$ i.e. $\eta(x) = \inf\{\alpha : \ x \in \alpha\,\overline{U}\}$.

From Theorem 13 and Remark 20 we have immediately the following result.

Theorem 21. *Let E be a normal locally convex topological vector space, U an open convex subset of E, $0 \in U$, $F : \overline{U} \to 2^E$ and there exists a convex compact subset K of E with $0 \in K$ and $F(\overline{U}) \subseteq K$. Also assume $F \in KKM(\overline{U}, E)$ is an upper semicontinuous map with compact values and suppose (2.1) holds. Then F has a fixed point in U.*

If we use Theorem 14 and Remark 20 we have immediately the following result.

Theorem 22. *Let E be a normal locally convex topological vector space, U an open convex subset of E, $0 \in U$, $F : \overline{U} \to 2^E$ and there exists a compact subset K of E with $F(\overline{U}) \subseteq K$. Also assume $F \in KKM(\overline{U}, E)$ is an upper semicontinuous map with compact values and suppose (2.1) holds. Then F has a fixed point in U.*

If we use Theorem 16 and Remark 20 we have immediately the following result.

Theorem 23. *Let E be a locally convex topological vector space, U an open convex subset of E, $0 \in U$, $F : \overline{U} \to 2^E$ and there exists a compact subset K of E with $F(\overline{U}) \subseteq K$. Also assume $F \in KKM(\overline{U}, E)$ is an upper semicontinuous map with compact values and suppose (2.1) holds. Then F has a fixed point in U.*

Remark 24. *There are also analogues of Theorem 17 and Theorem 18 (and see also Remark 19) when E is a locally convex topological vector space.*

Remark 25. *It is also of interest to present other approaches to obtaining Leray–Schauder alternatives. For example we now present two other proofs of Theorem 21 (and in both proofs we only need K to be a compact subset of E i.e. we do not need $0 \in K$ or K to be convex). However both these proofs rely on the form of the retraction given in Remark 20. A major advantage with the approach below is that*

Property 5 is **not** *needed (so our results hold for the KKM maps defined before Theorem 1).*

(i). Let r be as in Remark 20. Note $F \in KKM(\overline{U}, E)$, $r \in C(E, \overline{U})$ so from Theorem 4 (i) (note E is a normal space and $\overline{U}$ is admissible since a convex subset of a locally convex space is admissible) we have that $F\,r \in KKM(E, E)$ is an upper semicontinuous compact map (note $F\,r(E) \subseteq F(\overline{U}) \subseteq K$) with compact values. Now Theorem 2 (note E is admissible since a locally convex space is admissible) guarantees that there exists a $x \in E$ with $x \in F\,r(x)$. If $x \in E \backslash U$ then $r(x) = \frac{x}{\eta(x)} \in \partial U$ (note $\eta(x) = 1$ if $x \in \partial U$ and $\eta(x) > 1$ if $x \in E \backslash \overline{U}$) and $x \in F\left(\frac{x}{\eta(x)}\right)$ i.e. $\frac{x}{\eta(x)} \in \frac{1}{\eta(x)}\,F\left(\frac{x}{\eta(x)}\right)$. This contradicts (2.1) (here $\lambda = \frac{1}{\eta(x)} \in (0,1]$). Thus $x \in U$ so $x \in F\,r(x) = F(x)$.

In addition we note that we can remove the condition "E is a normal space" if we argue as follows (here we need K to be convex). Note from Theorem 4 (ii) that $F\,r \in KKM(E, K)$ and so $F\,r \in KKM(K, K)$. Now Theorem 2 guarantees that there exists a $x \in E$ with $x \in F\,r(x)$. Now argue as above to obtain the result.

(ii). Let r be as in Remark 20. From Theorem 1 (ii) note $r\,F \in KKM(\overline{U}, \overline{U})$ is an upper semicontinuous compact map (note $r\,F(\overline{U}) \subseteq r\,(K)$) with compact values. Now Theorem 2 guarantees that there exists a $x \in \overline{U}$ with $x \in r\,F(x)$. Now $x = r(y)$ for some $y \in F\,x$. We have two cases to consider, $x \in U$ and $x \in \partial U$. Suppose $x \in \partial U$. Then $\eta(x) = 1$ so $1 = \eta(x) = \eta(r(y)) = \frac{\eta(y)}{\max\{1, \eta(y)\}}$. If $\eta(y) = 1$ then $r(y) = y$ and $x = r(y) = y \in F(x)$, which contradicts (2.1). If $\eta(y) > 1$ then $x = r(y) = \frac{y}{\eta(y)}$ i.e. $x \in \frac{1}{\eta(y)}\,F(x)$, which contradicts (2.1). If $\eta(y) < 1$ then $y \in U$ so $r(y) = y$ and $x = r(y) = y$, a contradiction. It remains to consider the case $x \in U$. Then $\eta(x) < 1$ so $1 > \eta(x) = \eta(r(y)) = \frac{\eta(y)}{\max\{1, \eta(y)\}}$ which implies $\eta(y) < 1$. Thus $r(y) = y$ so $x = r(y) = y \in F(x)$, and we are finished. In addition note in (ii) we do not need E to be a normal space.

We now use Theorem 2, Theorem 4 and Theorem 22 (or Theorem 23) to present a generalization of Theorem 2 of Furi–Pera [4] type. This fixed point result is an extension and a generalization of the Schauder–Tychonoff fixed point theorem and the proof relies on being able to establish a Leray-Schauder alternative for KKM maps (which we established above).

Theorem 26. *Let E be a metrizable locally convex topological vector space, Q a closed convex subset of E, $0 \in Q$, $G : Q \to 2^E$ and there exists a compact subset K of E with $G(Q) \subseteq K$. Also assume $G \in KKM(Q, E)$ is an upper semicontinuous map with compact values and suppose the following condition is satisfied:*

$$\begin{cases} \text{if } \{(x_j, \lambda_j)\}_{j=1}^{\infty} \text{ is a sequence in } \partial Q \times [0,1] \text{ converging} \\ \text{to } (x, \lambda) \text{ with } x \in \lambda\, G(x) \text{ and } 0 \le \lambda < 1, \text{ then} \\ \{\lambda_j\, G(x_j)\} \subseteq Q \text{ for } j \text{ sufficiently large.} \end{cases} \tag{2.2}$$

Then there exists a $x \in Q$ with $x \in G(x)$.

Proof. Let $r : E \to Q$ be a retraction with $r(x) \in \partial Q$ if $x \in E\backslash Q$ (From Dugundji's extension theorem we know there exists a retraction $r : E \to Q$. If $int\,(Q) = \emptyset$ then $\partial Q = Q$ so $r(x) \in Q = \partial Q$ if $z \in E\backslash Q$. Now suppose $int\,(Q) \neq \emptyset$. Without loss of generality assume $0 \in int\,(Q)$. Then we can take the retraction $r : E \to Q$ as

$$r(x) = \frac{x}{\max\{1, \eta(x)\}} \quad \text{for} \quad x \in E$$

where η is the Minkowski functional on Q (i.e. $\eta(x) = \inf\{\alpha > 0 : \; x \in \alpha\, Q\}$) and note $r(x) \in \partial Q$ if $x \in E\backslash Q$). Now let

$$\Omega = \{x \in E : \; x \in G\,r(x)\}\,.$$

Note $G \in KKM(Q,E)$, $r \in C(E,Q)$ and Theorem 4 (i) (note Q is admissible and E is a normal space) guarantees that $G\,r \in KKM(E,E)$ is an upper semicontinuous compact map (note $G\,r(E) \subseteq G(Q) \subseteq K$) with compact values. Now Theorem 2 guarantees that there exists a $x \in E$ with $x \in G\,r(x)$ so $\Omega \neq \emptyset$. Also note Ω is closed and compact. We claim $\Omega \cap Q \neq \emptyset$. To show this we argue by contradiction. Suppose $\Omega \cap Q = \emptyset$. Then since Ω is compact and Q is closed there exists a $\delta > 0$ with $dist\,(Q,\Omega) > \delta$. Choose $m \in \{1, 2, ...\}$ with $1 < \delta\, m$ and let

$$U_i = \left\{x \in E : \; d(x,Q) < \frac{1}{i}\right\} \quad \text{for} \quad i \in \{m, m+1, ...\};$$

here d is the metric associated with E. Fix $i \in \{m, m+1, ...\}$ and since $dist\,(Q,\Omega) > \delta$ we have that $\Omega \cap \overline{U_i} = \emptyset$. Now Theorem 22 (with $F = G\,r$, $U = U_i$, note $F(\overline{U_i}) = G\,r(\overline{U_i}) \subseteq G\,r(E) \subseteq G(Q) \subseteq K$, $G\,r \in KKM(E,E)$ so $G\,r \in KKM(\overline{U_i}, E)$) guarantees that there exists $\lambda_i \in (0,1)$ and $y_i \in \partial U_i$ with $y_i \in \lambda_i\, G\,r(y_i)$. Since $y_i \in \partial U_i$ we have $\{\lambda_i\, Gr(y_i)\} \not\subseteq Q$ for $i \in \{m, m+1, ...\}$ and so

$$\{\lambda_i\, G\,r(y_i)\} \not\subseteq Q \quad \text{for} \quad i \in \{m, m+1, ...\}. \tag{2.3}$$

Let

$$D = \{x \in E : \; x \in \lambda\, G\,r(x) \quad \text{for some} \quad \lambda \in [0,1]\}.$$

Now $D \neq \emptyset$ (see Theorem 2 and take $\lambda = 1$) and D is closed and compact. This together with

$$d(y_j, Q) = \frac{1}{j} \quad \text{and} \quad |\lambda_j| \leq 1 \quad \text{for} \quad j \in \{m, m+1, ...\}$$

implies that we may assume without loss of generality that $\lambda_j \to \lambda^\star \in [0,1]$ and $y_j \to y^\star \in \partial Q$. Since $G\,r$ is upper semicontinuous with compact values and $y_j \in \lambda_j\, G\,r(y_j)$ for $j \in \{m, m+1, ...\}$ then $y^\star \in \lambda^\star\, Gr(y^\star)$. Then since $r(y^\star) = y^\star$ we have $y^\star \in \lambda^\star G(y^\star)$. If $\lambda^\star = 1$ then $y^\star \in G(y^\star)$, which contradicts $\Omega \cap Q = \emptyset$. Thus $0 \le \lambda^\star < 1$. Now (2.2) with $x_j = r(y_j)$ (note $y_j \in \partial U_j$ so $r(y_j) \in \partial Q$) and $x = y^\star = r(y^\star)$ and $y^\star \in \lambda^\star G(y^\star)$ implies

$$\{\lambda_j \, G\, r(y_j)\} \subseteq Q \quad \text{for} \quad j \quad \text{sufficiently large}.$$

This contradicts (2.3). Thus $\Omega \cap Q \neq \emptyset$, so there exists an $x \in Q$ with $x \in Gr(x) = G(x)$. □

Remark 27. *There is an analogue of Theorem 26 if E is a metrizable topological vector space if say we use Theorem 16. We just need to assume in addition that (i). there exists a retraction $r : E \to Q$ with $r(x) \in \partial Q$ if $x \in E \backslash Q$, (ii). Q, E and $\overline{U_i}$ (for each i) are admissible subsets of E, and (iii). $\overline{U_i}$ (for each i) is a retract of E. The proof is exactly the same as in Theorem 26 except we apply Theorem 16 instead of Theorem 22. One could also consider Theorem 15 here if in addition to (i), (ii), (iii) we have $0 \in K$, K is convex and E admissible in (ii) is replaced by K admissible. We just need to note that $G\, r \in KKM(K, K)$ in the above proof.*

Theorem 26 immediately yields the following results (and there is also an anologue of Remark 27 in each situation).

Theorem 28. *Let E be a metrizable locally convex topological vector space, Q a closed convex subset of E, $0 \in Q$, $\Phi : Q \to 2^E$ and there exists a map $G : Q \to 2^E$ with $G(x) \subseteq \Phi(x)$ for $x \in Q$ and there exists a compact subset K of E with $G(Q) \subseteq K$ (or alternatively, $\Phi(Q) \subseteq K$). Also assume $G \in KKM(Q, E)$ is an upper semicontinuous map with compact values and suppose (2.2) holds. Then there exists a $x \in Q$ with $x \in G(x) \subseteq \Phi(x)$.*

Remark 29. *In Theorem 28 once could replace (2.2) with*

$$\begin{cases} \textit{if} \ \ \{(x_j, \lambda_j)\}_{j=1}^{\infty} \ \ \textit{is a sequence in} \ \ \partial Q \times [0,1] \ \ \textit{converging} \\ \textit{to} \ \ (x, \lambda) \ \ \textit{with} \ \ x \in \lambda\, \Phi(x) \ \ \textit{and} \ \ 0 \le \lambda < 1, \ \ \textit{then} \\ \{\lambda_j\, \Phi(x_j)\} \subseteq Q \ \ \textit{for} \ \ j \ \ \textit{sufficiently large}. \end{cases}$$

Theorem 30. *Let E be a metrizable locally convex topological vector space, Q a closed convex subset of E, $0 \in Q$, $\Phi : Q \to 2^E$ and there exists a map $G : Q \to 2^E$ with $G(x) \subseteq co\, \Phi(x)$ for $x \in Q$ and there exists a compact subset K of E with $G(Q) \subseteq K$ (or alternatively, $co\, \Phi(Q) \subseteq K$). Also assume $G \in KKM(Q, E)$ is an upper semicontinuous map with compact values and suppose (2.2) holds. Then there exists a $x \in Q$ with $x \in G(x) \subseteq co\, \Phi(x)$.*

Remark 31. *In Theorem 30 once could replace (2.2) with*

$$\begin{cases} \textit{if} \ \ \{(x_j, \lambda_j)\}_{j=1}^{\infty} \ \ \textit{is a sequence in} \ \ \partial Q \times [0,1] \ \ \textit{converging} \\ \textit{to} \ \ (x, \lambda) \ \ \textit{with} \ \ x \in \lambda\, co\, \Phi(x) \ \ \textit{and} \ \ 0 \le \lambda < 1, \ \ \textit{then} \\ \{\lambda_j\, co\, \Phi(x_j)\} \subseteq Q \ \ \textit{for} \ \ j \ \ \textit{sufficiently large}. \end{cases}$$

Declaration.
Ethical Approval: Not Applicable.
Competing Interests: The author declares no conflict of interest.
Authors Contribution: Not Applicable.
Funding: Not Applicable.
Availability of Data and Matherials: Not Applicable.

References

[1] T.H. Chang, Y.Y. Huang, J.C. Jeng, *Fixed point theorems for multifunctions in S–KKM class*, Nonlinear Analysis. **44**(2001), 1007–1017. MR1830856. Zbl 0986.47045. DOI: https://doi.org/10.1016/s0362-546x(99)00318-1.

[2] T.H. Chang, C.L. Yen, *KKM property and fixed point theorems*, Jour. Math. Anal. Appl. **203**(1996), 224–235. MR1412490. Zbl 0883.47067. DOI: https://doi.org/10.1006/jmaa.1996.0376.

[3] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer-Verlag. New York, 2003. MR1987179. Zbl 1025.47002. DOI: https://doi.org/10.1007/978-0-387-21593-8.

[4] M. Furi, P. Pera, *A continuation method on locally convex spaces and applications to ordinary differential equations on noncompact intervals*, Annales Polonici Mathematici. **47**(1987), 331-346. MR0927581. Zbl 0656.47052. DOI: https://doi.org/10.4064/ap-47-3-331-346.

[5] D. O'Regan, *Generalized Leray-Schauder principles for general classes of maps in completely regular topological spaces*, Applicable Analysis. **93**(2014), 1674–1690. MR3219676. Zbl 1297.47063. DOI: https://doi.org/10.1080/00036811.2013.842231.

[6] D. O'Regan, *A homotopy approach for maps with a continuous inclusion property*, Topology and its Applications. **368**(2025), Paper No. 109384, 11pp. MR4891677. Zbl 08037426. DOI: https://doi.org/10.1016/j.topol.2025.109384.

Donal O'Regan, ORCID: https://orcid.org/0000-0002-4096-1469
School of Mathematical and Statistical Sciences,
University of Galway,
Galway, Ireland.
e-mail: donal.oregan@nuigalway.ie

License

Received: July 08, 2026; Accepted: September 21, 2026
Published electronically: September 24, 2026