

HARMONIC FUNCTIONS WITH POSITIVE COEFFICIENTS INVOLVING TWO-PARAMETER MITTAG-LEFFLER TYPE POISSON DISTRIBUTION

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Abstract. Making use of a special case of the two-parameter Mittag-Leffler type Poisson distribution, we introduce a new class of complex valued harmonic functions which are orientation preserving and univalent in the open unit disc and obtained the coefficient bounds. In addition to the distortion results and extreme points, there is a thorough discussion of inclusion relations, integral means, convolution properties, and partial sum results for this generalized class of functions.

1 Introduction and preliminaries

The study of the geometric behaviour of analytic functions has become one of the most interesting topics in recent years. The main objective in geometric function theory is to study and characterize the properties of analytic functions through geometrical and topological methods. One seeks to connect the analytical properties of functions with geometrical and topological insights, providing a deeper understanding of how analytic functions behave. Now we introduce some basics in geometric function theory and subclasses of analytic and harmonic functions that are in the scope of this study.

Let be $\mathcal{A}$ the class of analytic(holomorphic) in the open unit disk $\mathbf{U} = \{z : |z| < 1\}$ functions $h(z)$, normalized by $h(0) = 0 = h'(0) - 1$ of the form

$$h(z) = z + a_2 z^2 + a_3 z^3 + \dots + a_n z^n + \dots = z + \sum_{n \geq 2} a_n z^n \quad (1.1)$$

and $\mathcal{S}$ denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathbf{U}$. Also, let $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ denote the well known subclasses of $\mathcal{A}$ consisting of functions which are,

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respectively, starlike and convex functions of order $\alpha \in [0, 1)$ in the open unit disk $\mathbf{U}$. By definition, we have (see for details, [2, 13])

$$\mathcal{S}^*(\varpi) = \left\{ h \in \mathcal{S} : \Re \left(\frac{zh'(z)}{h(z)} \right) > \alpha, z \in \mathbf{U}; \alpha \in [0, 1) \right\},$$

and

$$\mathcal{C}(\alpha) = \left\{ f \in \mathcal{S} : \Re \left(1 + \frac{zh''(z)}{h'(z)} \right) > \alpha, z \in \mathbf{U}; \alpha \in [0, 1) \right\}.$$

We recall the following two subclasses of analytic functions with positive coefficients $\mathcal{V}$, defined and studied by Uralegaddi et al., [39] extensively.

A function $f \in \mathcal{V}$ is said to be starlike of order ϖ ($0 \leq \varpi < \frac{4}{3}$), if and only if

$$\Re \left(\frac{zh'(z)}{h(z)} \right) < \varpi \quad (z \in \mathbf{U}).$$

This function class is denoted by $\mathcal{VS}^*(\varpi)$.

A function $f \in \mathcal{V}$ is said to be convex of order ϖ ($0 \leq \varpi < \frac{4}{3}$) if and only if

$$\Re \left(1 + \frac{zh''(z)}{h'(z)} \right) < \varpi \quad (z \in \mathbf{U}).$$

This class is denoted by $\mathcal{VK}(\varpi)$. It is an established fact that $f \in \mathcal{VK}(\varpi) \iff zh' \in \mathcal{VS}^*(\varpi)$. For functions $h \in \mathcal{A}$ given by (1.1) and $\mathbf{h} \in \mathcal{A}$ given by $\mathbf{h}(z) = z + \sum_{n \geq 2} c_n z^n$, the Hadamard product (or convolution) of h and g is given by

$$(h * \mathbf{h})(z) = z + \sum_{n \geq 2} a_n c_n z^n, \quad z \in \mathbf{U}.$$

A new class $\mathcal{T}$ of analytic functions of the form $h(z) = z - \sum_{n \geq 2} |a_n| z^n$ was introduced by Silverman [30] in 1975. This opened up a new line of research in the theory of univalent functions as well as harmonic functions with negative coefficients [31]. Similar categories were introduced by Uralegaddi et al. [39] of convex, star-like functions with positive coefficients and provided a fresh, intriguing line of investigation. Actually, they took into account functions with positive coefficients as opposed to those with negative real numbers.

1.1 Harmonic univalent functions- $\mathcal{H}$:

A continuous function $f = u + iv$ is a complex-valued harmonic function in a complex domain $\mathcal{G}$ if both u and v are real and harmonic in $\mathcal{G}$. In any simply-connected domain $\mathcal{D} \subset \mathcal{G}$, we can write $f = h + \bar{g}$, where h and g are analytic in $\mathcal{D}$. We call h the analytic part and g the co-analytic part of f . A necessary and sufficient condition

for f to be locally univalent and orientation preserving in $\mathcal{D}$ is that $|h'(z)| > |g'(z)|$ in $\mathcal{D}$ (see [5]).

Denote by $\mathcal{H}$ the family of functions

$$f = h + \bar{g} \quad (1.2)$$

which are harmonic, univalent and orientation preserving in the open unit disc so that f is normalized by $f(0) = f_z(0) - 1 = 0$. Thus, for $f = h + \bar{g} \in \mathcal{H}$, the functions h and g are analytic in $\mathbf{U}$ can be expressed in the following forms:

$$h(z) = z + \sum_{n \geq 2} a_n z^n, \quad g(z) = \sum_{n \geq 1} b_n z^n \quad (0 \leq b_1 < 1),$$

and $f(z)$ is then given by

$$f(z) = z + \sum_{n \geq 2} a_n z^n + \sum_{n \geq 1} \overline{b_n z^n} \quad (0 \leq |b_1| < 1). \quad (1.3)$$

We note that the family $\mathcal{H}$ of orientation preserving, normalized harmonic univalent functions reduces to the well known class $\mathcal{S}$ of normalized univalent functions if the co-analytic part of f is identically zero, i.e. $g \equiv 0$.

For functions $f \in \mathcal{H}$ given by (1.3) and $F \in \mathcal{H}$ given by

$$F(z) = H(z) + \overline{G(z)} = z + \sum_{n \geq 2} A_n z^n + \sum_{n \geq 1} \overline{B_n z^n}, \quad (1.4)$$

we recall the Hadamard product (or convolution) of f and F by

$$(f * F)(z) = z + \sum_{n \geq 2} a_n A_n z^n + \sum_{n \geq 1} \overline{b_n B_n z^n} \quad (z \in \mathbf{U}). \quad (1.5)$$

1.2 Two-parameter Mittag-Leffler- type Poisson distribution MLPD

Influenced by Srivastava's methodology of embedding Pochhammer symbols into Mittag-Leffler types, the special case of the two-parameter Mittag-Leffler function $E_{\nu, \wp}(z)$ and its much more general multi-parameter ${}_p\Psi_q(z)$ have been and are being thoroughly studied by Srivastava et al., [37], and Miller and Ross [18], who proposed the special function as the foundation for the solution of the fractional order initial value problem, $E_{\nu, \wp}(z) = z^\nu e^{\wp z} \Upsilon^*(\nu, \wp z)$, where Υ^* is the incomplete gamma function [20, 21, 37]. It is simple to write the particular case of the two-parameter Mittag-Leffler function (p. 314, [18]) using the characteristics of incomplete gamma functions as

$$\mathbf{E}_{\nu, \wp}(z) := z^\nu \sum_{n=0}^{\infty} \frac{(\wp z)^n}{\Gamma(n + \nu + 1)}, \quad \nu, \wp, z \in \mathbb{C}, \text{ with } \operatorname{Re} \nu > 0, \operatorname{Re} \wp > 0 \quad (1.6)$$

which can be stated as

$$\mathbf{E}_{\nu, \wp}(z) \equiv z^\nu \mathbf{E}_{1, 1+\nu}(\wp z)$$

where $\mathbf{E}_{1, 1+\nu}(\wp z)$ is the Mittag-Leffler function of two parameters [41]. Srivastava et al., [37] and Eker and Ece [38], shown that for $\wp > 0$ and if $\nu > 2\wp - 1$ then normalized form of $\mathbf{E}_{\nu, \wp}$ is univalent and starlike in $\mathbb{D}_{\frac{1}{2}} = \{z \in \mathbb{C} : |z| < \frac{1}{2}\}$. More over it has been proven that if $\nu > (2 + \sqrt{2})\wp - 1$ then normalized form of $\mathbf{E}_{\nu, \wp}$ is univalent and convex in $\mathbb{D}_{\frac{1}{2}}$ for more details refer to [1, 2, 18, 37, 38] and references cited therein.

In Geometric Function Theory the elementary distributions such as the Pascal, Poisson, logarithmic, binomial and beta negative binomial have been partially studied from theoretical point of view (for detailed study we refer the readers to see [3, 12, 24, 29, 42]). Srivastava pioneered mathematical tools that link special functions to probability distributions (see [37]), which led later researchers to formulate the probability mass function of the special case of the two-parameter Mittag-Leffler function Poisson distribution as given below:

$$\mathfrak{P}_{\nu, \wp}(m, k) := \frac{(m\wp)^k m^\nu}{\mathbf{E}_{\nu, \wp}(m)\Gamma(\nu + k + 1)}, \quad k = 0, 1, 2, 3, \dots \quad (1.7)$$

where $\nu > -1, \wp > 0$ and $\mathbf{E}_{\nu, \wp}$ is as given in (1.6). The normalized form of the two-parameter Mittag-Leffler- type Poisson distribution **MLPD**[37], is given by

$$\mathfrak{M}_{\nu, \wp}^{m, n}(z) = z + \sum_{n=2}^{\infty} \frac{(m\wp)^{n-1} m^\nu}{\mathbf{E}_{\nu, \wp}(m)\Gamma(\nu + n)} z^n$$

also

$$\mathfrak{T}_{\nu, \wp}^{m, n}(z) = 2z - \mathfrak{M}_{\nu, \wp}^{m, n}(z) = z - \sum_{n=2}^{\infty} \frac{(m\wp)^{n-1} m^\nu}{\mathbf{E}_{\nu, \wp}(m)\Gamma(\nu + n)} z^n.$$

1.3 Two-parameter Mittag-Leffler- type Poisson distribution MLPD on $\mathcal{H}$

For $h \in \mathcal{A}$ by the definition of Hadamard product (convolution) of two series, we consider the following operator (see [1, 2, 18, 37, 38]):

$$\mathfrak{m}_{\nu, \wp}^{m, n} h(z) = h(z) * \mathfrak{M}_{\nu, \wp}^{m, n}(z) = z + \sum_{n=2}^{\infty} \Upsilon_n a_n z^n, \quad (1.8)$$

where

$$\Upsilon_n = \frac{(m\wp)^{n-1} m^\nu}{\mathbf{E}_{\nu, \wp}(m)\Gamma(\nu + n)}. \quad (1.9)$$

Now, for functions $f \in \mathcal{H}$ by convolution on harmonic functions, we have

$$\mathfrak{m}_{\nu, \wp}^{m,n} f(z) = \mathfrak{m}_{\nu, \wp}^{m,n} h(z) + \overline{\mathfrak{m}_{\nu, \wp}^{m,n} g(z)} \quad (1.10)$$

and inspired by earlier studies on class of harmonic functions involving special functions and derivative operators [6, 19, 20, 21, 22], in this article we define the two-parameter Mittag-Leffler-type Poisson distribution **MLPD** operator on harmonic functions as below

$$\mathfrak{D}_{\mathcal{H}}(\mathfrak{m}_{\nu, \wp}^{m,n} f(z)) = z(\mathfrak{m}_{\nu, \wp}^{m,n} h(z))' - \overline{z(\mathfrak{m}_{\nu, \wp}^{m,n} g(z))'} \quad (1.11)$$

where

$$\mathfrak{m}_{\nu, \wp}^{m,n} h(z) = z + \sum_{n \geq 2} \Upsilon_n a_n z^n \quad \text{and} \quad \mathfrak{m}_{\nu, \wp}^{m,n} g(z) = \sum_{n \geq 1} \Upsilon_n b_n z^n. \quad (1.12)$$

Many researchers were inspired by Uralegaddi et al.'s original work (see [8, 9, 10, 27, 38]) presented and examined a number of novel subclasses of analytic functions with positive coefficients involving special functions; however, the literature has not yet examined analogous results for harmonic univalent functions related to the special case of the two-parameter Mittag-Leffler type Poisson distribution **MLPD**. In this article we attempt to fill this gap by introducing a new subclass of harmonic univalent functions with positive coefficients related to **MLPD**.

Denote by $\mathcal{V}_{\mathcal{H}}$ the subfamily of $\mathcal{H}$ consisting of harmonic functions $f = h + \bar{g}$ of the form

$$f(z) = z + \sum_{n \geq 2} a_n z^n - \overline{\sum_{n \geq 1} b_n z^n} \quad (a_n \geq 0; b_n \geq 0 \text{ and } 0 \leq |b_1| < 1). \quad (1.13)$$

Inspired by previous research on harmonic functions with differential operators and special functions (see [6, 14, 15, 19, 20, 21, 22, 28],) now we define the following subclass of harmonic functions $f \in \mathcal{V}_{\mathcal{H}}$ related to the **MLPD** as below:

For $1 < \varpi \leq \frac{4}{3}$ we let, $\mathcal{P}_{\mathcal{H}}^v(\wp, \varpi)$ a new subclass of $\mathcal{H}$, consist of all functions of the form (1.3) satisfying the condition

$$\Re \left(\frac{\mathfrak{D}_{\mathcal{H}}(\mathfrak{m}_{\nu, \wp}^{m,n} f(z))}{\mathfrak{m}_{\nu, \wp}^{m,n} f(z)} \right) < \varpi \quad (1.14)$$

where $\mathfrak{D}_{\mathcal{H}}(\mathfrak{m}_{\nu, \wp}^{m,n} f(z))$ is given by (1.11). Also let, $\mathcal{V}_{\mathcal{H}}^v(\wp, \varpi) = \mathcal{P}_{\mathcal{H}}^v(\wp, \varpi) \cap \mathcal{V}_{\mathcal{H}}$. Additionally, using (1.3), we derive an adequate coefficient condition for functions $f \in \mathcal{H}$, which is also demonstrated that $f \in \mathcal{V}_{\mathcal{H}}$, the class of harmonic functions with positive coefficients, require this coefficient necessary. In addition to the distortion results and extreme points, there is a thorough discussion of inclusion relations, integral means, convolution properties, and partial sum results.

2 Coefficient bounds

In our first theorem, we obtain a sufficient coefficient condition for harmonic functions in $\mathcal{P}_{\mathcal{H}}^v(\wp, \varpi)$.

Theorem 1. Let $f = h + \bar{g}$ be given by (1.3). If

$$\sum_{n \geq 2} \Upsilon_n \frac{(n - \varpi)}{\varpi - 1} |a_n| + \sum_{n \geq 1} \Upsilon_n \frac{(n + \varpi)}{\varpi - 1} |b_n| \leq 1. \quad (2.1)$$

where $a_1 = 1$ and $1 < \varpi \leq \frac{4}{3}$, then $f \in \mathcal{P}_{\mathcal{H}}^v(\wp, \varpi)$.

Proof. We let (2.1) holds for the coefficients of $f = h + \bar{g}$. It suffices to show that,

$$\left| \frac{\frac{A(z)}{B(z)} - 1}{\frac{A(z)}{B(z)} - (2\varpi - 1)} \right| < 1, z \in \mathbf{U} \quad (2.2)$$

where

$$A(z) = z(\mathbf{m}_{\nu, \wp}^{m, n} h(z))' - \overline{z(\mathbf{m}_{\nu, \wp}^{m, n} g(z))'} = z + \sum_{n \geq 2} n \Upsilon_n a_n z^n - \sum_{n \geq 1} n \Upsilon_n \bar{b}_n \bar{z}^n \quad (2.3)$$

and

$$B(z) = \mathbf{m}_{\nu, \wp}^{m, n} f(z) = z + \sum_{n \geq 2} \Upsilon_n a_n z^n + \sum_{n \geq 1} \Upsilon_n \bar{b}_n \bar{z}^n. \quad (2.4)$$

Substituting for $A(z)$ and $B(z)$ in (2.2), we get

$$\begin{aligned} & \left| \frac{\frac{A(z)}{B(z)} - 1}{\frac{A(z)}{B(z)} - (2\varpi - 1)} \right| \\ & \leq \frac{\sum_{n \geq 2} \Upsilon_n (n-1) |a_n| |z|^{n-1} - \sum_{n \geq 1} \Upsilon_n (n+1) |b_n| |z|^{n-1}}{2(\varpi - 1) - \sum_{n \geq 2} \Upsilon_n [n - (2\varpi - 1)] |a_n| |z|^{n-1} - \sum_{n \geq 1} \Upsilon_n [n + (2\varpi - 1)] |b_n| |z|^{n-1}} \\ & \leq \frac{\sum_{n \geq 2} \Upsilon_n (n-1) |a_n| - \sum_{n \geq 1} \Upsilon_n (n+1) |b_n|}{2(\varpi - 1) - \sum_{n \geq 2} \Upsilon_n [n - (2\varpi - 1)] |a_n| - \sum_{n \geq 1} \Upsilon_n [n + (2\varpi - 1)] |b_n|}. \end{aligned}$$

The last expression is bounded by 1, if

$$2 \sum_{n \geq 2} \Upsilon_n (n-1) |a_n| + 2 \sum_{n \geq 1} \Upsilon_n (n+1) |b_n| \leq 2(\varpi - 1) - \sum_{n \geq 2} \Upsilon_n [n - (2\varpi - 1)] |a_n| - \sum_{n \geq 1} \Upsilon_n [n + (2\varpi - 1)] |b_n|,$$

which is equivalent to

$$\sum_{n \geq 2} \Upsilon_n \frac{(n - \varpi)}{\varpi - 1} |a_n| + \sum_{n \geq 1} \Upsilon_n \frac{(n + \varpi)}{\varpi - 1} |b_n| \leq 1.$$

But (2.1) is true by hypothesis. Hence, $\left| \frac{\frac{A(z)}{B(z)} - 1}{\frac{A(z)}{B(z)} - (2\varpi - 1)} \right| < 1, z \in \mathbf{U}$ and the theorem is proved for $A(z)$ and $B(z)$ is given by (2.3) and (2.4) respectively. $\square$

Theorem 2. For $a_1 = 1$ and $1 < \varpi \leq \frac{4}{3}$, $f = h + \bar{g} \in \mathcal{V}_{\mathcal{H}}^{\nu}(\wp, \varpi)$ if and only if

$$\sum_{n \geq 2} \frac{(n - \varpi) \Upsilon_n}{\varpi - 1} |a_n| + \sum_{n \geq 1} \frac{(n + \varpi) \Upsilon_n}{\varpi - 1} |b_n| \leq 1. \quad (2.5)$$

Proof. Since $\mathcal{V}_{\mathcal{H}}^{\nu}(\wp, \varpi) \subset \mathcal{P}_{\mathcal{H}}^{\nu}(\wp, \varpi)$, we only need to prove the "only if" part of the theorem. To this end, for functions f of the form (1.13), we notice that the condition

$$\Re \left(\frac{\mathfrak{D}_{\mathcal{H}}(m_{\nu, \wp}^{m, n} f(z))}{m_{\nu, \wp}^{m, n} f(z)} \right) < \varpi$$

Equivalently,

$$\Re \left(\frac{(1 - \varpi)z - \sum_{n \geq 2} \Upsilon_n(n - \varpi)a_n z^n - \sum_{n \geq 1} \Upsilon_n(n + \varpi)b_n \bar{z}^n}{z - \sum_{n \geq 2} \Upsilon_n a_n z^n + \sum_{n \geq 1} \Upsilon_n b_n \bar{z}^n} \right) \geq 0.$$

All values of $z \in \mathbf{U}$ must satisfy the aforementioned necessary condition. When selecting the values of z on the positive real axis, where $0 \leq z = r < 1$, we must have

$$\frac{1 - \varpi - \sum_{n \geq 2} \Upsilon_n(n - \varpi)a_n r^{n-1} - \sum_{n \geq 1} \Upsilon_n(n + \varpi)b_n r^{n-1}}{1 - \sum_{n \geq 2} \Upsilon_n a_n r^{n-1} + \sum_{n \geq 1} \Upsilon_n b_n r^{n-1}} \geq 0. \quad (2.6)$$

The numerator in (2.6) is negative for r that is close enough to 1 if the criterion (2.5) is not true. Therefore, $z_0 = r_0$ exists in $(0, 1)$ for which the (2.6) has a negative quotient. The necessary condition for $f \in \mathcal{V}_{\mathcal{H}}^{\nu}(\wp, \varpi)$ is in conflict with this. This concludes the theorem's proof. $\square$

3 Distortion bounds and Extreme Points

By routine procedure (see [9, 14, 20, 21]) we can easily prove the following results hence we state the following theorems without proof for functions in $\mathcal{V}_{\mathcal{H}}^{\nu}(\wp, \varpi)$.

Theorem 3. (Distortion Bounds:) Let $f \in \mathcal{V}_{\mathcal{H}}^{\nu}(\wp, \varpi)$. Then for $|z| = r < 1$, we have

$$\begin{aligned} (1 - |b_1|)r - \frac{1}{\Upsilon_2} \left(\frac{\varpi - 1}{2 - \varpi} - \frac{1 + \varpi}{2 - \varpi} |b_1| \right) r^2 &\leq |f(z)| \\ &\leq (1 + |b_1|)r + \frac{1}{\Upsilon_2} \left(\frac{\varpi - 1}{2 - \varpi} - \frac{1 + \varpi}{2 - \varpi} |b_1| \right) r^2 \end{aligned}$$

where

$$\Upsilon_2 = \frac{(m_{\wp}) m^{\nu}}{\mathbf{E}_{\nu, \wp}(m) \Gamma(\nu + 2)} \quad (3.1)$$

Corollary 4. (covering result:) If $f(z) \in \mathcal{V}_{\mathcal{H}}^v(\wp, \varpi)$, then

$$\left\{ w : |w| < \frac{2\Upsilon_2(1+\varpi)+1-[\Upsilon_2(1+\varpi)+1]\varpi}{\Upsilon_2(2-\varpi)} - \frac{2\Upsilon_2(1+\varpi)-1-[\Upsilon_2(1+\varpi)+1]\varpi}{\Upsilon_2(2-\varpi)} b_1 \right\} \subset f(\mathbf{U})$$

where Υ_2 as given in (3.1).

Next we state the extreme points of closed convex hulls of $\mathcal{V}_{\mathcal{H}}^v(\wp, \varpi)$ denoted by $clco\mathcal{V}_{\mathcal{H}}^v(\wp, \varpi)$.

Theorem 5. A function $f(z) \in \mathcal{V}_{\mathcal{H}}^v(\wp, \varpi)$ if and only if $f(z) = \sum_{n \geq 1} (X_n h_n(z) + Y_n g_n(z))$ where $h_1(z) = z$, $h_n(z) = z + \frac{\varpi-1}{\Upsilon_n(n-\varpi)} z^n$, ($n \geq 2$) and $g_n(z) = z + \frac{\varpi-1}{\Upsilon_n(n+\varpi)} \bar{z}^n$, ($n \geq 1$) also $\sum_{n \geq 1} (X_n + Y_n) = 1$, $X_n \geq 0$ and $Y_n \geq 0$. In particular, the extreme points of $\mathcal{V}_{\mathcal{H}}^v(\wp, \varpi)$ are $\{h_n\}$ and $\{g_n\}$.

Theorem 6. The family $\mathcal{V}_{\mathcal{H}}^v(\wp, \varpi)$ is closed under convex combinations.

4 Inclusion Results

Now, we will examine the closure properties of the class $\mathcal{V}_{\mathcal{H}}^v(\wp, \varpi)$ under the generalized Bernardi-Libera -Livingston integral operator $\mathcal{L}_c(f)$ which is defined by $\mathcal{L}_c(f) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt$, $c > -1$.

Theorem 7. Let $f(z) \in \mathcal{V}_{\mathcal{H}}^v(\wp, \varpi)$. Then $\mathcal{L}_c(f(z)) \in \mathcal{V}_{\mathcal{H}}^v(\wp, \varpi)$.

Lemma 8. [40] Let $f = h + \bar{g}$ be given by (1.3). If

$$\sum_{n \geq 2} \frac{(n-\hbar)}{1-\hbar} |a_n| + \sum_{n \geq 1} \frac{(n+\hbar)}{1-\hbar} |b_n| \leq 1. \quad (4.1)$$

where $a_1 = 1$ and $0 \leq \hbar < 1$, then $f \in \mathcal{M}_{\mathcal{H}}^*(\hbar)$.

Theorem 9. Let $f = h + \bar{g} \in \mathcal{V}_{\mathcal{H}}(\hbar)$ be given by (1.13). Then $f \in \mathcal{V}_{\mathcal{H}}^v(\wp, \frac{4-3\varpi}{3-2\varpi})$.

Proof. Since $f \in \mathcal{V}_{\mathcal{H}}^v(\wp, \varpi)$ then by Theorem 1, we must have

$$\sum_{n \geq 2} \frac{\Upsilon_2(n-\varpi)}{\varpi-1} |a_n| + \sum_{n \geq 1} \frac{\Upsilon_2(n+\varpi)}{\varpi-1} |b_n| \leq 1. \quad (4.2)$$

To show that $f \in \mathcal{V}_{\mathcal{H}}^v(\wp, \frac{4-3\varpi}{3-2\varpi})$, by virtue of Lemma 8 we have to show that

$$\sum_{n \geq 2} \frac{\Upsilon_2[n - (\frac{4-3\varpi}{3-2\varpi})]}{1 - \frac{4-3\varpi}{3-2\varpi}} |a_n| + \sum_{n \geq 1} \frac{\Upsilon_2[n + (\frac{4-3\varpi}{3-2\varpi})]}{1 - \frac{4-3\varpi}{3-2\varpi}} |b_n| \leq 1. \quad (4.3)$$

where $0 \leq \frac{4-3\varpi}{3-2\varpi} < 1$. For this, it is sufficient to prove that

$$\frac{n-\varpi}{\varpi-1} \geq \frac{[n - (\frac{4-3\varpi}{3-2\varpi})]}{1 - \frac{4-3\varpi}{3-2\varpi}}, (n = 2, 3, \dots)$$

and

$$\frac{n+\varpi}{\varpi-1} \geq \frac{[n + (\frac{4-3\varpi}{3-2\varpi})]}{1 - \frac{4-3\varpi}{3-2\varpi}}, (n = 1, 2, 3, \dots)$$

or equivalently $(\varpi-1)(n-\varpi) \geq 0$, $(n = 2, 3, \dots)$ and $(\varpi-1)(n+\varpi) \geq 0$, $(n = 1, 2, 3, \dots)$, which is true and the theorem is proved. $\square$

Corollary 10. $\mathcal{V}_{\mathcal{H}}^v(\wp, \varpi) \subset \mathcal{V}_{\mathcal{H}}^v(\wp, \frac{4}{3}) \subset \mathcal{V}_{\mathcal{H}}^v$.

5 Integral Means Inequalities

In this section, we obtain integral means inequalities for the functions in the family $\mathcal{V}_{\mathcal{H}}^v(\wp, \varpi)$ due to Dziok [7] and Silverman [34].

Lemma 11. [16] *If the functions ϕ and φ are analytic in $\mathbb{U}$ with $\varphi \prec \phi$, then for $\eta > 0$, and $0 < r < 1$,*

$$\int_0^{2\pi} |\varphi(re^{i\theta})|^\eta d\theta \leq \int_0^{2\pi} |\phi(re^{i\theta})|^\eta d\theta. \quad (5.1)$$

Due to recent work of Dziok [7], we suppose $\mathfrak{h} \in \mathcal{V}_{\mathcal{H}}^v(\wp, \varpi)$ $\eta > 0$, $0 \leq \varpi < 1$, $0 \leq \mathfrak{N} \leq 1$, and $\xi_{12}(z)$ is defined by

$$\begin{aligned} \xi_{12}(z) &= z - \frac{\varpi-1}{(n-\varpi)\Upsilon_2} z^2 \quad (n \geq 2), \\ \xi_{22}(z) &= z + \frac{\varpi-1}{(n+\varpi)\Upsilon_2} \bar{z}^2; \quad (n \geq 2). \end{aligned}$$

where Υ_2 as given in (3.1). Since

$$\frac{\xi_{1n}^*(z)}{z} \prec \frac{\xi_{12}^*(z)}{z} \quad \text{and} \quad \frac{\bar{\xi}_2(z)}{z} \prec \frac{\xi_{12}^*(z)}{z}$$

by Lemma 11 we have

$$\begin{aligned} \int_0^{2\pi} \left| \frac{\xi_{1n}^*(z)}{z} \right|^\eta d\theta &\leq \int_0^{2\pi} \left| \frac{\xi_{12}^*(z)}{z} \right|^\eta d\theta, \quad (z = re^{i\theta}) \\ \int_0^{2\pi} \left| \frac{\xi_{2n}^*(z)}{z} \right|^\eta d\theta &= \int_0^{2\pi} \left| \frac{\bar{\xi}_2(z)}{z} \right|^\eta d\theta \leq \int_0^{2\pi} \left| \frac{\xi_{12}^*(z)}{z} \right|^\eta d\theta, \quad (z = re^{i\theta}). \end{aligned}$$

Thus we have the following result:

Lemma 12. Let $0 < r < 1, \eta > 0$. Then

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} |\xi_{1n}^*(re^{i\theta})|^\eta d\theta &\leq \frac{1}{2\pi} \int_0^{2\pi} |\xi_{12}^*(re^{i\theta})|^\eta d\theta, \quad (n = 1, 2, 3, \cdot) \\ \frac{1}{2\pi} \int_0^{2\pi} |\xi_{2n}^*(re^{i\theta})|^\eta d\theta &= \frac{1}{2\pi} \int_0^{2\pi} |\xi_{12}^*(re^{i\theta})|^\eta d\theta, \quad (n = 2, 3, 4, \cdot). \end{aligned}$$

where ξ_{1n}^* and ξ_{2n}^* are defined as in Theorem (5).

By Lemma 12 and Theorem 5 we have the following :

Theorem 13. Let $0 < r < 1, \eta > 0$. Then

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^\eta d\theta &\leq \frac{1}{2\pi} \int_0^{2\pi} |\xi_{12}^*(re^{i\theta})|^\eta d\theta, \quad (n = 1, 2, 3, \cdot) \\ \frac{1}{2\pi} \int_0^{2\pi} |\mathfrak{D}_{\mathcal{H}}(\mathfrak{m}_{\nu, \varphi}^{m, n} f(re^{i\theta}))|^\eta d\theta &= \frac{1}{2\pi} \int_0^{2\pi} |\mathfrak{D}_{\mathcal{H}}(\mathfrak{m}_{\nu, \varphi}^{m, n} \xi_{12}^*(re^{i\theta}))|^\eta d\theta, \quad (n = 2, 3, 4, \cdot). \end{aligned}$$

where ξ_{1n}^* and ξ_{2n}^* are defined as in Theorem (5).

6 Convolution Properties

Let $F_j(z) \in \mathcal{V}_{\mathcal{H}}^v(\wp, \varpi), (j = 1, 2, 3, \dots, p)$ be given by

$$F_j(z) = z - \sum_{n \geq 2} |a_{n,j}| z^n - \overline{\sum_{n \geq 1} |b_{n,j}| z^n}, \quad (6.1)$$

then the convolution is defined by

$$(F_1 * \dots F_p)(z) = z - \sum_{n \geq 2} \prod_{j=1}^p |a_{n,j}| z^n - \sum_{n \geq 1} \prod_{j=1}^p |b_{n,j}| \overline{z^n}, \quad (6.2)$$

Theorem 14. Let $F_j(z) \in \mathcal{V}_{\mathcal{H}}^v(\wp, \varpi_j), (j = 1, 2, 3, \dots, p)$ then $(F_1 * \dots F_p)(z) \in \mathcal{V}_{\mathcal{H}}^v(\wp, \vartheta)$, where

$$\vartheta = 1 + \frac{\prod_{j=1}^p (\varpi_j - 1)}{\Upsilon_2 \prod_{j=1}^p (2 + \varpi_j) - \prod_{j=1}^p (\varpi_j - 1)}. \quad (6.3)$$

Proof. We use the principle of mathematical induction in our proof. Let $F_1 \in \mathcal{V}_{\mathcal{H}}^v(\wp, \varpi_1)$, and $F_2 \in \mathcal{V}_{\mathcal{H}}^v(\wp, \varpi_2)$. By using Theorem 2 we have

$$\sum_{n \geq 2} \Upsilon_n \frac{(n - \varpi_j)}{\varpi_j - 1} |a_{n,j}| + \sum_{n \geq 1} \Upsilon_n \frac{(n + \varpi_j)}{\varpi_j - 1} |b_{n,j}| \leq 1 \quad (6.4)$$

then,

$$\begin{aligned} & \left[\sum_{n \geq 2} \left(\sqrt{\Upsilon_n \frac{(n - \varpi_1)}{\varpi_1 - 1}} |a_{n,1}| \right)^2 \times \sum_{n \geq 2} \left(\sqrt{\Upsilon_n \frac{(n - \varpi_2)}{\varpi_2 - 1}} |a_{n,2}| \right)^2 \right]^{\frac{1}{2}} \\ & + \left[\sum_{n \geq 1} \left(\sqrt{\Upsilon_n \frac{(n + \varpi_1)}{\varpi_1 - 1}} |b_{n,1}| \right)^2 \times \sum_{n \geq 1} \left(\sqrt{\Upsilon_n \frac{(n + \varpi_2)}{\varpi_2 - 1}} |b_{n,2}| \right)^2 \right]^{\frac{1}{2}} \leq 1. \end{aligned} \quad (6.5)$$

Thus by applying Cauchy-Schwarz inequality, we have

$$\begin{aligned} & \sum_{n \geq 2} \left(\sqrt{\Upsilon_n^2 \frac{(n - \varpi_1)(n - \varpi_2)}{(\varpi_1 - 1)(\varpi_2 - 1)}} |a_{n,1} a_{n,2}| \right)^2 \\ & \leq \left[\sum_{n \geq 2} \left(\sqrt{\Upsilon_n \frac{(n - \varpi_1)}{\varpi_1 - 1}} |a_{n,1}| \right)^2 \right]^{\frac{1}{2}} \times \left[\sum_{n \geq 2} \left(\sqrt{\Upsilon_n \frac{(n - \varpi_2)}{\varpi_2 - 1}} |a_{n,2}| \right)^2 \right]^{\frac{1}{2}} \end{aligned}$$

and

$$\begin{aligned} & \sum_{n \geq 1} \left(\sqrt{\Upsilon_n^2 \frac{(n + \varpi_1)(n + \varpi_2)}{(\varpi_1 - 1)(\varpi_2 - 1)}} |b_{n,1} b_{n,2}| \right)^2 \\ & \leq \left[\sum_{n \geq 1} \left(\sqrt{\Upsilon_n \frac{(n + \varpi_1)}{\varpi_1 - 1}} |b_{n,1}| \right)^2 \right]^{\frac{1}{2}} \times \left[\sum_{n \geq 1} \left(\sqrt{\Upsilon_n \frac{(n + \varpi_2)}{\varpi_2 - 1}} |b_{n,2}| \right)^2 \right]^{\frac{1}{2}}. \end{aligned}$$

Then, we get

$$\sum_{n \geq 2} \sqrt{\Upsilon_n \frac{(n - \varpi_1)(n - \varpi_2)}{(\varpi_1 - 1)(\varpi_2 - 1)}} |a_{n,1} a_{n,2}| + \sum_{n \geq 1} \sqrt{\Upsilon_n^2 \frac{(n + \varpi_1)}{\varpi_1 - 1}} |b_{n,1} b_{n,2}| \leq 1.$$

Therefore, if

$$\sum_{n \geq 2} \Upsilon_n \frac{(n - \hbar)}{\hbar - 1} |a_{n,1} a_{n,2}| \leq \sum_{n \geq 2} \sqrt{\Upsilon_n^2 \frac{(n - \varpi_1)(n - \varpi_2)}{(\varpi_1 - 1)(\varpi_2 - 1)}} |a_{n,1} a_{n,2}|$$

$$\sum_{n \geq 1} \Upsilon_n \frac{(n + \hbar)}{\hbar - 1} |b_{n,1} b_{n,2}| \leq \sum_{n \geq 1} \sqrt{\Upsilon_n^2 \frac{(n + \varpi_1)(n + \varpi_2)}{(\varpi_1 - 1)(\varpi_2 - 1)}} |b_{n,1} b_{n,2}|$$

that is, if

$$\sqrt{|a_{n,1} a_{n,2}|} \leq \frac{\hbar - 1}{n - \hbar} \sqrt{\frac{(n - \varpi_1)(n - \varpi_2)}{(\varpi_1 - 1)(\varpi_2 - 1)}}$$

$$\sqrt{|b_{n,1} b_{n,2}|} \leq \frac{\hbar - 1}{n + \hbar} \sqrt{\frac{(n + \varpi_1)(n + \varpi_2)}{(\varpi_1 - 1)(\varpi_2 - 1)}}$$

then $(F_1 * F_2)(z) \in \mathcal{V}_{\mathcal{H}}^v(\wp, \hbar)$. By (6.4) we have

$$\sum_{n \geq 2} \sqrt{\Upsilon_n \frac{(n - \varpi_j)}{\varpi_j - 1}} |a_{n,j}| \leq 1$$

and

$$\sum_{n \geq 1} \sqrt{\Upsilon_n \frac{(n + \varpi_j)}{\varpi_j - 1}} |b_{n,j}| \leq 1.$$

Hence we get

$$\sqrt{|a_{n,j}|} \leq \frac{\varpi_j - 1}{(n - \varpi_j)}$$

and

$$\sqrt{|b_{n,j}|} \leq \frac{\varpi_j - 1}{(n + \varpi_j)}.$$

Consequently, if

$$\sqrt{\frac{(\varpi_1 - 1)(\varpi_2 - 1)}{n^{2\ell}(n - \varpi_1)(n - \varpi_2)}} \leq \frac{\hbar - 1}{n - \hbar} \sqrt{\frac{(n - \varpi_1)(n - \varpi_2)}{(\varpi_1 - 1)(\varpi_2 - 1)}}$$

$$\sqrt{\frac{(\varpi_1 - 1)(\varpi_2 - 1)}{n^{2\ell}(n + \varpi_1)(n + \varpi_2)}} \leq \frac{\hbar - 1}{n + \hbar} \sqrt{\frac{(n + \varpi_1)(n + \varpi_2)}{(\varpi_1 - 1)(\varpi_2 - 1)}}.$$

That is, if

$$\frac{n - \hbar}{\hbar - 1} \leq \frac{(n - \varpi_1)(n - \varpi_2)}{(\varpi_1 - 1)(\varpi_2 - 1)}$$

$$\frac{n + \hbar}{\hbar - 1} \leq \frac{(n + \varpi_1)(n + \varpi_2)}{(\varpi_1 - 1)(\varpi_2 - 1)}$$

then $(F_1 * F_2)(z) \in \mathcal{V}_{\mathcal{H}}^v(\wp, \hbar)$. Then we see that

$$\hbar \geq 1 + \frac{(n - 1)(\varpi_1 - 1)(\varpi_2 - 1)}{\Upsilon_n(n + \varpi_1)(n + \varpi_2) + (\varpi_1 - 1)(\varpi_2 - 1)} = \phi(n)$$

and

$$\hbar \geq 1 + \frac{(n+1)(\varpi_1-1)(\varpi_2-1)}{\Upsilon_n(n+\varpi_1)(n+\varpi_2) - (\varpi_1-1)(\varpi_2-1)} = \psi(n).$$

Since $\phi(n)$ for $n \geq 2$ and $\psi(n)$ for $n \geq 1$ increasing

$$\hbar \geq 1 + \frac{(\varpi_1-1)(\varpi_2-1)}{\Upsilon_2(2+\varpi_1)(2+\varpi_2) + (\varpi_1-1)(\varpi_2-1)}$$

and

$$\hbar \geq 1 + \frac{2(\varpi_1-1)(\varpi_2-1)}{(1+\varpi_1)(1+\varpi_2) - (\varpi_1-1)(\varpi_2-1)}.$$

and also

$$\frac{(\varpi_1-1)(\varpi_2-1)}{\Upsilon_2(2+\varpi_1)(2+\varpi_2) + (\varpi_1-1)(\varpi_2-1)} \leq \frac{2(\varpi_1-1)(\varpi_2-1)}{(1+\varpi_1)(1+\varpi_2) - (\varpi_1-1)(\varpi_2-1)}$$

then $(F_1 * F_2)(z) \in \mathcal{V}_{\mathcal{H}}^v(\wp, \hbar)$ where

$$\hbar \geq 1 + \frac{(\varpi_1-1)(\varpi_2-1)}{\Upsilon_2(2+\varpi_1)(2+\varpi_2) + (\varpi_1-1)(\varpi_2-1)}.$$

Next, we suppose that, $(F_1 * F_2 * \dots * F_p)(z) \in \mathcal{V}_{\mathcal{H}}^v(\wp, \vartheta)$, where

$$\vartheta = 1 + \frac{\prod_{j=1}^p (\varpi_j - 1)}{\Upsilon_2 \prod_{j=1}^p (2 + \varpi_j) - \prod_{j=1}^p (\varpi_j - 1)}.$$

We can show that, $(F_1 * F_2 * \dots * F_{p+1})(z) \in \mathcal{V}_{\mathcal{H}}^v(\wp, \mathfrak{w})$, where

$$\mathfrak{w} \geq 1 + \frac{(\vartheta-1)(\varpi_{p+1}-1)}{\Upsilon_2(2+\vartheta)(2+\varpi_{p+1}) + (\vartheta-1)(\varpi_{p+1}-1)}.$$

Since,

$$(\vartheta-1)(\hbar_{p+1}-1) = \frac{\prod_{j=1}^{p+1} (\varpi_j - 1)}{\Upsilon_2 \prod_{j=1}^p (2 + \varpi_j) - \prod_{j=1}^p (\varpi_j - 1)}$$

and

$$(\vartheta-2)(\hbar_{p+1}-2) = \frac{\prod_{j=1}^{p+1} (\varpi_j - 1)}{\Upsilon_2 \prod_{j=1}^p (2 + \varpi_j) - \prod_{j=1}^p (\varpi_j - 1)}$$

we have

$$\mathfrak{w} = 1 + \frac{\prod_{j=1}^{p+1} (\varpi_j - 1)}{\Upsilon_2 \prod_{j=1}^{p+1} (2 + \varpi_j) - \prod_{j=1}^{p+1} (\varpi_j - 1)}.$$

□

Corollary 15. Let $F_j(z) \in \mathcal{V}_{\mathcal{H}}^v(\wp, \varpi)$, ($j = 1, 2, 3, \dots, p$) then $(F_1 * \dots * F_p)(z) \in \mathcal{V}_{\mathcal{H}}^v(\wp, \vartheta)$, where

$$\vartheta = 1 + \frac{(\varpi - 1)^p}{\Upsilon_2(2 + \varpi)^p - (\varpi - 1)^p}. \quad (6.6)$$

7 Partial Sums results

In 1985, Silvia [35] studied the partial sums of convex functions of order $\hbar$ ($0 \leq \hbar < 1$). Later on, Silverman [32], several researchers studied and generalized the results on partial sums for various classes of analytic functions only but few authors studied analogues results on harmonic functions (see [25, 26]). Now in this section we discussed the partial sums results for the class of harmonic functions with positive coefficients $\mathcal{P}_{\mathcal{H}}^v(\wp, \varpi)$ with $(1 < \varpi \leq \frac{4}{3})$ on lines similar to Porwal [25].

Let $\mathcal{P}_{\mathcal{H}}^v(\frac{\mathbf{L}_n}{\mathfrak{w}}, \frac{\mathbf{M}_n}{\mathfrak{w}})$ denote the subclass of $\mathcal{H}$ consisting of functions $f = h + \bar{g}$ of the form (1.3) which satisfy the inequality

$$\sum_{n \geq 2} \frac{\mathbf{L}_n}{\mathfrak{w}} |a_n| + \sum_{n \geq 1} \frac{\mathbf{M}_n}{\mathfrak{w}} |b_n| \leq 1 \quad (7.1)$$

where

$$\frac{\mathbf{L}_n}{\mathfrak{w}} = \frac{\Upsilon_n(n - \varpi)}{\varpi - 1} \text{ and } \frac{\mathbf{M}_n}{\mathfrak{w}} = \frac{\Upsilon_n(n + \varpi)}{\varpi - 1}$$

and $\mathfrak{w} = \varpi - 1$ unless otherwise stated.

Now, we discuss the ratio of a function of the form (1.13) with $b_1 = 0$ are

$$f_m(z) = z + \sum_{n=2}^m a_n z^n + \overline{\sum_{n \geq 1} b_n z^n}$$

,

$$f_k(z) = z + \sum_{n \geq 2} a_n z^n + \overline{\sum_{n=2}^k b_n z^n}$$

,

$$f_{m,k}(z) = z + \sum_{n=2}^m a_n z^n + \overline{\sum_{n=2}^k b_n z^n}.$$

We first obtain the sharp bounds for $\Re \left\{ \frac{f(z)}{f_m(z)} \right\}$.

Theorem 16. If f of the form (1.13) with $b_1 = 0$ satisfies the condition (7.1), then

$$\Re \left\{ \frac{f(z)}{f_m(z)} \right\} \geq \frac{\mathbf{L}_{m+1} - \mathfrak{w}}{\mathbf{L}_{m+1}}, \quad (z \in \mathbf{U}) \quad (7.2)$$

where

$$\mathbf{L}_n \geq \begin{cases} \mathfrak{w}, & \text{if } n = 2, 3, \dots, m \\ \mathbf{L}_{m+1}, & \text{if } n = m+1, m+2, \dots \end{cases}$$

$$\mathbf{M}_n \geq \mathfrak{w}, \quad \text{if } n = 2, 3, \dots$$

The result (7.2) is sharp with the function given by

$$f(z) = z + \frac{\mathfrak{w}}{\mathbf{L}_{m+1}} z^{m+1}. \quad (7.3)$$

Proof. Define the function $w(z)$ by

$$\begin{aligned} \frac{1+w(z)}{1-w(z)} &= \frac{\mathbf{L}_{m+1}}{\mathfrak{w}} \left[\frac{f(re^{i\theta})}{f_m(re^{i\theta})} - \frac{\mathbf{L}_{m+1} - \mathfrak{w}}{\mathbf{L}_{m+1}} \right] \\ &= \frac{1 + \sum_{n=2}^m a_n r^{n-1} e^{i(n-1)\theta} + \sum_{n=2}^{\infty} \overline{b_n} r^{n-1} e^{-i(n+1)\theta} - \frac{\mathbf{L}_{m+1}}{\mathfrak{w}} \left(\sum_{n=m+1}^{\infty} a_n r^{n-1} e^{i(n-1)\theta} \right)}{1 + \sum_{n=2}^m a_n r^{n-1} e^{i(n-1)\theta} + \sum_{n=2}^{\infty} \overline{b_n} r^{n-1} e^{-i(n+1)\theta}}. \end{aligned} \quad (7.4)$$

It suffices to show that $|w(z)| \leq 1$. Now, from (7.4) we can write

$$w(z) = \frac{\frac{\mathbf{L}_{m+1}}{\mathfrak{w}} \left(\sum_{n=m+1}^{\infty} a_n r^{n-1} e^{i(n-1)\theta} \right)}{2 + 2 \left(\sum_{n=2}^{\infty} a_n r^{n-1} e^{i(n-1)\theta} + \sum_{n=2}^{\infty} \overline{b_n} r^{n-1} e^{-i(n+1)\theta} \right) + \frac{\mathbf{L}_{m+1}}{\mathfrak{w}} \left(\sum_{n=m+1}^{\infty} a_n r^{n-1} e^{i(n-1)\theta} \right)}.$$

Hence we obtain

$$|w(z)| \leq \frac{\frac{\mathbf{L}_{m+1}}{\mathfrak{w}} \left(\sum_{n=m+1}^{\infty} |a_n| \right)}{2 - 2 \left[\sum_{n=2}^m |a_n| + \sum_{n=2}^{\infty} |b_n| \right] - \frac{\mathbf{L}_{m+1}}{\mathfrak{w}} \sum_{n=m+1}^{\infty} |a_n|}.$$

Now $|w(z)| \leq 1$ if

$$\sum_{n=2}^{\infty} |a_n| + \sum_{n=2}^{\infty} |b_n| + \frac{\mathbf{L}_{m+1}}{\mathfrak{w}} \sum_{n=m+1}^{\infty} |a_n| \leq 1.$$

From the condition (7.1), it is sufficient to show that

$$\sum_{n=2}^{\infty} |a_n| + \sum_{n=2}^{\infty} |b_n| + \frac{\mathfrak{L}_{m+1}}{\mathfrak{w}} \sum_{n=m+1}^{\infty} |a_n| \leq \sum_{n=2}^{\infty} \frac{\mathbf{L}_n}{\mathfrak{w}} |a_n| + \sum_{n=2}^{\infty} \frac{\mathbf{M}_n}{\mathfrak{w}} |a_n|$$

which is equivalently to

$$\sum_{n=2}^m \left(\frac{\mathbf{L}_n - \mathfrak{w}}{\mathfrak{w}} \right) |a_n| + \sum_{n=2}^{\infty} \left(\frac{\mathbf{M}_n - \mathfrak{w}}{\mathfrak{w}} \right) |b_n| + \sum_{n=m+1}^{\infty} \left(\frac{\mathbf{L}_n - \mathbf{L}_{n+1}}{\mathfrak{w}} \right) |a_n| \geq 0.$$

To see that the function given by (7.3) gives the sharp result, we observe that for $z = re^{i\pi/n}$

$$\begin{aligned} \frac{f(z)}{f_m(z)} &= 1 + \frac{\mathfrak{w}}{\mathbf{L}_{m+1}} z^m \rightarrow 1 - \frac{\mathfrak{w}}{\mathbf{L}_{m+1}} \\ &= \frac{\mathbf{L}_{m+1} - \mathfrak{w}}{\mathbf{L}_{m+1}} \quad \text{when } r \rightarrow 1^-. \end{aligned}$$

□

We next determine bounds for $\Re \{f_m(z)/f(z)\}$.

Theorem 17. *If f of the form (1.13) with $b_1 = 0$ satisfies the condition (7.1), then*

$$\Re \left\{ \frac{f_m(z)}{f(z)} \right\} \geq \frac{\mathbf{L}_{m+1}}{\mathbf{L}_{m+1} + \mathfrak{w}}, \quad (z \in \mathbf{U}) \quad (7.5)$$

where

$$\begin{aligned} \mathbf{L}_n &\geq \begin{cases} \mathfrak{w}, & \text{if } n = 2, 3, \dots, m \\ \mathbf{L}_{m+1}, & \text{if } n = m+1, m+2, \dots \end{cases} \\ \mathbf{M}_n &\geq \mathfrak{w}, \quad \text{if } n = 2, 3, \dots \end{aligned} \quad (7.6)$$

The result (7.5) is sharp with the function given by

$$f(z) = z + \frac{\mathfrak{w}}{\mathbf{L}_{m+1}} z^{m+1}. \quad (7.7)$$

Proof. Define the function $w(z)$ by

$$\begin{aligned} \frac{1+w(z)}{1-w(z)} &= \frac{\mathbf{L}_{m+1} + \mathfrak{w}}{\mathfrak{w}} \left[\frac{f_m(re^{i\theta})}{f(re^{i\theta})} - \frac{\mathbf{L}_{m+1}}{\mathbf{L}_{m+1} + \mathfrak{w}} \right] \\ &= \frac{1 + \sum_{n=2}^m a_n r^{n-1} e^{i(n-1)\theta} + \sum_{n=2}^{\infty} \overline{b_n} r^{n-1} e^{-i(n+1)\theta} - \frac{\mathbf{L}_{m+1}}{\mathfrak{w}} \left(\sum_{n=m+1}^{\infty} a_n r^{n-1} e^{i(n-1)\theta} \right)}{1 + \sum_{n=2}^m a_n r^{n-1} e^{i(n-1)\theta} + \sum_{n=2}^{\infty} \overline{b_n} r^{n-1} e^{-i(n+1)\theta}}. \end{aligned}$$

Hence we obtain

$$|w(z)| \leq \frac{\frac{\mathbf{L}_{m+1}+\mathfrak{w}}{\mathfrak{w}} \left(\sum_{n=m+1}^{\infty} |a_n| \right)}{2 - 2 \left[\sum_{n=2}^m |a_n| + \sum_{n=2}^{\infty} |b_n| \right] - \frac{\mathbf{L}_{m+1}-\mathfrak{w}}{\mathfrak{w}} \sum_{n=m+1}^{\infty} |a_n|} \leq 1.$$

The last inequality is equivalent to

$$\sum_{n=2}^{\ell} |a_n| + \sum_{n=2}^{\infty} |b_n| + \frac{\mathbf{L}_{m+1}}{\mathfrak{w}} \sum_{n=m+1}^{\infty} |a_n| \leq 1.$$

Making use of (7.1) and the condition (7.6) we obtain (7). Finally equality holds in (7.5) for the extremal function $f(z)$ given by (7.7). $\square$

We next turns to ratios for the for $\Re \left\{ f'(z)/f'_m(z) \right\}$ and $\Re \left\{ f'_m(z)/f'(z) \right\}$.

Theorem 18. *If f of the form (1.13) with $b_1 = 0$ satisfies the condition (7.1), then*

$$\Re \left\{ \frac{f'(z)}{f'_m(z)} \right\} \geq \frac{\mathbf{L}_{m+1} - (m+1)\mathfrak{w}}{\mathbf{L}_{m+1}}, \quad (z \in \mathbf{U}) \quad (7.8)$$

where

$$\mathbf{L}_n \geq \begin{cases} \mathfrak{w}, & \text{if } n = 2, 3, \dots, m \\ \mathbf{L}_{m+1}, & \text{if } n = m+1, m+2, \dots \end{cases}$$

$$\mathbf{M}_n \geq \mathfrak{w}, \quad \text{if } n = 2, 3, \dots$$

The result (7.8) is sharp for f given (7.7).

Proof. Define the function $w(z)$ by

$$\frac{1+w(z)}{1-w(z)} = \frac{\mathbf{L}_{m+1}}{(m+1)\mathfrak{w}} \left[\frac{f'(z)}{f'_m(z)} - \frac{\mathbf{L}_{m+1} - (m+1)\mathfrak{w}}{\mathbf{L}_{m+1}} \right]$$

$$= \frac{1 + \sum_{n=2}^m na_n r^{n-1} e^{i(n-1)\theta} + \sum_{n=2}^{\infty} n\overline{b_n} r^{n-1} e^{-i(n+1)\theta} - \frac{\mathbf{L}_{m+1}}{(m+1)\mathfrak{w}} \left(\sum_{n=m+1}^{\infty} na_n r^{n-1} e^{i(n-1)\theta} \right)}{1 + \sum_{n=2}^m na_n r^{n-1} e^{i(n-1)\theta} - \sum_{n=2}^{\infty} n\overline{b_n} r^{n-1} e^{-i(n+1)\theta}}.$$

The result(7.8) follows by using the techniques as used in Theorem 16. $\square$

Proceeding exactly as in the proof of Theorem 17, we can prove the following theorem .

Theorem 19. *If f of the form (1.13) with $b_1 = 0$ satisfies the condition (7.1), then*

$$\Re \left\{ \frac{f'_m(z)}{f'(z)} \right\} \geq \frac{\mathbf{L}_{m+1}}{\mathbf{L}_{m+1} + (m+1)\mathfrak{w}}, \quad (z \in \mathbf{U}). \quad (7.9)$$

The result (7.8) is sharp for f given (7.7).

We next determine bounds for $\Re \{f(z)/f_k(z)\}$ and $\Re \{f_k(z)/f(z)\}$.

Theorem 20. *If f of the form (1.13) with $b_1 = 0$ satisfies the condition (7.1), then*

$$\Re \left\{ \frac{f(z)}{f_k(z)} \right\} \geq \frac{\mathbf{M}_{k+1} - \mathfrak{w}}{\mathbf{M}_{k+1}}, \quad (z \in \mathbf{U}) \quad (7.10)$$

where

$$\mathbf{M}_n \geq \begin{cases} \mathfrak{w}, & \text{if } n = 2, 3, \dots, k \\ \mathbf{M}_{k+1}, & \text{if } n = k+1, k+2, \dots \end{cases}$$

$$\mathbf{L}_n \geq \mathfrak{w}, \quad \text{if } n = 2, 3, \dots$$

. The result (7.10) is sharp with the function given by

$$f(z) = z + \frac{\mathfrak{w}}{\mathbf{M}_{k+1}} \bar{z}^{k+1}. \quad (7.11)$$

Theorem 21. *If f of the form (1.13) with $b_1 = 0$ satisfies the condition (7.1), then*

$$\Re \left\{ \frac{f_k(z)}{f(z)} \right\} \geq \frac{\mathbf{M}_{k+1}}{\mathbf{M}_{k+1} + \mathfrak{w}}, \quad (z \in \mathbf{U}) \quad (7.12)$$

where

$$\mathbf{M}_n \geq \begin{cases} \mathfrak{w}, & \text{if } n = 2, 3, \dots, k \\ \mathbf{M}_{k+1}, & \text{if } n = k+1, k+2, \dots \end{cases}$$

$$\mathbf{L}_n \geq \mathfrak{w}, \quad \text{if } n = 2, 3, \dots$$

The result (7.12) is sharp for f given (7.11).

Proof. Define the function $w(z)$ by

$$\frac{1+w(z)}{1-w(z)} = \frac{\mathbf{M}_{k+1} + \mathfrak{w}}{\mathfrak{w}} \left[\frac{f_k(re^{i\theta})}{f(re^{i\theta})} - \frac{\mathbf{M}_{k+1}}{\mathbf{M}_{k+1} + \mathfrak{w}} \right]$$

$$= \frac{1 + \sum_{n=2}^{\infty} a_n r^{n-1} e^{i(n-1)\theta} + \sum_{n=2}^k \bar{b}_n r^{n-1} e^{-i(n+1)\theta} - \frac{\mathbf{M}_{k+1}}{\mathbf{M}_{k+1} + \mathfrak{w}} \sum_{n=2}^{k+1} \bar{b}_n r^{n-1} e^{-i(n-1)\theta}}{1 + \sum_{n=2}^{\infty} a_n r^{n-1} e^{i(n-1)\theta} + \sum_{n=2}^k \bar{b}_n r^{n-1} e^{-i(n+1)\theta}}.$$

We omit the details of proof, because it runs parallel to that from Theorem 17. $\square$

Theorem 22. If f of the form (1.13) with $b_1 = 0$ satisfies the condition (7.1), then

$$\Re \left\{ \frac{f(z)}{f_{m,k}(z)} \right\} \geq \frac{\mathbf{L}_{m+1} - \mathfrak{w}}{\mathbf{L}_{m+1}}, \quad (z \in \mathbf{U}) \quad (7.13)$$

where

$$\mathbf{L}_n \geq \begin{cases} \mathfrak{w}, & \text{if } n = 2, 3, \dots, m, m+1 \\ \mathbf{L}_{m+1}, & \text{if } n = m+1, m+2, \dots \end{cases}$$

$$\mathbf{M}_n \geq \begin{cases} \mathfrak{w}, & \text{if } n = 2, 3, \dots, m \\ \mathbf{L}_{m+1}, & \text{if } n = m+1, m+2, \dots \end{cases}$$

The result (7.13) is sharp for f given (7.7).

Theorem 23. If f of the form (1.13) with $b_1 = 0$ satisfies the condition (7.1), then

$$\Re \left\{ \frac{f(z)}{f_{m,k}(z)} \right\} \geq \frac{\mathbf{M}_{k+1} - \mathfrak{w}}{\mathbf{M}_{k+1}}, \quad (z \in \mathbf{U}) \quad (7.14)$$

where

$$\mathbf{M}_n \geq \begin{cases} \mathfrak{w}, & \text{if } n = 2, 3, \dots, k \\ \mathbf{M}_{k+1}, & \text{if } n = k+1, k+2, \dots \end{cases}$$

$$\mathbf{L}_n \geq \begin{cases} \mathfrak{w}, & \text{if } n = 2, 3, \dots, k \\ \mathbf{M}_{k+1}, & \text{if } n = k+1, k+2, \dots \end{cases}$$

Theorem 24. If f of the form (1.13) with $b_1 = 0$ satisfies the condition (7.1), then

$$\Re \left\{ \frac{f_{m,k}(z)}{f(z)} \right\} \geq \frac{\mathbf{L}_{m+1}}{\mathbf{L}_{m+1} + \mathfrak{w}}, \quad (z \in \mathbf{U}). \quad (7.15)$$

Theorem 25. If f of the form (1.13) with $b_1 = 0$ satisfies the condition (7.1), then

$$\Re \left\{ \frac{f_{m,k}(z)}{f(z)} \right\} \geq \frac{\mathbf{M}_{k+1}}{\mathbf{M}_{k+1} + \mathfrak{w}}, \quad (z \in \mathbf{U}). \quad (7.16)$$

The result (7.16) is sharp for f given (7.11).

Theorem 26. If f of the form (1.13) with $b_1 = 0$ satisfies the condition (7.1), then

$$\Re \left\{ \frac{f'(z)}{f'_m(z)} \right\} \geq \frac{\mathbf{L}_{m+1} - (m+1)\mathfrak{w}}{\mathbf{L}_{m+1}}, \quad (z \in \mathbf{U}) \quad (7.17)$$

where

$$\mathbf{L}_n \geq \begin{cases} \mathfrak{w}, & \text{if } n = 2, 3, \dots, m \\ \mathbf{L}_{m+1}, & \text{if } n = m+1, m+2, \dots \end{cases}$$

$$\mathbf{M}_n \geq \begin{cases} \mathfrak{w}, & \text{if } n = 2, 3, \dots, m \\ \mathbf{L}_{m+1}, & \text{if } n = m+1, m+2, \dots \end{cases}$$

The result (7.17) is sharp for f given (7.7).

Theorem 27. If f of the form (1.13) with $b_1 = 0$ satisfies the condition (7.1), then

$$\Re \left\{ \frac{f'_{m,k}(z)}{f'(z)} \right\} \geq \frac{\mathbf{L}_{m+1}}{\mathbf{L}_{m+1} + (m+1)\mathfrak{w}}, \quad (z \in \mathbf{U}). \quad (7.18)$$

The result (7.18) is sharp for f given (7.7).

Concluding Remarks: By choosing the values of the parameters the various results presented in this paper would provide interesting extensions and generalizations of the subclasses of harmonic starlike functions with positive coefficients of order $\varpi(1 < \varpi \leq \frac{4}{3})$ based on Miller-Ross-type Poisson distribution and similarly for convex functions. The details involved in the derivations of such specializations of the results presented in this paper are fairly straight-forward, hence omitted. Moreover the study can be extended to meromorphically harmonic functions convoluting with Poisson distribution [?], Pascal distribution [12] and Borel distribution[43]. **Data Availability.** No data were used in this paper.

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