

A CLASS OF ANALYTIC FUNCTIONS ASSOCIATED WITH A MITTAG–LEFFLER-TYPE POISSON DISTRIBUTION

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Abstract. In this paper, we introduce and investigate a new subclass of analytic functions with negative coefficients associated with a Poisson distribution generated by a special case of the classical two-parameter Mittag–Leffler function. A convolution operator corresponding to the associated Poisson distribution series is first defined and employed to construct the desired function class. For this newly introduced class, we establish coefficient estimates, necessary and sufficient conditions, distortion inequalities, extreme point representations, closure properties, convolution and integral transform results, neighbourhood properties, integral mean inequalities, and various radius problems. The results obtained in this work extend several known investigations concerning probability distribution operators in geometric function theory and demonstrate further applications of Mittag–Leffler-type Poisson distribution series in the study of analytic functions.

1 Introduction

Let $\mathcal{A}$ denote the class of all functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1.1)$$

in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. Let S denote the subclass of $\mathcal{A}$ consisting of all functions which are univalent in U and satisfy the normalization conditions $f(0) = 0$ and $f'(0) = 1$. A function $f \in \mathcal{A}$ is a starlike function of the order ξ , $0 \leq \xi < 1$, if it satisfies

$$\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \xi, \quad z \in U. \quad (1.2)$$

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We denote this class with $S^*(\xi)$. A function $f \in \mathcal{A}$ is a convex function of the order ξ , $0 \leq \xi < 1$, if it satisfies

$$\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \xi, \quad z \in U. \quad (1.3)$$

We denote this class with $K(\xi)$. Note that $S^*(0) = S^*$ and $K(0) = K$ are the usual classes of starlike and convex functions in U respectively. For $f \in \mathcal{A}$ given by (1.1) and $g(z)$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad (1.4)$$

their convolution (or Hadamard product), denoted by $(f * g)$, is defined as

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z), \quad (z \in U). \quad (1.5)$$

Note that $f * g \in \mathcal{A}$.

Let T denote the class of functions analytic in U of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0 \quad (z \in U) \quad (1.6)$$

and let $T^*(\xi) = T \cap S^*(\xi)$, $C(\xi) = T \cap K(\xi)$. The class $T^*(\xi)$ and allied classes possess some interesting properties and have been extensively studied by Silverman [16].

The theory of analytic and univalent functions has been one of the central areas of geometric function theory for many decades. Various subclasses of $\mathcal{A}$ have been introduced and investigated through differential operators, convolution operators, integral transforms, and special functions. These studies have produced numerous geometric properties, including coefficient estimates, distortion theorems, growth inequalities, inclusion relationships, subordination results, neighbourhood properties, and radius problems.

During the last decade, probability distributions have become an important source for constructing new subclasses of analytic functions. In particular, operators associated with Poisson, Pascal, Borel, and other discrete probability distributions have attracted considerable attention because they provide a natural connection between probability theory and geometric function theory. Such distribution-generated operators have proved useful in studying coefficient inequalities, convolution properties, neighbourhood results, closure theorems, and integral mean inequalities for various subclasses of analytic functions.

The classical two-parameter Mittag-Leffler function is defined by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad \Re(\alpha) > 0, \Re(\beta) > 0. \quad (1.7)$$

This function and its numerous generalizations have been extensively studied because of their important applications in fractional calculus, fractional differential equations, integral transforms, probability theory, and geometric function theory.

For the particular choice $\alpha = 1$ and $\beta = \nu + 1$, together with the replacement z by cz , one obtains the function

$$E_{\nu,c}(z) = z^\nu \sum_{n=0}^{\infty} \frac{(cz)^n}{\Gamma(n + \nu + 1)}, \quad \nu > -1, \quad c > 0. \quad (1.8)$$

Equivalently,

$$E_{\nu,c}(z) = z^\nu E_{1,\nu+1}(cz), \quad (1.9)$$

which shows that $E_{\nu,c}(z)$ is simply a special case of the classical two-parameter Mittag–Leffler function.

Since the function defined by (1.9) is not normalized in the sense of the class $\mathcal{A}$, we introduce the normalized function

$$\mathcal{E}_{\nu,c}(z) = z^{1-\nu} \Gamma(\nu + 1) E_{\nu,c}(z) = z + \sum_{n=2}^{\infty} \frac{c^{n-1} \Gamma(\nu + 1)}{\Gamma(n + \nu)} z^n, \quad (1.10)$$

which belongs to the class $\mathcal{A}$.

Motivated by recent developments on probability distribution series associated with special functions, we consider the Poisson distribution generated by $E_{\nu,c}(z)$. Its probability mass function is defined by

$$P_{\nu,c}(m, n) = \frac{m^\nu (cm)^n}{E_{\nu,c}(m) \Gamma(n + \nu + 1)}, \quad n = 0, 1, 2, \dots, \quad (1.11)$$

where $m > 0$, $c > 0$, and $\nu > -1$.

The corresponding distribution series is

$$\mathbb{F}_{\nu,c}^m(z) = z + \sum_{n=2}^{\infty} \frac{m^\nu (cm)^{n-1}}{\Gamma(n + \nu) E_{\nu,c}(m)} z^n, \quad z \in U. \quad (1.12)$$

For $f \in \mathcal{A}$, we define the convolution operator

$$\begin{aligned} \mathcal{K}_{\nu,c}^m f(z) &= \mathbb{F}_{\nu,c}^m(z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \frac{m^\nu (cm)^{n-1}}{\Gamma(n + \nu) E_{\nu,c}(m)} a_n z^n \\ &= z + \sum_{n=2}^{\infty} \Theta(n) a_n z^n, \end{aligned} \quad (1.13)$$

where

$$\Theta(n) = \frac{m^\nu (cm)^{n-1}}{\Gamma(n + \nu) E_{\nu,c}(m)}. \quad (1.14)$$

and $*$ denotes the Hadamard product.

Using this operator, we introduce a new subclass of analytic functions with negative coefficients associated with the above Poisson distribution series. The principal objective of this paper is to investigate the geometric properties of this class. In particular, we establish coefficient estimates, necessary and sufficient conditions, distortion inequalities, extreme point representations, closure properties, neighbourhood results, convolution properties, integral transforms, integral mean inequalities, and radius problems. The results presented here extend several known investigations on probability distribution operators in geometric function theory and provide further applications of Mittag-Leffler-type Poisson distribution series in the study of analytic functions (see for example [8, 9, 10, 11, 12].)

Inspired by the recent investigations on probability distribution operators associated with special cases of the two-parameter Mittag-Leffler function (see [1, 2, 3, 7, 14, 15, 19, 20, 21],) we now introduce a new subclass of analytic functions defined by means of the operator $\mathcal{K}_{\nu,c}^m$.

Definition 1. Let $0 \leq \varepsilon \leq 1$, $0 \leq \hbar < 1$, and $k \geq 0$. A function $f \in T$ of the form (1.6) is said to belong to the class $\mathbb{L}_{\nu,c}^m(\varepsilon, \hbar, k)$ if it satisfies the analytic criterion

$$\Re \left\{ \frac{F_\varepsilon(z)}{zF'_\varepsilon(z)} - \hbar \right\} > k \left| \frac{F_\varepsilon(z)}{zF'_\varepsilon(z)} - 1 \right|, \quad z \in U, \quad (1.15)$$

where the auxiliary function F_ε is defined by

$$\frac{F_\varepsilon(z)}{zF'_\varepsilon(z)} = \frac{(1-\varepsilon)\mathcal{K}_{\nu,c}^m f(z) + \varepsilon z(\mathcal{K}_{\nu,c}^m f(z))'}{z(\mathcal{K}_{\nu,c}^m f(z))' + \varepsilon z^2(\mathcal{K}_{\nu,c}^m f(z))''}, \quad z \in U, \quad (1.16)$$

and the operator $\mathcal{K}_{\nu,c}^m$ is defined by Equation (1.13).

2 Characteristic Properties of the Class $\mathbb{L}_{\nu,c}^m(\varepsilon, \hbar, k)$

We recall the following lemmas in order to establish our main results.

Lemma 2. If χ is a real number and w is a complex number, then

$$\Re(w) \geq \chi \iff |w + (1 - \chi)| - |w - (1 + \chi)| \geq 0.$$

Lemma 3. If w is a complex number and χ, k are real numbers, then

$$\Re(w) \geq k|w - 1| + \chi \iff \Re \left\{ w(1 + ke^{i\theta}) - ke^{i\theta} \right\} \geq \chi, \quad -\pi \leq \theta \leq \pi.$$

Theorem 4. A function f of the form (1.6) belongs to the class $\mathbb{L}_{\nu,c}^m(\varepsilon, \hbar, k)$ if and only if

$$\sum_{n=2}^{\infty} (1 + n\varepsilon - \varepsilon) |(1 + k) - n(\hbar + k)| \Theta(n) |a_n| \leq 1 - \hbar, \quad (2.1)$$

where $0 \leq \varepsilon \leq 1$, $0 \leq \hbar < 1$, $k \geq 0$, and $\Theta(n)$ is given by (1.14).

Proof. Necessity. Assume that $f \in \mathbb{L}_{\nu,c}^m(\varepsilon, \hbar, k)$. By Definition 1.1 and Lemma 2.2, it is sufficient to show that

$$\Re \left\{ \frac{F_\varepsilon(z)}{zF'_\varepsilon(z)} (1 + ke^{i\theta}) - ke^{i\theta} \right\} > \hbar, \quad -\pi \leq \theta \leq \pi. \quad (2.2)$$

Substituting the series representation of $K_{\nu,c}^m f(z)$ into the above expression, we obtain

$$\Re \left(\frac{(1 - \hbar) - \sum_{n=2}^{\infty} [(1 + ke^{i\theta}) - n(\hbar + ke^{i\theta})] (1 + n\varepsilon - \varepsilon) \Theta(n) a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} n(1 + n\varepsilon - \varepsilon) \Theta(n) a_n z^{n-1}} \right) > 0.$$

Let $z = r$, where $0 < r < 1$. Since $\Re(-e^{i\theta}) \geq -1$, we obtain

$$\Re \left(\frac{(1 - \hbar) - \sum_{n=2}^{\infty} (1 + n\varepsilon - \varepsilon) [(1 + k) - n(\hbar + k)] \Theta(n) a_n r^{n-1}}{1 - \sum_{n=2}^{\infty} n(1 + n\varepsilon - \varepsilon) \Theta(n) a_n r^{n-1}} \right) \geq 0.$$

Finally, letting $r \rightarrow 1^-$ gives

$$\sum_{n=2}^{\infty} (1 + n\varepsilon - \varepsilon) |(1 + k) - n(\hbar + k)| \Theta(n) |a_n| \leq 1 - \hbar,$$

which proves (2.1).

Sufficiency. Conversely, suppose that condition (2.1) holds. By Lemma 2.1,

$$\Re(w) > \hbar \iff |w - (1 + \hbar)| < |w + (1 - \hbar)|.$$

Therefore, it is sufficient to verify that

$$\left| \frac{P(z)}{Q(z)} - \left(1 + k \left| \frac{P(z)}{Q(z)} - 1 \right| + \hbar \right) \right| < \left| \frac{P(z)}{Q(z)} + \left(1 - k \left| \frac{P(z)}{Q(z)} - 1 \right| - \hbar \right) \right|,$$

where

$$P(z) = (1 - \varepsilon) K_{\nu,c}^m f(z) + \varepsilon z (K_{\nu,c}^m f(z))',$$

and

$$Q(z) = z(K_{\nu,c}^m f(z))' + \varepsilon z^2 (K_{\nu,c}^m f(z))''.$$

Using the series representations,

$$P(z) = z - \sum_{n=2}^{\infty} (1 + n\varepsilon - \varepsilon) \Theta(n) a_n z^n,$$

and

$$Q(z) = z - \sum_{n=2}^{\infty} n(1 + n\varepsilon - \varepsilon) \Theta(n) a_n z^n.$$

Let

$$L = \left| \frac{P(z)}{Q(z)} - \left(1 + k \left| \frac{P(z)}{Q(z)} - 1 \right| + \hbar \right) \right|,$$

and

$$R = \left| \frac{P(z)}{Q(z)} + \left(1 - k \left| \frac{P(z)}{Q(z)} - 1 \right| - \hbar \right) \right|.$$

Using the coefficient condition (2.1), together with the triangle inequality, we obtain

$$R - L \geq (1 - \hbar) - \sum_{n=2}^{\infty} (1 + n\varepsilon - \varepsilon) |(1 + k) - n(\hbar + k)| \Theta(n) |a_n| \geq 0.$$

Hence $L < R$, which implies

$$\Re \left(\frac{F_{\varepsilon}(z)}{z F'_{\varepsilon}(z)} \right) > k \left| \frac{F_{\varepsilon}(z)}{z F'_{\varepsilon}(z)} - 1 \right| + \hbar.$$

Therefore,

$$f \in \mathbb{L}_{\nu,c}^m(\varepsilon, \hbar, k),$$

which completes the proof. □

Corollary 5. *If $f \in \mathbb{L}_{\nu,c}^m(\varepsilon, \hbar, k)$, then*

$$|a_n| \leq \frac{1 - \hbar}{\Psi(\varepsilon, \hbar, k, n)}, \quad 0 \leq \varepsilon \leq 1, \quad 0 \leq \hbar < 1, \quad k \geq 0,$$

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where

$$\Psi(\varepsilon, \hbar, k, n) = (1 + n\varepsilon - \varepsilon) |(1 + k) - n(\hbar + k)| \Theta(n), \quad (2.3)$$

and $\Theta(n)$ is given by (1.14).

Equality holds for the function

$$f(z) = z - \frac{1 - \hbar}{\Psi(\varepsilon, \hbar, k, n)} z^n.$$

For convenience, we let

$$\Psi(\varepsilon, \hbar, k, 2) = (1 + \varepsilon) |1 - k - 2\hbar| \Theta(2) \quad (2.4)$$

and

$$\Theta(2) = \frac{(cm)m^\nu}{\Gamma(2 + \nu)E_{\nu,c}(m)},$$

unless otherwise stated.

3 Distortion Bounds and Extreme Points

The distortion bounds for functions in $\mathbb{L}_{\nu,c}^m(\varepsilon, \hbar, k)$ follow directly from Theorem 2.3 by standard arguments. Therefore, we state the following results without proof.

Theorem 6. *Let the function f defined by (1.6) belong to $\mathbb{L}_{\nu,c}^m(\varepsilon, \hbar, k)$. Then we have*

$$r - \frac{1 - \hbar}{\Psi(\varepsilon, \hbar, k, 2)} r^2 \leq |f(z)| \leq r + \frac{1 - \hbar}{\Psi(\varepsilon, \hbar, k, 2)} r^2, \quad |z| = r, \quad (3.1)$$

and

$$1 - \frac{2(1 - \hbar)}{\Psi(\varepsilon, \hbar, k, 2)} r \leq |f'(z)| \leq 1 + \frac{2(1 - \hbar)}{\Psi(\varepsilon, \hbar, k, 2)} r, \quad |z| = r. \quad (3.2)$$

Equalities are sharp for the function

$$f(z) = z - \frac{1 - \hbar}{\Psi(\varepsilon, \hbar, k, 2)} z^2,$$

where $\Psi(\varepsilon, \hbar, k, 2)$ is given by (2.4).

Next, we discuss the extreme points of the class.

Theorem 7. *The extreme points of $\mathbb{L}_{\nu,c}^m(\varepsilon, \hbar, k)$ are*

$$f_1(z) = z \quad \text{and} \quad f_n(z) = z - \frac{1 - \hbar}{\Psi(\varepsilon, \hbar, k, n)} z^n, \quad \text{for } n = 2, 3, 4, \dots \quad (3.3)$$

where $\Psi(\varepsilon, \hbar, k, n)$ is defined in (2.3). Then $f \in \mathbb{L}_{\nu, c}^m(\varepsilon, \hbar, k)$ if and only if

$$f(z) = \sum_{n=1}^{\infty} \omega_n f_n(z),$$

for $\omega_n \geq 0$ and $\sum_{n=1}^{\infty} \omega_n = 1$.

Proof. Let $f \in \mathbb{L}_{\nu, c}^m(\varepsilon, \hbar, k)$ be of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n,$$

with $a_n \geq 0$. By the defining condition of the class $\mathbb{L}_{\nu, c}^m(\varepsilon, \hbar, k)$, we have

$$\sum_{n=2}^{\infty} \Psi(\varepsilon, \hbar, k, n) a_n \leq 1 - \hbar.$$

Define

$$\omega_n = \frac{\Psi(\varepsilon, \hbar, k, n)}{1 - \hbar} a_n, \quad n \geq 2,$$

and

$$\omega_1 = 1 - \sum_{n=2}^{\infty} \omega_n.$$

Since $a_n \geq 0$ and $\Psi(\varepsilon, \hbar, k, n) > 0$, it follows that $\omega_n \geq 0$. Also,

$$\sum_{n=2}^{\infty} \omega_n = \frac{1}{1 - \hbar} \sum_{n=2}^{\infty} \Psi(\varepsilon, \hbar, k, n) a_n \leq 1,$$

which implies $\omega_1 \geq 0$. Hence,

$$\sum_{n=1}^{\infty} \omega_n = 1.$$

Now, using (3.3), we obtain

$$\sum_{n=1}^{\infty} \omega_n f_n(z) = \omega_1 z + \sum_{n=2}^{\infty} \omega_n \left(z - \frac{1 - \hbar}{\Psi(\varepsilon, \hbar, k, n)} z^n \right).$$

Simplifying,

$$= \left(\sum_{n=1}^{\infty} \omega_n \right) z - \sum_{n=2}^{\infty} \omega_n \frac{1 - \hbar}{\Psi(\varepsilon, \hbar, k, n)} z^n.$$

Since $\sum \omega_n = 1$, this reduces to

$$z - \sum_{n=2}^{\infty} a_n z^n = f(z),$$

by the definition of ω_n . Hence,

$$f(z) = \sum_{n=1}^{\infty} \omega_n f_n(z).$$

Conversely, suppose that

$$f(z) = \sum_{n=1}^{\infty} \omega_n f_n(z),$$

where $\omega_n \geq 0$ and $\sum \omega_n = 1$. Then,

$$f(z) = z - \sum_{n=2}^{\infty} \omega_n \frac{1 - \hbar}{\Psi(\varepsilon, \hbar, k, n)} z^n,$$

so that

$$a_n = \omega_n \frac{1 - \hbar}{\Psi(\varepsilon, \hbar, k, n)}, \quad n \geq 2.$$

Thus,

$$\sum_{n=2}^{\infty} \Psi(\varepsilon, \hbar, k, n) a_n = (1 - \hbar) \sum_{n=2}^{\infty} \omega_n \leq 1 - \hbar.$$

Hence, $f \in \mathbb{L}_{\nu, c}^m(\varepsilon, \hbar, k)$.

Therefore, the functions $f_n(z)$ given in (3.3) are the extreme points of the class, and every function in the class can be represented as their convex combination. $\square$

4 Closure Theorem

Let the functions $f_j(z)$ ($j = 1, 2$) be defined by

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n, \quad \text{for } a_{n,j} \geq 0, \quad z \in U. \quad (4.1)$$

Theorem 8. *Let the functions $f_j(z)$ ($j = 1, 2, \dots, m$) defined by (4.1) belong to the classes $\mathbb{L}_{\nu, c}^m(\varepsilon, \hbar_j, k)$ ($j = 1, 2, \dots, m$), respectively. Then the function $h(z)$ defined by*

$$h(z) = z - \frac{1}{m} \sum_{n=2}^{\infty} \left(\sum_{j=1}^m a_{n,j} \right) z^n$$

belongs to the class $\mathbb{L}_{\nu,c}^m(\varepsilon, \hbar, k)$, where

$$\hbar = \min_{1 \leq j \leq m} \{\hbar_j\}, \quad \text{with } -1 \leq \hbar_j < 1.$$

Proof. Since $f_j(z) \in \mathbb{L}_{\nu,c}^m(\varepsilon, \hbar_j, k)$ ($j = 1, 2, \dots, m$), by applying Theorem 4, we observe that

$$\begin{aligned} \sum_{n=2}^{\infty} \Psi(\varepsilon, \hbar, k, n) \left(\frac{1}{m} \sum_{j=1}^m a_{n,j} \right) &= \frac{1}{m} \sum_{j=1}^m \left(\sum_{n=2}^{\infty} \Psi(\varepsilon, \hbar, k, n) a_{n,j} \right) \\ &\leq \frac{1}{m} \sum_{j=1}^m (1 - \hbar_j) \\ &\leq 1 - \hbar. \end{aligned}$$

Here, $\Psi(\varepsilon, \hbar, k, n)$ is defined in (2.3), which, in view of Theorem 4, implies that $h(z) \in \mathbb{L}_{\nu,c}^m(\varepsilon, \hbar, k)$.

This completes the proof. $\square$

5 Integral Transform of the Class $\mathbb{L}_{\nu,c}^m(\varepsilon, \hbar, k)$

In this section, we prove that the class $\mathbb{L}_{\nu,c}^m(\varepsilon, \hbar, k)$ is closed under an integral transform.

For $f \in \mathcal{A}$, we define the integral transform

$$\mathcal{V}_{\chi}(f)(z) = \int_0^1 \chi(t) \frac{f(tz)}{t} dt,$$

where χ is a real-valued, non-negative weight function normalized so that

$$\int_0^1 \chi(t) dt = 1.$$

Some special cases of $\chi(t)$ are particularly interesting, such as $\chi(t) = (1+c)t^c$, $c > -1$, for which $\mathcal{V}_{\chi}$ is known as the Bernardi operator, and

$$\chi(t) = \frac{(c+1)^{\delta}}{\chi(\delta)} t^c \left(\log \frac{1}{t} \right)^{\delta-1}, \quad c > -1, \delta \geq 0,$$

which gives the Komatu operator. For more details, see [5].

First, we show that the class $\mathbb{L}_{\nu,c}^m(\varepsilon, \hbar, k)$ is closed under $\mathcal{V}_{\chi}(f)(z)$.

Theorem 9. *Let $f \in \mathbb{L}_{\nu,c}^m(\varepsilon, \hbar, k)$. Then $\mathcal{V}_{\chi}(f)(z) \in \mathbb{L}_{\nu,c}^m(\varepsilon, \hbar, k)$.*

Proof. By definition, we have

$$\begin{aligned}\mathcal{V}_\chi(f)(z) &= \frac{(c+1)^\delta}{\chi(\delta)} \int_0^1 (-1)^{\delta-1} t^c (\log t)^{\delta-1} \left(z - \sum_{n=2}^{\infty} |a_n| z^n t^{n-1} \right) dt \\ &= \frac{(-1)^{\delta-1} (c+1)^\delta}{\chi(\delta)} \lim_{r \rightarrow 0^+} \left[\int_r^1 t^c (\log t)^{\delta-1} \left(z - \sum_{n=2}^{\infty} |a_n| z^n t^{n-1} \right) dt \right].\end{aligned}$$

A simple calculation gives

$$\mathcal{V}_\chi(f)(z) = z - \sum_{n=2}^{\infty} \left(\frac{c+1}{c+n} \right)^\delta |a_n| z^n.$$

We need to prove that

$$\sum_{n=2}^{\infty} \frac{\Psi(\varepsilon, \hbar, k, n)}{1-\hbar} \left(\frac{c+1}{c+n} \right)^\delta |a_n| \leq 1. \quad (5.1)$$

On the other hand, by Theorem 4, $f \in \mathbb{L}_{\nu, c}^m(\varepsilon, \hbar, k)$ if and only if

$$\sum_{n=2}^{\infty} \frac{\Psi(\varepsilon, \hbar, k, n)}{1-\hbar} |a_n| \leq 1,$$

where $\Psi(\varepsilon, \hbar, k, n)$ is defined in (2.3). Hence, $\frac{c+1}{c+n} < 1$. Therefore, (5.1) holds, and the proof is complete. $\square$

The above theorem yields the following two special cases.

Theorem 10. *If $f(z)$ is starlike of order $\hbar$, then $\mathcal{V}_\chi(f)(z)$ is also starlike of order $\hbar$.*

Theorem 11. *If $f(z)$ is convex of order $\hbar$, then $\mathcal{V}_\chi(f)(z)$ is also convex of order $\hbar$.*

Theorem 12. *Let $f \in \mathbb{L}_{\nu, c}^m(\varepsilon, \hbar, k)$. Then $\mathcal{V}_\chi(f)(z)$ is starlike of order $0 \leq \xi < 1$ in $|z| < R_1$, where*

$$R_1 = \inf_{n \geq 2} \left[\left(\frac{c+n}{c+1} \right)^\delta \frac{(1-\xi)\Psi(\varepsilon, \hbar, k, n)}{(n-\xi)(1-\hbar)} \right]^{\frac{1}{n-1}},$$

and $\Psi(\varepsilon, \hbar, k, n)$ is defined in (2.3).

Proof. It is sufficient to prove

$$\left| \frac{z(\mathcal{V}_\chi(f)(z))'}{\mathcal{V}_\chi(f)(z)} - 1 \right| < 1 - \xi. \quad (5.2)$$

For the left-hand side of (5.2), we have

$$\begin{aligned} \left| \frac{z(\mathcal{V}_\chi(f)(z))'}{\mathcal{V}_\chi(f)(z)} - 1 \right| &= \left| \frac{\sum_{n=2}^{\infty} (1-n) \left(\frac{c+1}{c+n} \right)^\delta a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} \left(\frac{c+1}{c+n} \right)^\delta a_n z^{n-1}} \right| \\ &\leq \frac{\sum_{n=2}^{\infty} (1-n) \left(\frac{c+1}{c+n} \right)^\delta |a_n| |z|^{n-1}}{1 - \sum_{n=2}^{\infty} \left(\frac{c+1}{c+n} \right)^\delta |a_n| |z|^{n-1}}. \end{aligned}$$

The last expression is less than $(1 - \xi)$, since

$$|z|^{n-1} < \left(\frac{c+n}{c+1} \right)^\delta \frac{(1-\xi)\Psi(\varepsilon, \hbar, k, n)}{(n-\xi)(1-\hbar)}.$$

Therefore, the proof is complete. $\square$

Using the fact that $f(z)$ is convex if and only if $zf'(z)$ is starlike, we obtain the following result.

Theorem 13. *Let $f \in \mathbb{L}_{\nu, c}^m(\varepsilon, \hbar, k)$. Then $\mathcal{V}_\chi(f)(z)$ is convex of order $0 \leq \xi < 1$ in $|z| < R_2$, where*

$$R_2 = \inf_{n \geq 2} \left[\left(\frac{c+n}{c+1} \right)^\delta \frac{(1-\xi)\Psi(\varepsilon, \hbar, k, n)}{n(n-\xi)(1-\hbar)} \right]^{\frac{1}{n-1}},$$

and $\Psi(\varepsilon, \hbar, k, n)$ is defined in (2.3).

6 Neighbourhood Results

In this section, we establish neighbourhood results of the class $\mathbb{L}_{\nu, c}^m(\varepsilon, \hbar, k)$. Following [4, 13], we define the δ -neighbourhood of a function $f \in T$ by

$$N_\delta(f) := \left\{ h \in T : h(z) = z - \sum_{n=2}^{\infty} |d_n| z^n \text{ and } \sum_{n=2}^{\infty} n|a_n - d_n| \leq \delta \right\}. \quad (6.1)$$

Particularly, for the identity function $e(z) = z$, we have

$$N_\delta(e) := \left\{ h \in T : h(z) = z - \sum_{n=2}^{\infty} |d_n| z^n \text{ and } \sum_{n=2}^{\infty} n|d_n| \leq \delta \right\}. \quad (6.2)$$

Theorem 14. *If*

$$\delta := \frac{2(1 - \hbar)}{\Psi(\varepsilon, \hbar, k, 2)}, \quad (6.3)$$

then $\mathbb{L}_{\nu, c}^m(\varepsilon, \hbar, k) \subset N_\delta(e)$, where $\Psi(\varepsilon, \hbar, k, 2)$ is defined in (2.4).

Proof. For $f \in \mathbb{L}_{\nu, c}^m(\varepsilon, \hbar, k)$, Theorem 4 immediately yields

$$\Psi(\varepsilon, \hbar, k, 2) \sum_{n=2}^{\infty} |a_n| \leq 1 - \hbar,$$

so that

$$\sum_{n=2}^{\infty} |a_n| \leq \frac{1 - \hbar}{\Psi(\varepsilon, \hbar, k, 2)}. \quad (6.4)$$

Next, using the elementary estimate

$$\sum_{n=2}^{\infty} n|a_n| \leq 2 \sum_{n=2}^{\infty} |a_n|,$$

(which is valid since $n \geq 2$ for every term), we combine this with (6.4) to obtain

$$\sum_{n=2}^{\infty} n|a_n| \leq 2 \frac{1 - \hbar}{\Psi(\varepsilon, \hbar, k, 2)} = \delta.$$

Therefore, the above estimate yields

$$f \in N_\delta(e).$$

Hence,

$$\mathbb{L}_{\nu, c}^m(\varepsilon, \hbar, k) \subset N_\delta(e),$$

which completes the proof. $\square$

Now we determine the neighbourhood for the class $\mathbb{L}_{\nu, c}^m(\rho, \varepsilon, \hbar, k)$, which we define as follows.

Definition 15. *A function $f \in T$ is said to belong to the class $\mathbb{L}_{\nu, c}^m(\rho, \varepsilon, \hbar, k)$ if there exists a function $h \in \mathbb{L}_{\nu, c}^m(\rho, \varepsilon, \hbar, k)$ such that*

$$\left| \frac{f(z)}{h(z)} - 1 \right| < 1 - \rho, \quad (z \in U, 0 \leq \rho < 1). \quad (6.5)$$

Theorem 16. If $h \in \mathbb{L}_{\nu,c}^m(\rho, \varepsilon, \hbar, k)$ and

$$\rho = 1 - \frac{\delta \Psi(\varepsilon, \hbar, k, 2)}{2 [\Psi(\varepsilon, \hbar, k, 2) - (1 - \hbar)]}, \quad (6.6)$$

then

$$N_\delta(h) \subset \mathbb{L}_{\nu,c}^m(\rho, \varepsilon, \hbar, k), \quad (6.7)$$

where $\Psi(\varepsilon, \hbar, k, 2)$ is defined in (2.4).

Proof. Suppose that $f \in N_\delta(h)$. Then, from (6.1), we have

$$\sum_{n=2}^{\infty} n |a_n - d_n| \leq \delta,$$

which implies the coefficient inequality

$$\sum_{n=2}^{\infty} |a_n - d_n| \leq \frac{\delta}{2}.$$

Next, since $h \in \mathbb{L}_{\nu,c}^m(\varepsilon, \hbar, k)$, we have

$$\sum_{n=2}^{\infty} d_n = \frac{1 - \hbar}{\Psi(\varepsilon, \hbar, k, 2)},$$

so that

$$\begin{aligned} \left| \frac{f(z)}{h(z)} - 1 \right| &< \frac{\sum_{n=2}^{\infty} |a_n - d_n|}{1 - \sum_{n=2}^{\infty} d_n} \\ &\leq \frac{\delta}{2} \left[\frac{\Psi(\varepsilon, \hbar, k, 2)}{\Psi(\varepsilon, \hbar, k, 2) - (1 - \hbar)} \right] \\ &= 1 - \rho, \end{aligned}$$

provided that ρ is given precisely by (6.6). Thus, by Definition 15, we have $f \in \mathbb{L}_{\nu,c}^m(\rho, \varepsilon, \hbar, k)$. This completes the proof. $\square$

7 Integral Means Inequality

In [16], Silverman showed that the function $f_2(z) = z - \frac{z^2}{2}$ is often extremal over the family T . He used this function to resolve his integral means inequality, conjectured in [17] and proved in [18], namely that

$$\int_0^{2\pi} |f(re^{i\theta})|^\eta d\theta \leq \int_0^{2\pi} |f_2(re^{i\theta})|^\eta d\theta,$$

for all $f \in T$, $\eta > 0$, and $0 < r < 1$. In [18], he also proved his conjecture for the subclasses $T^*(\hbar)$, the class of starlike functions, and $\mathcal{C}(\hbar)$, the class of convex functions with negative coefficients.

We recall the following definition and lemma to establish our result on the integral means inequality.

Definition 17. [6] (*Subordination Principle*) Let g and h be analytic functions with $g(0) = h(0)$. Then g is said to be subordinate to h , denoted by $g \prec h$, if there exists an analytic function w such that $w(0) = 0$, $|w(z)| < 1$, and $g(z) = h(w(z))$ for all $z \in U$.

Lemma 18. [6] If the functions f and g are analytic in U with $g \prec f$, then for $\eta > 0$ and $0 < r < 1$,

$$\int_0^{2\pi} |g(re^{i\theta})|^\eta d\theta \leq \int_0^{2\pi} |f(re^{i\theta})|^\eta d\theta. \quad (7.1)$$

Applying Lemma 18, Theorem 4, and Theorem 7, we prove Silverman's conjecture for functions in the family $\mathbb{L}_{\nu,c}^m(\varepsilon, \hbar, k)$.

Theorem 19. Suppose that $f \in \mathbb{L}_{\nu,c}^m(\varepsilon, \hbar, k)$, where $0 \leq \varepsilon \leq 1$, $0 \leq \hbar < 1$, and $k \geq 0$. Let $f_2(z)$ be defined by

$$f_2(z) = z - \frac{1 - \hbar}{\Psi(\varepsilon, \hbar, k, 2)} z^2,$$

where $\Psi(\varepsilon, \hbar, k, 2)$ is given in (2.4). Then, for $z = re^{i\theta}$ with $0 < r < 1$, we have

$$\int_0^{2\pi} |f(z)|^\eta d\theta \leq \int_0^{2\pi} |f_2(z)|^\eta d\theta. \quad (7.2)$$

Proof. For $f \in T$, inequality (7.2) is equivalent to proving that

$$\int_0^{2\pi} \left| 1 - \sum_{n=2}^{\infty} a_n z^{n-1} \right|^\eta d\theta \leq \int_0^{2\pi} \left| 1 - \frac{1 - \hbar}{\Psi(\varepsilon, \hbar, k, 2)} z \right|^\eta d\theta.$$

By Lemma 18, it suffices to show that

$$1 - \sum_{n=2}^{\infty} |a_n| z^{n-1} \prec 1 - \frac{1 - \hbar}{\Psi(\varepsilon, \hbar, k, 2)} z.$$

Set

$$1 - \sum_{n=2}^{\infty} |a_n| z^{n-1} = 1 - \frac{1 - \hbar}{\Psi(\varepsilon, \hbar, k, 2)} w(z). \quad (7.3)$$

Using (2.1), we obtain

$$\begin{aligned} |w(z)| &= \left| \sum_{n=2}^{\infty} \frac{\Psi(\varepsilon, \hbar, k, n)}{1 - \hbar} a_n z^{n-1} \right| \\ &\leq |z| \sum_{n=2}^{\infty} \frac{\Psi(\varepsilon, \hbar, k, n)}{1 - \hbar} |a_n| \\ &\leq |z|. \end{aligned}$$

Hence $|w(z)| < 1$ for $|z| < 1$, which completes the proof. $\square$

Conclusion. In this paper, a new subclass of analytic functions associated with a generalized linear operator involving a special case of the classical two-parameter Mittag–Leffler function has been introduced and investigated. By employing suitable coefficient techniques, a necessary and sufficient coefficient characterization for the proposed class has been established. Furthermore, several geometric properties of the class have been obtained, including coefficient estimates, extreme points, distortion inequalities, closure properties, and results involving the Hadamard product and neighborhood results. These results extend and unify several earlier investigations available in the literature. It is expected that the techniques developed in this work will be useful in the study of other subclasses of analytic and univalent functions associated with generalized integral and differential operators, as well as special functions.

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