GENERATING ELLIPSES BY USING MECHANISMS RELYING ON SIMPLE GEOMETRICAL ASPECTS

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ABSTRACT: An ellipse obtained as the intersection of a cone with an inclined plane. Starting from the ordinary equation of an ellipse, one performs the analogy with a basic equation from Trigonometry. The coordinates of the ellipse leading point are determined, building a simple geometrical figure. This figure allows for the synthesis of the leading mechanism. Three mechanisms generating the same ellipses were achieved in this way. Geometric construction, the generating ellipses, the mechanism structure, the kinematics of the mechanism are presented along with the succesice positions of the generating mechanisms.

KEY WORDS: ellipse generation, mechanisms used to generate ellipse

1. INTRODUCTION

Ellipses are curves known since antiquity. The easiest way to generate them consists in using a rope linke to 2 fixed nails (foci) which is controlled by a mobile nail and therefore can plot the gardnerțs ellipse. Other methods used to generate ellipses are known as well. The best contributions in this sense belong to Prof. Artobolevskii [1], otherwise authoring many volumes where the generation of ellipses with mechanisms from modern technique is approached. All the aforementioned mechanism rely on geometrical considerations. They have a variable degree of complexity. General rules applicable to mechanisms able to generate many other curves are applied. A mechanism relying on the gardner's ellipse is presented in [2]. It makes use of an arc whose coordinates are computed. An historical study on the mechanisms generating curves (including ellipses) is provided by [3]. Several examples for ellipses'generating mechanisms are presented in [4]. One can find detalis on ellipses, accompanied by equations, graphical examples, generation modality, in [5]. The conical curves are studied in detail in [6], where animations are provided as well. Further on we are dealing with three mechanisms used to generate ellipses by using simple geometrical considerations.

2. THE FIRST ELLIPSOGRAPH

Starting from the ellipse equation (1) and the trigonometrical equation (2), Eqs. (3) and (4) are obtained. From them on can calculate the coordinates of the points x and y and the leading point.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad (1)$$

$$sin^2\varphi+cos^2\varphi=1 \qquad (2)$$

$$sin^2\varphi = \frac{x^2}{a^2} \quad (3)$$

$$cos^2\varphi = \frac{y^2}{b^2} \quad (4)$$

Two alike rectangular triangles are built $\Delta ACB \sim \Delta AME$, (Figure 1.a). Furtheron the following equations can be written, while considering a=AB, b=AE:

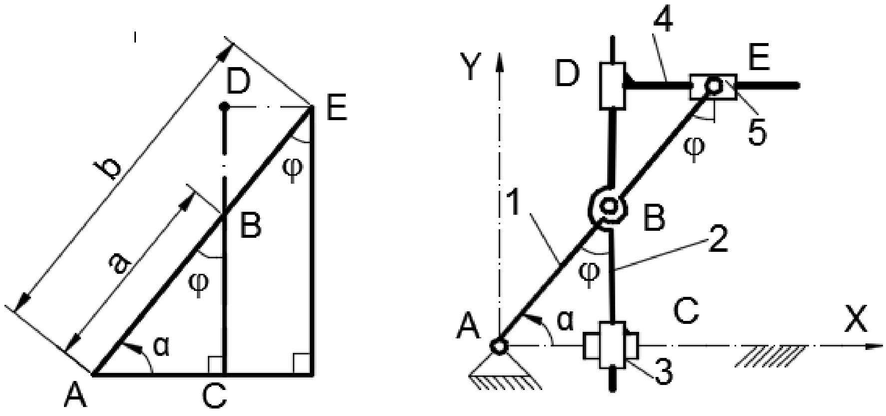
$$\begin{cases} x_B = AB.\cos\alpha = x_C \\ y_B = AB.\sin\alpha \end{cases} \quad (5)$$

$$\begin{cases} x_E = AE.\cos\alpha \\ y_E = AE.\sin\alpha \end{cases} \quad (6)$$

$$\begin{cases} x_D = x_C = a.\sin\varphi \\ y_D = y_E = b.\cos\varphi \end{cases} \quad (7)$$

$$\varphi = 90^0 - \alpha \quad (8)$$

The coordinates for B are obtained from the equation (5), whilst eq. (6) provides the coordinates of E and eq. (7) is used to determine the coordinates for the leading point D. Therefore one can get the angles correlation from (8).
Based on this, the generating mechanism from Figure 1.b. could be achieved.



a. b.

Figure 1. a. Generation Geometry; b. The generating mechanism.

Given the lenghts a and b one can get the point B and E. The abscissa for the ellipse's generating point is provided by the segment AC, computed from (7). The corresponding ordinate is given by (7), corresponding to the equations (3) and (4). In order to have the element CD moving along the abscissa and D moving along a line parallel to the ordinate, a coulisse was introduced in C whilst a 90^0 angle is obtained in D. The coulisse from E allows a sliding of this point along the element no. 4, pushing D along a vertical direction.
Figure 2 provides a structure for the mechanism, whose type is R-RPP-RPP.

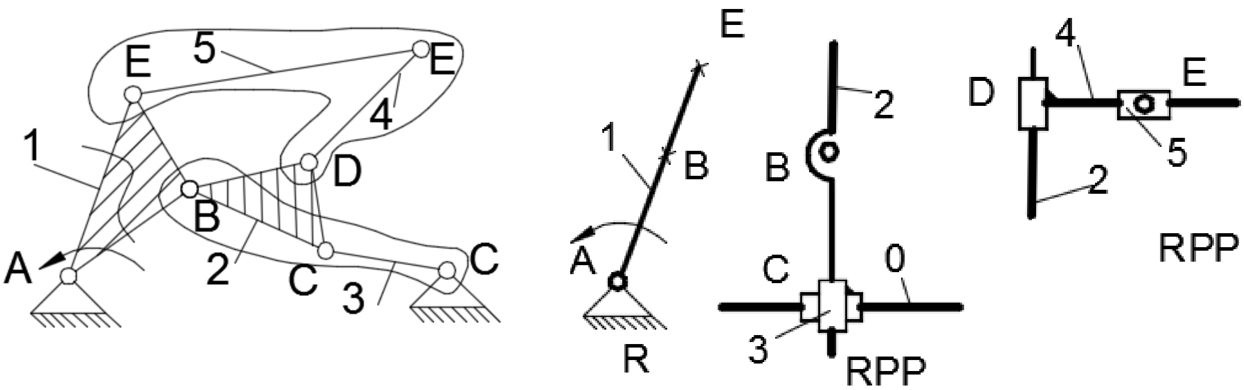
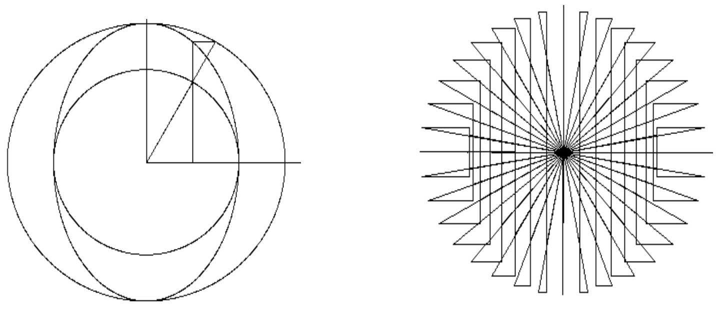


Figure 2. Structure of an ellipsograph mechanism.

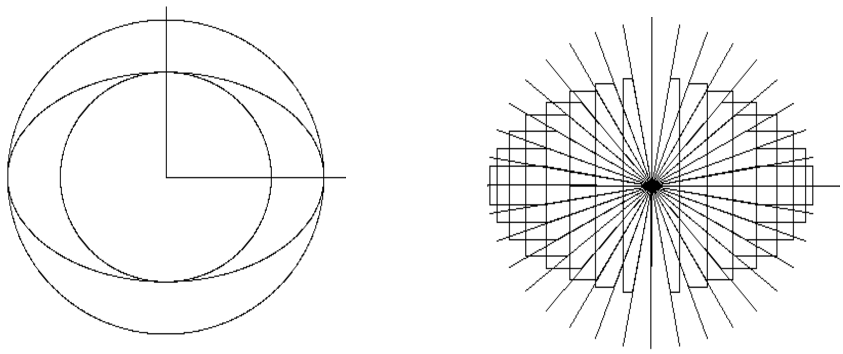
The ellipse depicted by Figure 3.a was obtained for a=40 and b=60 (mm). The mechanism is also presented in a certain position in this figure. Figure 3.b depicts successive position of the same mechanism. The circles plotted by B and E are described as well. The ellipse's longer axis is along the ordinate whilst the shortest axis is along the abscissa.



a. b.

Figure 3. a. Ellipse generated by the leading point D, for a=40; b=60;
b. The mechanism in successive positions

Figure 4.a depicts the ellipse obtained for a=60 and b=40 (mm). Similar to the previous case, the mechanism is also presented in a certain position in this figure. Figure 4.b depicts successive position of the same mechanism. The circles plotted by B and E are described as well and again the ellipse's longer axis is along the ordinate

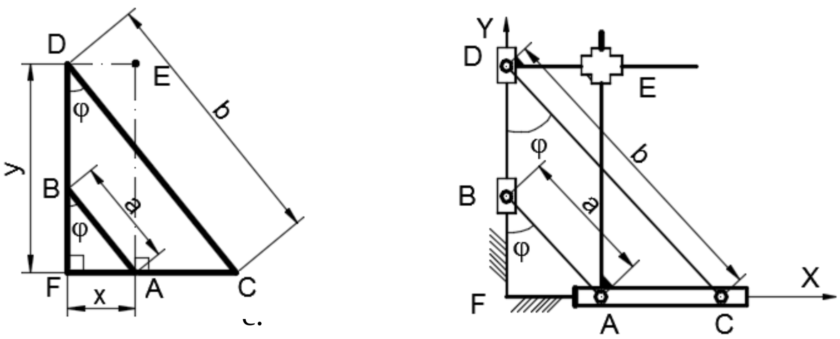


a. b.

Figure 4. a. Ellipse generated by the leading point D, for a=60; b=40;
b. The mechanism in successive positions

3. THE SECOND ELLIPSOGRAPH

Figure 5.a. was built based on the equations (1)…(4), where two alike triangles can be noticed. Based on these relations, one can see in Figure 5.a that the point E is plotting an ellipse. The synthesis of the generating mechanism from Figure. 5.b was performed. The leading element is the coulisse AC. B is sliding along the ordinate and requires the coulisse from this point. A similar constraint is valide for the coulisse from the point D. The leading point of the ellipse will be the point E. Its abscissa is the same with the abscissa of A and its ordinate is identic to the ordinate of D. Therefore it is placed at the intersection between the directions AE and DE. From a structural point of view, the mechanism consists of the leading element with a translation movement AC, the dyade AB of type RRP, the dyade CD of type RRP and the double coulisse from E which has a parasitic structure and forces the ordinate y to have the length x. In other words, it materializes the intersection of two lines.



a. b.

Figure 5. a. Geometry of generation; b. Generating mechanism

Based on Figure 5.b one can write the following equations:

$$\begin{cases} \sin\varphi = \frac{x}{a} \\ \cos\varphi = \sqrt{1-\sin^2\varphi} \end{cases} \quad (9)$$

$$y_B = a.\cos\varphi \quad (10)$$

$$y_D = y_E = b.\cos\varphi \quad (11)$$

$$x_C = b.\sin\varphi \quad (12)$$

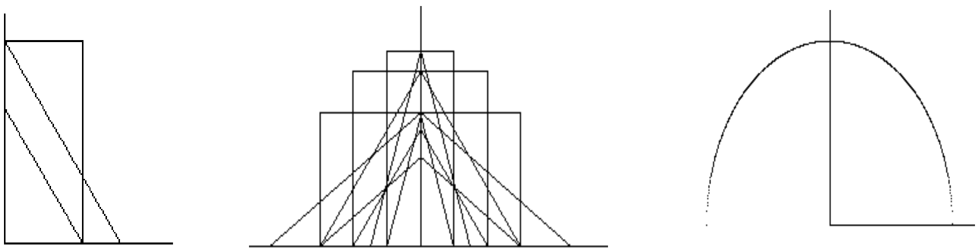
Is cycling $x=x_A$. As long as the sign $S=\pm 1$, Eqs. (3), two solutions are possible.

Figure 6.a depicts the ellipse obtained for a=40 and b=60 (mm). Similar to the previous case, the mechanism is also presented in a certain position in this figure and Figure 6.b depicts successive positions.

Only the positions corresponding to the mechanism operating on top of the abscissa were obtained for S=+1.

Actually the operating cycle is limitted to the value of „a", which occurs in the equation (9). Therefore the domain of x is [-a, a].

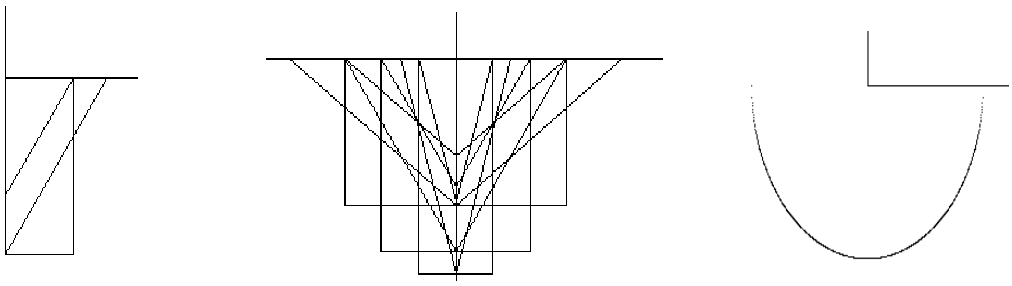
Figure 6.c depicts half of the ellipse for S=+1.



a. b. c.

Figure 6. For S=+1. a. The mechanism in a particular position;
b. Successive positions of the mechanism; c. Generated ellipse.

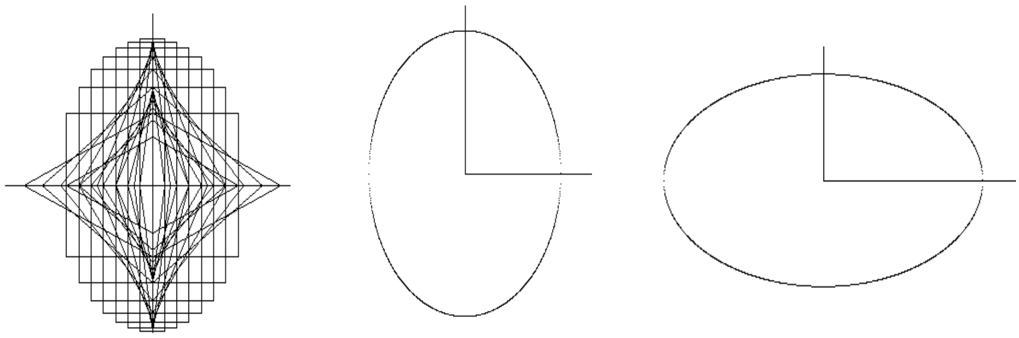
When S=–1 one obtained: the mechanism in the position from Figure 7.a, the succesive positions from Figure 7.b and the curve from Figure 7.c.



a. b. c.

Figure 7. For S=–1. a. The mechanism in a particular position;
b. Successive positions of the mechanism; c. Generated ellipse.

When the point B is alligned along the same line with A and C, the inertia allows for a top-down crossing. In other words, it crosses from one solution toward the other, S=± 1. Figure 8.a depicts the successive solutions in this case whilst Figure 8.b depicts the full ellipse.



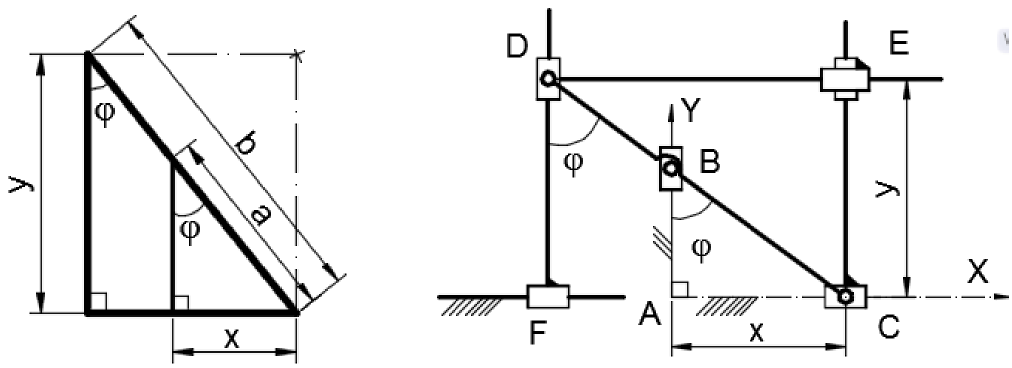
a. b. c.

Figure 8 . The case S=± 1. a. Succesive positions of the mechanism;
b. Ellipse generated for a < b; c. Horizontal ellipse when a > b

One can notice that when φ is around 0^0 and 180^0 the curve presents certain discontinuities and B is aligned with A and C. If a=60>b=40, an horizontal ellipse is obtained (Figure 8.c).

4. THE THIRD ELLIPSOGRAPH

Similar to the previous cases, the ellipse equations is used as starting point, but now "b" is considered in prolongation of a, as depicted in Figure 9.a. The coulisse from C represents the leading element. Other coulisses were placed in B and D such as to allow these points to move along directions parallel to the ordinate. A double coulisse was placed in E in order to perform a materialization of the point E as intersection of two variable lines. From a structural point of view, the mechanism consists in: the leading element with translation movement of C, the dyade CB (of type RRP), the dyade DF (of type RPP) and the element E which has a parasitic structure and performs the materialization of the directions whose intersection is represented by the leading point E. The mechanism provides the coordinate x (that is AC) along with the coordinate $y=y_D$ which is forced by the double coulisse E to slide for a given x.



a. b.

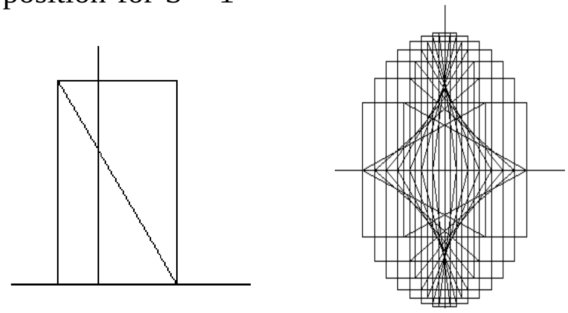
Figure 9. a. Generation geometry; b. Generating mechanism.

One has to consider Figure 9.b and the equations (9)…(12). The following equations have to be considered too:

$$x_E = x \qquad (13)$$

$$x_D = -(b-a).sin\varphi \qquad (14)$$

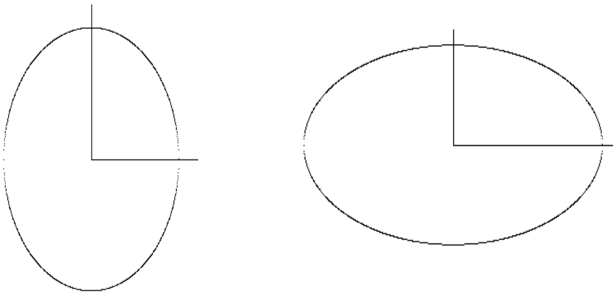
Figure 10.a depicts the ellipsograph obtained for a=60 and b=40 (mm). Figure 10.a depicts the mechanism in a certain position for S=+1 Figure 10.b depicts successive positions when S= ± 1.



a. b.

Figure 10. a. The mechanism in a certain position for S=+1;
b. Succesive positions of the mechanism for S= ± 1.

The full ellipses generated for S= ± 1 are depicted by Figure 11.a, when a=40 and b=60. Figure 11.b is dedicated to the case when a=60; b=40.



a. b.

Figure 11. Ellipse generated for S= ± 1: a. when a=40; b=60; b. when a=60; b=40

5. CONCLUSION

Starting from the ellipse's equation and a trigonometric relation, one deduced the coordinated of the leading point for ellipses by considering a simple geometric figure. Based on it, an ellipses' generating mechanism could be built. The second generating mechanism was obtained in a similar manner. It is unlike the first mechanism but can be used to generate the same types of ellipses. The third mechanism, different from the other two was also built to generate the same ellipses. It was demonstrated that ellipses generating mechanisms can be obtained easily, starting from simple geometrical considerations. Ellipses and successive positions of the generating mechanisms were generated.

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